# A Method of Determination of an Acquisition Program in Order to Maximize the Total Utility Using Linear Programming in Integer Numbers

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**Abstract.** This paper solves in a different way the problem of maximization of the total utility using the linear programming in integer numbers. The author uses the diofantic equations (equations in integers numbers) and after a decomposing in different cases, he obtains the maximal utility.

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## A method of maximization the total utility

Let a consumer which has a budget of acquision of r goods $G_1,...,G_r$, in value of S u.m. The prices of the r goods $G_i$, $i=\overline{1,r}$ are $p_i$, $i=\overline{1,r}$. The marginal utlities corresponding to an arbitrary number of doses are in the following table:

| No. of dose | $U_{m1}$ | … | $U_{mr}$ |
|---|---|---|---|
| 1 | $u_{11}$ | … | $u_{1r}$ |
| … | … | … | … |
| i | $u_{i1}$ | … | $u_{ir}$ |
| … | … | … | … |
| n | $u_{n1}$ | … | $u_{nr}$ |

We propose, in what follows, the determination of the number of doses $a_i$ from the good $G_i$, $i=\overline{1,r}$ such that the total utility: $U_t=\sum_{j=1}^{r}\sum_{i=1}^{a_j}u_{ij}$ be maximal.

Let note: $x_{ij}=\begin{cases}1 \text{ if the i-th dose from the good j is used}\\0 \text{ if the i-th dose from the good j is not used}\end{cases}$

Because the impossibility of using the (i-1)-th dose involved the existence's impossibility of the i-th dose, we shall put the condition that: $x_{ij}\in\mathbf{N}$, $0\le x_{ij}\le x_{i-1,j}$ for $i>1$ and $j=\overline{1,r}$.

We have also: $\sum_{i=1}^{n}\sum_{j=1}^{m}p_j x_{ij}\le S$.

The problem consists in the determination of $x_{ij}$ such that to have max $\sum_{j=1}^{r}\sum_{i=1}^{n}u_{ij}x_{ij}$.

The problem is therefore:

$$(1)\qquad \begin{cases}\max\sum_{j=1}^{r}\sum_{i=1}^{n}u_{ij}x_{ij}\\ \sum_{i=1}^{n}\sum_{j=1}^{m}p_j x_{ij}\le S\\ x_{ij}\le x_{i-1,j},\ i=\overline{2,n},\ j=\overline{1,r}\\ x_{ij}\le 1,\ i=\overline{1,n},\ j=\overline{1,r}\\ x_{ij}\ge 0,\ i=\overline{1,n},\ j=\overline{1,r}\end{cases}$$

Finally we shall have: $a_j=\sum_{i=1}^{n}x_{ij}$, $j=\overline{1,r}$.

Because the problem (1) is in integer numbers, we shall apply the algorithm of Gomory.

After the solving of (1) using the Simplex algorithm, we shall have two cases:

<u>Case 1</u>

If $\overline{x}_{ij}\in\mathbf{N}$, $i=\overline{1,n}$, $j=\overline{1,r}$ the problem is completely solved.

<u>Case 2</u>

If $\exists\,\overline{x}_{kp}\notin\mathbf{N}$, $k=\overline{1,n}$, $p=\overline{1,r}$ the variable $\overline{x}_{kp}$ is obvious in the basis.

In this case, let note $y_{kpts}$ the element of the Simplex table at the intersection of $x_{kp}$-row with $x_{ts}$-column. In order to simplify the notations, let: $v_{kpts}=\{y_{kpts}\}\in[0,1)$, $v_{kp}=\{\overline{x}_{kp}\}\in[0,1)$ the fractional part of these quantities, $B=\{(g,h)\mid x_{gh}$ is a basis variable$\}$ and $S=\{(t,s)\mid x_{ts}$ is not a basis variable$\}$.

We have now, from: $x_{gh}=\overline{x}_{gh}-\sum_{(t,s)\in S}y_{ghts}x_{ts}\ \ \forall(g,h)\in B$:

(2) $$x_{kp}=\overline{x}_{kp}-\sum_{(t,s)\in S}y_{kpts}x_{ts}=\left[\overline{x}_{kp}\right]+v_{kp}-\sum_{(t,s)\in S}\left[y_{kpts}\right]x_{ts}-\sum_{(t,s)\in S}v_{kpts}x_{ts}$$

We can write (2) also in the form:

(3) $$x_{kp}-\left[\overline{x}_{kp}\right]+\sum_{(t,s)\in S}\left[y_{kpts}\right]x_{ts}=v_{kp}-\sum_{(t,s)\in S}v_{kpts}x_{ts}$$

In order that the problem has integer solution it therefore necessary and sufficient that: $x_{kp}-\left[\overline{x}_{kp}\right]+\sum_{(t,s)\in S}\left[y_{kpts}\right]x_{ts}\in\mathbf{Z}$ or, in other words: $v_{kp}-\sum_{(t,s)\in S}v_{kpts}x_{ts}\in\mathbf{Z}$.

Let now:

(4) $$v=v_{kp}-\sum_{(t,s)\in S}v_{kpts}x_{ts}$$

from where:

(5) $$\sum_{(t,s)\in S}v_{kpts}x_{ts}=v_{kp}-v,\ v\in\mathbf{Z}$$

From the hypotesis, $v_{kpts}$, $v_{kp}\in[0,1)$ and $\sum_{(t,s)\in S}v_{kpts}x_{ts}\geq0$ from the positive character of variables.

We have now three cases:

<u>Case 2.1</u>

If $v>0$ we have $v\in\mathbf{N}^*$ therefore $0\leq\sum_{(t,s)\in S}v_{kpts}x_{ts}=v_{kp}-v$. From this: $v_{kp}\geq v\geq1$ – contradiction with the choice of $v_{kp}$.

<u>Case 2.2</u>

If $v=0$ we have that $\sum_{(t,s)\in S}v_{kpts}x_{ts}=v_{kp}\geq v_{kp}$.

Case 2.3

If v<0 we have from the condition that v is integer: v≤-1 which implies: -v≥1. Finally: $\sum_{(t,s)\in S} v_{kpts} x_{ts}$ =$v_{kp}$-v≥$v_{kp}$+1>$v_{kp}$>0.

From these cases, we have that the condition to be integer for $x_{kp}$ is: $\sum_{(t,s)\in S} v_{kpts} x_{ts}$ ≥$v_{kp}$.

After all these considerations, making the notation: y= $\sum_{(t,s)\in S} v_{kpts} x_{ts}$ -$v_{kp}$ we shall obtain the new problem:

$$(6)\quad \begin{cases} \max \sum_{j=1}^{r} \sum_{i=1}^{n} u_{ij} x_{ij} \\ \sum_{i=1}^{n} \sum_{j=1}^{m} p_j x_{ij} \le S \\ y - \sum_{(t,s)\in S} v_{kpts} x_{ts} = -v_{kp} \\ x_{ij} \le x_{i-1,j},\ i = \overline{2,n},\ j = \overline{1,r} \\ x_{ij} \le 1,\ i = \overline{1,n},\ j = \overline{1,r} \\ x_{ij} \ge 0,\ i = \overline{1,n},\ j = \overline{1,r},\ y \ge 0 \end{cases}$$

If the problem (6) will has at finally an integer solution the problem will be completely solved. If not, we shall resume the upper steps.

**Example**

| No. of dose | $U_{mx}$ | $U_{m3}$ | $U_{mz}$ |
|---|---|---|---|
| 1 | 10 | 12 | 15 |
| 2 | 8 | 10 | 12 |
| 3 | 7 | 5 | 10 |
| 4 | 6 | 2 | 7 |

$p_x$=6, $p_y$=5, $p_z$=4, S=50.

The linear programming problem is:

$$\begin{cases} \max(10x_{11}+8x_{21}+7x_{31}+6x_{41}+12x_{12}+10x_{22}+5x_{32}+2x_{42}+15x_{13}+12x_{23}+10x_{33}+7x_{43}) \\ 6(x_{11}+x_{21}+x_{31}+x_{41})+5(x_{12}+x_{22}+x_{32}+x_{42})+4(x_{13}+x_{23}+x_{33}+x_{43}) \le 50 \\ x_{ij} \le x_{i-1,j}, i=\overline{2,4}, j=\overline{1,3} \\ x_{ij} \le 1, i=\overline{1,4}, j=\overline{1,3} \\ x_{ij} \ge 0, i=\overline{1,4}, j=\overline{1,3} \end{cases}$$

After the application of the Simplex algorithm we obtain:

$x_{11}$=1, $x_{21}$=1, $x_{31}$=1, $x_{41}$=1/6, $x_{12}$=1, $x_{22}$=1, $x_{32}$=1, $x_{42}$=0, $x_{13}$=1, $x_{23}$=1, $x_{33}$=1, $x_{43}$=1

We shall add the restriction:

$y-0,8x_{42}=-1/6$

and we obtain now the problem:

$$\begin{cases} \max(10x_{11}+8x_{21}+7x_{31}+6x_{41}+12x_{12}+10x_{22}+5x_{32}+2x_{42}+15x_{13}+12x_{23}+10x_{33}+7x_{43}) \\ 6(x_{11}+x_{21}+x_{31}+x_{41})+5(x_{12}+x_{22}+x_{32}+x_{42})+4(x_{13}+x_{23}+x_{33}+x_{43}) \le 50 \\ y-0,8x_{42}=-1/6 \\ x_{ij} \le x_{i-1,j}, i=\overline{2,4}, j=\overline{1,3} \\ x_{ij} \le 1, i=\overline{1,4}, j=\overline{1,3} \\ x_{ij} \ge 0, i=\overline{1,4}, j=\overline{1,3} \end{cases}$$

Finally, we have:

$x_{11}$=1, $x_{21}$=1, $x_{31}$=1, $x_{41}$=1, $x_{12}$=1, $x_{22}$=1, $x_{32}$=0, $x_{42}$=0, $x_{13}$=1, $x_{23}$=1, $x_{33}$=1, $x_{43}$=1

and: $a_1=x_{11}+x_{21}+x_{31}+x_{41}=4$, $a_2=x_{12}+x_{22}+x_{32}+x_{42}=2$, $a_3=x_{13}+x_{23}+x_{33}+x_{43}=4$ and the maximal utility will be $U_t$=97 for 4 goods x, 2 goods y and 4 goods z.