A Method of Determination of an Acquisition Program of N Goods in Order to Maximize the Total Utility

Catalin Angelo Ioan
Danubius University of Galati
catalin_angelo_ioan@univ-danubius.ro

Abstract. This paper solves in a different way the problem of maximization of the total utility for n goods. The author uses the diophantic equations (equations in integer numbers) and after a decomposing in different cases, he obtains the maximal utility.

Keywords: utility; maximization; diophantic

JEL Classification: C02; C5

A method of maximization the total utility for n goods

Let a consumer which has a budget of acquisition of r goods, r\geq2, in value of S \in \mathbb{N} u.m. The prices of the r goods \(x_i, i=1, r\) are \(p_i \in \mathbb{N}, i=1, r\). The marginal utilities corresponding to an arbitrary number of doses are in the following table:

<table>
<thead>
<tr>
<th>No. of dose</th>
<th>U_{m1}</th>
<th>\ldots</th>
<th>U_{mr}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>u_{11}</td>
<td>\ldots</td>
<td>u_{1r}</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>i</td>
<td>u_{i1}</td>
<td>\ldots</td>
<td>u_{ir}</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>n</td>
<td>u_{n1}</td>
<td>\ldots</td>
<td>u_{nr}</td>
</tr>
</tbody>
</table>
We want in what follows to determine the number of doses $a_i$ for the good $x_i$, $i=1,r$ such that the total utility: $U_i = \sum \sum u_{ij}$ to be maximal.

Let therefore $S_1 \leq S$ and the equation:

$$\sum_{i=1}^r a_i p_i = S_1.$$  

(1)

Let denote with $d=(p_1,\ldots,p_r)$ the greatest common divisor of $p_i$, $i=1,r$. We well know the fact that in order the equation has entire solutions we have necessarily $d \mid S_1$.

Also, we shall consider:

$S_1 > S\text{-min}\{p_1,\ldots,p_r\}$ because if $S_1 \leq S\text{-min}\{p_1,\ldots,p_r\}$ with a supplementary unit of one good $i$, where $1 \leq i \leq r$, the total utility will grow.

Dividing (1) at $d$, we have:

$$\sum_{i=1}^r a_i \frac{p_i}{d} = \frac{S_1}{d}$$

and with the notation: $q_i = \frac{p_i}{d}$ follows:

$$\sum_{i=1}^r a_i q_i = \frac{S_1}{d}.$$  

(3)

where the greatest common divisor of $q_i$, $i=1,r$ is $(q_1,\ldots,q_r)=1$.

It is well known that for any relative prime numbers $A,B \in \mathbb{N}$ it exists $\alpha$ and $\beta \in \mathbb{Z}$ (determined eventually with the Euclid algorithm) such that: $\alpha A + \beta B = 1$.

Let therefore $d_i = q_1$ and $d_i = (d_{i-1},q_i)$, $i=2,r$. Because $(q_1,\ldots,q_r)=1$ it follows obviously that $d_i=1$.

We have now $\exists \alpha, \beta \in \mathbb{N}$, $i=2,r$, such that:

$$\alpha_i d_{i-1} + \beta_i q_i = d_i,$$  

(4)

In what follows we shall denote: $d_0=0$, $\alpha_0=1$, $\beta_0=1$, $\alpha_1=1$, $\beta_1=1$ such that: $\alpha_1 \beta_0 + \beta_1 = d_1$. 

107
Writing in detail the relation (4), we obtain:

\[(5) \quad \alpha_1 d_0 + \beta_1 q_1 = d_1 \]
\[\alpha_2 d_1 + \beta_2 q_2 = d_2 \]
\[\alpha_3 d_2 + \beta_3 q_3 = d_3 \]

... 
\[\alpha_r d_{r-1} + \beta_r q_r = d_r \]

Substituting the first of (5) in the second:

\[(6) \quad \alpha_2 \alpha_1 d_0 + \alpha_2 \beta_1 q_1 + \beta_2 q_2 = d_2 \]

after, the second in the third:

\[(7) \quad \alpha_3 \alpha_2 \alpha_1 d_0 + \alpha_3 \alpha_2 \beta_1 q_1 + \alpha_3 \beta_2 q_2 + \beta_3 q_3 = d_3 \]

we shall obtain, by induction:

\[(8) \quad \sum_{i=1}^{r-1} \prod_{j=i+1} \alpha_j \beta_j q_i + \beta_r q_r = d_r \]

We have therefore:

\[(9) \quad \sum_{i=1}^r \sigma_i q_i = 1 \]

with the obvious notations: \( \sigma_i = \beta_i \prod_{j=1}^i \alpha_j \) for \( i \leq r-1 \) and \( \sigma_r = \beta_r \).

From (3), (9) we have now:

\[(10) \quad \sum_{i=1}^r a_i q_i = \frac{S}{d} \sum_{i=1}^r \sigma_i q_i \]

or, in other words:

\[(11) \quad \sum_{i=1}^r q_i (a_i - \frac{S}{d} \sigma_i) = 0 \]

For a fixed \( k = 1, r \) we can write (11):
Let now $\delta_k = \{q_1, \ldots, q_k, \ldots, q_i\}$ where the sign $\wedge$ means that the term is missing.

Because $(\delta, q_0) = 1$ follows:

$$\delta_i \frac{S_i}{d} \sigma_i$$

therefore:

$$a_k = \frac{S_k}{d} \sigma_k = \zeta_k \delta_k, \quad k = \overline{1, r}$$

From (11), (14) we have that:

$$\sum_{i=1}^{k} \zeta_i \delta_i q_i = 0$$

We can write (14) also like:

$$a_k = \frac{S_k}{d} \sigma_k + \zeta_k \delta_k, \quad k = \overline{1, r}$$

Because $a_k \geq 0, \quad k = \overline{1, r}$, we obtain that:

$$S_k \sigma_k + \zeta_k \delta_k d \geq 0, \quad k = \overline{1, r}$$

From (1) we can see easily that:

$$a_k \leq \min \left\{ \frac{S_k}{p_k}, n \right\}, \quad k = \overline{1, r}$$

From (16), (17) and (18) we find that:

$$\begin{cases} 
\zeta_k \geq -\frac{S_k \sigma_k}{\delta_k d} \\
\zeta_k \leq \min \left( \frac{S_k (d - p_k \sigma_k)}{p_k \delta_k d}, \frac{nd - S_k \sigma_k}{\delta_k d} \right), \quad k = \overline{1, r}
\end{cases}$$

We have therefore:
\[ \zeta_k \in \left[-\frac{S_{\sigma_k}}{\delta_k d}, \min \left( \frac{S_{\sigma_k} (d-p_{\sigma_k})}{p_k \delta_k d}, \frac{nd-S_{\sigma_k}}{\delta_k d} \right) \right] \cap \mathbb{N}, \; k=1, r \]

The length of the range is less than or equal with \( \frac{S_k}{p_k \delta_k} \), therefore there exist at most \( \left\lfloor \frac{S_k}{p_k \delta_k} \right\rfloor + 1 \) integer values of \( \zeta_k \) (where \([z]\) denotes the integer part of \(z\)) that verifies the acceptability conditions.

**Example**

<table>
<thead>
<tr>
<th>No. of dose</th>
<th>( U_{mx} )</th>
<th>( U_{my} )</th>
<th>( U_{mz} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>16</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

\( p_x=4, \; p_y=6, \; p_z=10, \; S=50. \)

**Solution**

We have \( d=(4,6,10)=2 \) therefore \( S_1>50-\min(4,6,10)=46 \). Because \( d \mid S_1 \) we shall have \( S_1 \in \{48,50\} \).

Dividing by 2 the reduced prices become: \( q_1=2, \; q_2=3, \; q_3=5. \)

Let now: \( d_1=2, \; d_2=(d_1,q_2)=(2,3)=1, \; d_3=(d_2,q_3)=(1,5)=1. \)

We have: \( \alpha_1=1, \; \beta_1=1 \) and from (4) the equation: \( 2\alpha_2+3\beta_2=1 \) implies: \( \alpha_2=-1 \) and \( \beta_2=1. \) Also, the equation: \( \alpha_3+5\beta_3=1 \) implies: \( \alpha_3=-4 \) and \( \beta_3=1. \)
Let now: $\sigma_1 = \beta_1 \prod_{j=2}^{3} \alpha_j = \beta_1 \alpha_2 \alpha_3 = 4$, $\sigma_2 = \beta_2 \prod_{j=3}^{3} \alpha_j = \beta_2 \alpha_3 = -4$, $\sigma_3 = \beta_3 = 1$

and: $\delta_1 = (q_2, q_3) = (3, 5) = 1$, $\delta_2 = (q_1, q_3) = (2, 5) = 1$, $\delta_3 = (q_1, q_2) = (2, 3) = 1$.

The relation (15) becomes: $2\zeta_1 + 3\zeta_2 + 5\zeta_3 = 0$.

From (20):

$$\zeta_4 \in \left[ -\frac{\sigma_4 S_4}{2}, \min \left( S_4 \left( 2 - \frac{p_4 \sigma_4}{2} \right), -\frac{S_4 \sigma_4}{2} \right) \right] \cap \mathbb{N}$$

therefore:

$$\zeta_4 \in \left[ -2S_4, \min (\frac{7}{4} S_4, 8 - 2S_4) \right] \cap \mathbb{N}$$

$$\zeta_2 \in \left[ 2S_1, \min \left( \frac{13}{6} S_1, 8 + 2S_1 \right) \right] \cap \mathbb{N}$$

$$\zeta_3 \in \left[ -\frac{S_3}{2}, \min \left( -\frac{2}{5} S_3, 8 - \frac{S_3}{2} \right) \right] \cap \mathbb{N}$$

and also from (16):

$$a_1 = \frac{S_1 \sigma_1 + \zeta_1 \delta_1}{d} = \frac{4S_1 + 2\zeta_1}{2} = 2S_1 + \zeta_1$$

$$a_2 = \frac{S_2 \sigma_2 + \zeta_2 \delta_2}{d} = \frac{-4S_1 + 2\zeta_2}{2} = -2S_1 + \zeta_2$$

$$a_3 = \frac{S_3 \sigma_3 + \zeta_3 \delta_3}{d} = \frac{S_1 + 2\zeta_3}{2} = S_1 + \zeta_3$$

Finally we have the following cases:

$S_1 = 48 \Rightarrow \zeta_1 \in [-96, -88] \cap \mathbb{N}$

$\zeta_2 \in [96, 104] \cap \mathbb{N}$

$\zeta_3 \in [-24, -20] \cap \mathbb{N}$

$S_1 = 50 \Rightarrow \zeta_1 \in [-100, -92] \cap \mathbb{N}$

$\zeta_2 \in [100, 108] \cap \mathbb{N}$
\( \zeta \in [-25, -20] \cap \mathbb{N} \)

with \( 2\zeta + 3\zeta_2 + 5\zeta_3 = 0 \)

For \( S_1 = 48 \) and certainly: \( a_1 = 96 + \zeta_1, \ a_2 = -96 + \zeta_2, \ a_3 = 24 + \zeta_3 \) we have:

\( \zeta = -96, \ \zeta_2 = 99, \ \zeta_3 = -21 \Rightarrow a_1 = 0, \ a_2 = 3, \ a_3 = 3, \ U_i = 20 + 16 + 15 + 15 + 12 + 10 = 88 \)

\( \zeta = -96, \ \zeta_2 = 104, \ \zeta_3 = -24 \Rightarrow a_1 = 0, \ a_2 = 8, \ a_3 = 0, \ U_i = 20 + 16 + 15 + 14 + 13 + 10 + 8 + 7 = 103 \)

\( \zeta = -95, \ \zeta_2 = 100, \ \zeta_3 = -22 \Rightarrow a_1 = 1, \ a_2 = 4, \ a_3 = 2, \ U_i = 10 + 20 + 16 + 15 + 14 + 12 = 102 \)

\( \zeta = -94, \ \zeta_2 = 96, \ \zeta_3 = -20 \Rightarrow a_1 = 2, \ a_2 = 0, \ a_3 = 4, \ U_i = 10 + 8 + 15 + 12 + 10 + 7 = 62 \)

\( \zeta = -94, \ \zeta_2 = 101, \ \zeta_3 = -23 \Rightarrow a_1 = 2, \ a_2 = 5, \ a_3 = 1, \ U_i = 10 + 8 + 20 + 16 + 15 + 14 + 13 + 15 = 111 \)

\( \zeta = -93, \ \zeta_2 = 97, \ \zeta_3 = -21 \Rightarrow a_1 = 3, \ a_2 = 1, \ a_3 = 3, \ U_i = 10 + 8 + 7 + 20 + 15 + 12 + 10 = 82 \)

\( \zeta = -93, \ \zeta_2 = 102, \ \zeta_3 = -24 \Rightarrow a_1 = 3, \ a_2 = 6, \ a_3 = 0, \ U_i = 10 + 8 + 7 + 20 + 15 + 14 + 13 + 10 = 113 \)

\( \zeta = -92, \ \zeta_2 = 98, \ \zeta_3 = -22 \Rightarrow a_1 = 4, \ a_2 = 2, \ a_3 = 2, \ U_i = 10 + 8 + 7 + 6 + 20 + 16 + 15 + 12 = 94 \)

\( \zeta = -91, \ \zeta_2 = 99, \ \zeta_3 = -23 \Rightarrow a_1 = 5, \ a_2 = 3, \ a_3 = 1, \ U_i = 10 + 8 + 7 + 6 + 5 + 20 + 16 + 15 + 15 = 102 \)

\( \zeta = -90, \ \zeta_2 = 100, \ \zeta_3 = -24 \Rightarrow a_1 = 6, \ a_2 = 4, \ a_3 = 0, \ U_i = 10 + 8 + 7 + 20 + 16 + 15 + 14 + 10 = 105 \)

\( \zeta = -89, \ \zeta_2 = 96, \ \zeta_3 = -22 \Rightarrow a_1 = 7, \ a_2 = 0, \ a_3 = 2, \ U_i = 10 + 8 + 7 + 6 + 5 + 4 + 3 + 15 + 12 = 70 \)

\( \zeta = -88, \ \zeta_2 = 97, \ \zeta_3 = -23 \Rightarrow a_1 = 8, \ a_2 = 1, \ a_3 = 1, \ U_i = 10 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 20 + 15 = 80 \)

For \( S_1 = 50 \) and: \( a_1 = 100 + \zeta_1, \ a_2 = 100 + \zeta_2, \ a_3 = 25 + \zeta_3 \) we have:

\( \zeta = -100, \ \zeta_2 = 100, \ \zeta_3 = -20 \Rightarrow a_1 = 0, \ a_2 = 0, \ a_3 = 5, \ U_i = 15 + 12 + 10 + 7 + 5 = 49 \)

\( \zeta = -100, \ \zeta_2 = 105, \ \zeta_3 = -23 \Rightarrow a_1 = 5, \ a_2 = 2, \ a_3 = 2, \ U_i = 20 + 16 + 15 + 14 + 13 + 15 + 12 = 105 \)

\( \zeta = -99, \ \zeta_2 = 101, \ \zeta_3 = -21 \Rightarrow a_1 = 1, \ a_2 = 1, \ a_3 = 4, \ U_i = 10 + 20 + 15 + 12 + 10 + 7 = 74 \)

\( \zeta = -99, \ \zeta_2 = 106, \ \zeta_3 = -24 \Rightarrow a_1 = 1, \ a_2 = 6, \ a_3 = 1, \ U_i = 10 + 20 + 15 + 14 + 13 + 10 + 15 = 113 \)

\( \zeta = -98, \ \zeta_2 = 102, \ \zeta_3 = -22 \Rightarrow a_1 = 2, \ a_2 = 2, \ a_3 = 3, \ U_i = 10 + 8 + 20 + 16 + 15 + 12 + 10 = 91 \)

\( \zeta = -98, \ \zeta_2 = 107, \ \zeta_3 = -25 \Rightarrow a_1 = 2, \ a_2 = 7, \ a_3 = 0, \ U_i = 10 + 20 + 16 + 15 + 14 + 13 + 10 + 8 = 106 \)

\( \zeta = -97, \ \zeta_2 = 103, \ \zeta_3 = -23 \Rightarrow a_1 = 3, \ a_2 = 3, \ a_3 = 2, \ U_i = 10 + 8 + 7 + 20 + 16 + 15 + 14 + 12 = 103 \)

\( \zeta = -96, \ \zeta_2 = 104, \ \zeta_3 = -24 \Rightarrow a_1 = 4, \ a_2 = 4, \ a_3 = 1, \ U_i = 10 + 8 + 7 + 6 + 20 + 15 + 14 + 15 = 111 \)
\[ \zeta_1=-95, \ zeta_2=100, \ zeta_3=-22 \Rightarrow a_1=5, \ a_2=0, \ a_3=3, \ U_t=10+8+7+6+5+15+12+10=73 \]
\[ \zeta_1=-95, \ zeta_2=105, \ zeta_3=-25 \Rightarrow a_1=5, \ a_2=5, \ a_3=0, \]
\[ U_t=10+8+7+6+5+20+16+15+14+13=114 \]
\[ \zeta_1=-94, \ zeta_2=101, \ zeta_3=-23 \Rightarrow a_1=6, \ a_2=1, \ a_3=2, \ U_t=10+8+7+6+5+4+20+15+12=87 \]
\[ \zeta_1=-93, \ zeta_2=102, \ zeta_3=-24 \Rightarrow a_1=7, \ a_2=2, \ a_3=1, \ U_t=10+8+7+6+5+4+3+20+16+15=94 \]
\[ \zeta_1=-92, \ zeta_2=103, \ zeta_3=-25 \Rightarrow a_1=8, \ a_2=3, \ a_3=0, \]
\[ U_t=10+8+7+6+5+4+3+2+20+16+15=96 \]

Finally, the maximal utility will be \( U_t = 114 \) for 5 goods \( x \) and 5 goods \( y \).