Mathematical and Quantative Methods

The Consumer's Behavior after the Preferences Nature

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Abstract: The paper treats the consumer behavior after the nature of its preferences. There are analyzed in terms of Marshall demand, the perfectly substitutable, the perfectly complementary, the case of independently goods in the meaning of utility, the case of separable goods in the meaning of utility and the neutral goods. Significant for the results is that n goods are treated simultaneously with generalized utility functions instead the classical theory.

Keywords: consumer; demand; utility

JEL Classification: D01

1. Introduction

The classical theory of consumer's behavior in relation to the income analyze usually the choice's optimization from a basket of two goods situated in different preference relations to each other.

Although the current theory requires that this onset is sufficient, saying that for a good fixed, the basket of other goods can be considered as a whole, we will try to impose a new approach, treating each of them individually.

We believe that this approach is more realistic, because a change in the structure of consumption of a good influence on each other goods (with separate prices and specific dependency relations).

In the first part of the article we will briefly review known results on the application in Marshallian or Hicksian terms, then we customize and resolve these issues for five categories of goods, namely: perfectly substitutable goods, perfectly complementary goods, goods independent in the meaning of utility, separable goods in the meaning of utility and neutral goods.

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2. The Marshall Demand

Let a consumer faced with a choice of any number of quantities of goods $B_1,...,B_n$, SC - their space consumption and the sale prices: $p_1,...,p_n$. We assume that all available income V can be assigned to act consumer buying, his preferences being not affected by the size of V. We say, in this case, that the demand for goods is unmatched. Let also be a utility function $U:SC\to\mathbf{R}_+$. Considering the budget zone $ZB=\{(x_1,...,x_n)\in SC \mid \sum_{i=1}^{n} p_i x_i \le V\}$ we put the problem of determining the consumption basket so that utility is maximum.

The problem becomes:

$$\begin{cases} \max U(x_1,...,x_n) \\ \sum_{i=1}^{n} p_i x_i \le V \\ x_1,...,x_n \in SC \end{cases}$$

It can be show that while the function U is concave and SC – convex, then the optimal solution of the problem is located on the border zone of the budget, satisfying the conditions:

$$\begin{cases} \max U(x_1,...,x_n) \\ \sum_{i=1}^{n} p_i x_i = V \\ x_1,...,x_n \in SC \end{cases}$$

The new problem, is therefore to determine the function U extremes when the variables are subject to links. We will apply the Lagrange multiplier method.

Let therefore: $L(x_1,...,x_n,\lambda)=U(x_1,...,x_n)+\lambda\left(\sum_{i=1}^{n} p_i x_i - V\right)$. The extreme conditions are:

$$\begin{cases} \frac{\partial L}{\partial x_i} = 0, i = \overline{1,n} \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases}$$

from where:

$$\begin{cases} U_{m,i} + \lambda p_i = 0, i = \overline{1,n} \\ \sum_{i=1}^{n} p_i x_i - V = 0 \end{cases}$$

$U_{m,i}$ being the marginal utility corresponding to the i-th good.

From the first n relations, we deduce $\lambda = -\frac{U_{m,i}}{p_i}$, $i = \overline{1,n}$ and or otherwise:

$\frac{U_{m,1}}{p_1} = ... = \frac{U_{m,n}}{p_n}$ - the Second Law of Gossen

Solving now the characteristic system:

$$\begin{cases} \frac{U_{m,1}}{p_1} = ... = \frac{U_{m,n}}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

follows the solution of the problem:

$$\begin{cases} \overline{x}_1 = f_1(p_1,...,p_n, V) \\ \qquad ... \\ \overline{x}_n = f_n(p_1,...,p_n, V) \end{cases}$$

The restriction of the function U at the hyperplane $\sum_{i=1}^{n} p_i x_i = V$ has the same nature asU, therefore it is concave. As this result, the point $(\overline{x}_1,...,\overline{x}_n)$ is a local maximum.

3. The Hicks Demand

Let now the same consumer who wants a given level of utility in conditions that it is willing to allocate the lowest income to achieve its goals. We will say, in this case, that the demand for goods is compensated. Considering the utility function $U: SC \to \mathbf{R}_+$ and $\overline{u}$ the desired utility, the problem of determining the consumption basket so that allocated income be minimum is:

$$\begin{cases} \min \sum_{i=1}^{n} p_i x_i \\ U(x_1,...,x_n) \geq \overline{u} \\ x_1,...,x_n \in SC \end{cases}$$

As in the previous section, we obtain that, while the objective function is linear, it is convex, in particular, so the optimal solution of the problem is located on the boundary of the zone $U(x_1,...,x_n) \geq \overline{u}$.

The problem becomes:

$$\begin{cases} \min \sum_{i=1}^{n} p_i x_i \\ U(x_1,...,x_n) = \overline{u} \\ x_1,...,x_n \in SC \end{cases}$$

We apply the Lagrange multiplier method again and also, because the objective function is linear, it has null second differential and the preferred consumption zone of any $x \in SC$ is convex. The restriction of the objective function at $U(x_1,...,x_n)=\overline{u}$ is convex, therefore the stationary points of the Lagrangian will be points of local minimum.

Let therefore $L(x_1,...,x_n,\lambda)=\sum_{i=1}^{n} p_i x_i + \lambda\left(U(x_1,...,x_n) - \overline{u}\right)$. The extreme conditions are:

$$\begin{cases} \frac{\partial L}{\partial x_i} = 0, i = \overline{1,n} \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases}$$

or:

$$\begin{cases} p_i + \lambda U_{m,i} = 0, i = \overline{1,n} \\ U(x_1,...,x_n) - \overline{u} = 0 \end{cases}$$

From the first n relations, we deduce:

$$\lambda = -\frac{p_i}{U_{m,i}}, \ i = \overline{1,n}$$

and or otherwise:

$$\frac{U_{m,1}}{p_1} = \ldots = \frac{U_{m,n}}{p_n}$$ - the Second Law of Gossen

Solving now the characteristic system:

$$\begin{cases} \frac{U_{m,1}}{p_1} = \ldots = \frac{U_{m,n}}{p_n} \\ U(x_1,\ldots,x_n) = \overline{u} \end{cases}$$

follows the solution of the problem:

$$\begin{cases} \tilde{x}_1 = g_1(p_1,\ldots,p_n,\overline{u}) \\ \ldots \\ \tilde{x}_n = g_n(p_1,\ldots,p_n,\overline{u}) \end{cases}$$

The analysis of the two types of demands shows that income hyperplane must be tangent to the utility hypersurface.

Because the tangent hyperplane at an arbitrary point has parameters: $U_{m,1},\ldots,U_{m,n}$ and the hyperplane of income: $p_1,\ldots,p_n$, the condition of the problem leads to the proportionality of them, so to the Gossen's Second Law.

Another aspect that deserves to be considered is the economic interpretation of λ from the two methods of Lagrange multipliers.

In the case of Marshall demand, we have $dV=\sum_{i=1}^{n} p_i dx_i$. On the other hand, from Gossen's Second Law: $U_{m,i}=-\lambda p_i$, $i=\overline{1,n}$ therefore: $-\lambda dV=-\lambda\sum_{i=1}^{n} p_i dx_i = \sum_{i=1}^{n} U_{m,i} dx_i$ $=dU$ or $\lambda=-\frac{dU}{dV}$. Therefore, in the case of Marshall demand, λ multiplier is the opposite marginal utility of income.

In the Hicks case, we have $dU=\sum_{i=1}^{n} U_{m,i} dx_i$. Again, from Gossen's Second Law: $U_{m,i}=-\lambda p_i$, $i=\overline{1,n}$ which implies: $dU=-\lambda\sum_{i=1}^{n} p_i dx_i$ $=-\lambda dV$ hence the same meaning of λ.

4. The Consumer's Behavior after the Preferences Nature

4.1. The Consumer's Behavior for Perfectly Substitutable Goods

Let be a lot of goods perfect substitutes $B_1,...,B_n$, SC – their space of consumption and sale prices: $p_1,...,p_n$. If a consumer has an income V and directs his choice after the utility function $U(x_1,...,x_n)=a_1x_1+...+a_nx_n$, $a_i>0$, $i=\overline{1,n}$.

We put the question of Marshall optimization to maximize the utility.

We saw in that necessary and sufficient conditions for maximum are:

$$\begin{cases} \frac{U_{m,1}}{p_1} = ... = \frac{U_{m,n}}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

which leads, because $U_{m,i}=a_i$, $i=\overline{1,n}$ to:

$$\begin{cases} \frac{a_1}{p_1} = ... = \frac{a_n}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

Like a conclusion, if $\frac{a_1}{p_1} = ... = \frac{a_n}{p_n}$ then all the points of the budget hyperplane $(\overline{x}_1,...,\overline{x}_n)$ where $\sum_{i=1}^{n} p_i \overline{x}_i = V$ are optimal components of the consumer basket.

If $\exists i \neq j = \overline{1,n}$ so that: $\frac{a_i}{p_i} \neq \frac{a_j}{p_j}$ then the system is incompatible, so there are no solutions inside the budget hyperplane (*the zone bounded by the coordinates hyperplanes*).

In this case, we consider the comparison of the utility function on the intersection between the budget hyperplane and the coordinates hyperplanes.

Let therefore the partition $I=\{1,...,n\}$: $I=I_1\cup...\cup I_k$, $I_p\cap I_t=\emptyset$ such that $\forall u,v\in I_p$ we have: $\frac{a_u}{p_u} = \frac{a_v}{p_v}$ and $\forall u\in I_p$, $v\in I_t$, $p\neq t$: $\frac{a_u}{p_u} \neq \frac{a_v}{p_v}$. The partition of I consists of sets of indices for which ratios are equal.

Consider now that $\exists u\in I_p$ $\exists v\in I_t$ with $p\neq t$ such that $x_u\neq 0$ și $x_v\neq 0$. From the general problem of extremes with links, we have: $\frac{a_u}{p_u} = \frac{a_v}{p_v}$ that conflicts with $I_p\cap I_t=\emptyset$.

Following these considerations, it follows that if $\exists u\in I_p$ such that $x_u\neq 0$ then $x_v=0$ $\forall v\in I\text{-}I_p$. In this case, the problem becomes:

$$\begin{cases} \frac{a_u}{p_u}=\frac{a_v}{p_v}\ \forall u,v\in I_p \\ \sum_{u\in I_p} p_u x_u = V \end{cases}$$

with the optimal solution $(\overline{x}_u)_{u\in I_p}$ consisting of all data points locus given by $\sum_{u\in I_p} p_u x_u = V$, the maximum utility being: U= $\sum_{u\in I_p} a_u x_u$. Comparing the maximum utility values, corresponding to all elements of partition, we obtain the optimal consumption basket. This problem is very simple. Thus, noting $\lambda_p=\frac{a_u}{p_u}$ $\forall u\in I_p$ we have $a_u=\lambda_p p_u$ therefore U= $\sum_{u\in I_p}\lambda_p p_u x_u=\lambda_p V$. From these facts, we will get the maximum utility for λ_p=maximum. We then compare the values of λ_p for each of the elements of the partition of I, the corresponding locus being $\sum_{u\in I_p} p_u x_u = V$ corresponding to p such that λ_p=maximum.

In particular, for two perfectly substitutable goods, we have: $\frac{a_1}{p_1}=\frac{a_2}{p_2}$. If this condition occurs, then the optimal consumption basket is given by pairs: $(\overline{x}_1,\overline{x}_2)$ where $\sum_{i=1}^{n} p_i\overline{x}_i = V$. If $\frac{a_1}{p_1}\neq\frac{a_2}{p_2}$ we have the following situations:

- $\frac{a_1}{p_1}>\frac{a_2}{p_2}$ involves the optimal consumption basket: $(\overline{x}_1,\overline{x}_2)=\left(\frac{V}{p_1},0\right)$;
- $\frac{a_1}{p_1}<\frac{a_2}{p_2}$ involves optimal consumption basket: $(\overline{x}_1,\overline{x}_2)=\left(0,\frac{V}{p_2}\right)$.

4.2. The Consumer's Behavior for Perfectly Complementary Goods

In the case of perfectly complementary goods, we have: $U(x_1,...,x_n)=\min(a_1x_1,...,a_nx_n)$, $a_i>0$, $i=\overline{1,n}$.

Let the budget hyperplane: $\sum_{i=1}^{n} p_i x_i = V$ and $U_0>0$ – fixed.

Let consider now the point $\overline{x} = (\overline{x}_1,...,\overline{x}_n)$ for which: $a_1 \overline{x}_1 = ... = a_n \overline{x}_n = U_0$.

We have therefore $U(\overline{x}_1,...,\overline{x}_n) = U_0$. The condition that $\overline{x}$ to be on the budget hyperplane is: $\sum_{i=1}^{n} p_i \overline{x}_i = V$ or otherwise: $\sum_{i=1}^{n} \frac{p_i}{a_i} U_0 = V$ where: $U_0 = \frac{V}{\sum_{i=1}^{n} \frac{p_i}{a_i}}$.

Let then show that if $(x_1,...,x_n)$ belongs to the budget hyperplane, then the maximum utility is $U_0 = \frac{V}{\sum_{i=1}^{n} \frac{p_i}{a_i}}$ and is obtained for $\overline{x} = (\overline{x}_1,...,\overline{x}_n)$ where

$\overline{x}_j = \frac{V}{a_j \sum_{i=1}^{n} \frac{p_i}{a_i}}$, $j=\overline{1,n}$.

Let therefore $\sum_{i=1}^{n} p_i x_i = V$ or else: $\sum_{i=1}^{n} a_i \frac{p_i}{a_i} x_i = V$ and note, for the simplicity: $\frac{p_i}{a_i} = q_i$ and $a_i x_i = y_i$. We have: $\sum_{i=1}^{n} q_i y_i = V$. Suppose now that $\exists 1 \le i \le n$ such that $y_i \neq \frac{V}{\sum_{i=1}^{n} q_i}$

If $y_i \ge \frac{V}{\sum_{i=1}^{n} q_i}$, $i=\overline{1,n}$ then: $V = \sum_{i=1}^{n} q_i y_i \ge \sum_{i=1}^{n} q_i \frac{V}{\sum_{i=1}^{n} q_i} = \frac{1}{\sum_{i=1}^{n} q_i} \sum_{i=1}^{n} q_i V = V$ from where: $y_i = \frac{V}{\sum_{i=1}^{n} q_i}$, $i=\overline{1,n}$ - contradiction.

Therefore: $\exists 1 \le j \le n$ such that: $y_i < \frac{V}{\sum_{i=1}^{n} q_i}$. In this case: $a_i x_i = y_i < \frac{V}{\sum_{i=1}^{n} q_i}$ where:

$U(x_1,...,x_n) = \min(a_1 x_1,...,a_n x_n) < \frac{V}{\sum_{i=1}^{n} q_i} = U_0$.

After these facts, we obtain that any point on the budget hyperplane different from $\overline{x}$ will have a lower utility.

In particular, for two perfectly complementary goods, we have:

$$\overline{x}_1 = \frac{V}{a_1\left(\frac{p_1}{a_1} + \frac{p_2}{a_2}\right)} = \frac{a_2 V}{p_1 a_2 + p_2 a_1} \text{ and } \overline{x}_2 = \frac{V}{a_2\left(\frac{p_1}{a_1} + \frac{p_2}{a_2}\right)} = \frac{a_1 V}{p_1 a_2 + p_2 a_1}$$

4.3. The Consumer's Behavior in the Case of Goods Independent in the Meaning of Utility

In the case of this type of goods, the utility function is:

$U(x_1,...,x_n)=f_1(x_1)+...+f_n(x_n)$ with $f_i \in C^2(0,\infty)$, $f_i'' \le 0$, $i=\overline{1,n}$ and $f_1(0)+...+f_n(0)=0$

Because $U_{m,i}= f_i'(x_i)$, $i=\overline{1,n}$, the necessary and sufficient conditions are:

$$\begin{cases} \frac{f_1'(x_1)}{p_1} = ... = \frac{f_n'(x_n)}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

In the particular case of a utility function of the form:

$U(x_1,x_2,...,x_n)= x_1^{\alpha_1} + x_2^{\alpha_2} + ... + x_n^{\alpha_n}$ with $\alpha_i \in (0,1)$, $i=\overline{1,n}$

we have:

$$\begin{cases} \frac{\alpha_1 x_1^{\alpha_1 -1}}{p_1} = ... = \frac{\alpha_n x_n^{\alpha_n -1}}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

Noting with λ the common values of the ratios, we get: $x_i=\left(\frac{p_i}{\alpha_i}\lambda\right)^{\frac{1}{\alpha_i-1}}$, $i=\overline{1,n}$. From the income relationship: $\sum_{i=1}^{n} p_i\left(\frac{p_i}{\alpha_i}\lambda\right)^{\frac{1}{\alpha_i-1}} = V$ where:

$$\sum_{i=1}^{n} \frac{p_i^{\frac{\alpha_i}{\alpha_i-1}}}{\alpha_i^{\frac{1}{\alpha_i-1}}} \lambda^{\frac{1}{\alpha_i-1}} = V$$

If $\overline{\lambda}$ is a strictly positive solution of this equation, then the final solution is: $x_i = \left(\frac{p_i}{\alpha_i}\overline{\lambda}\right)^{\frac{1}{\alpha_i - 1}}$, $i=\overline{1,n}$.

4.4. The Consumer's Behavior in the case of Separable Goods in the Meaning of Utility

The **s**eparable utility function for such goods is:

$U(x_1,...,x_n)=f_1(x_1)\cdot...\cdot f_n(x_n)$ with $f_i \in C^2(0,\infty)$, $f_i(x)>0$ $\forall x>0$,

$f_i'' \le 0$, $i=\overline{1,n}$, $f_1(0)\cdot...\cdot f_n(0)=0$

and the quadratic form: $H=\sum_{i=1}^{n}\frac{f_i''}{f_i}\xi_i^2+\sum_{i,j=1}^{n}\frac{f_i' f_j'}{f_i f_j}\xi_i\xi_j$ is negatively defined.

The necessary and sufficient conditions for maximum are:

$$\begin{cases} \dfrac{U\frac{f_1'}{f_1}}{p_1} = ... = \dfrac{U\frac{f_n'}{f_n}}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

or otherwise:

$$\begin{cases} \dfrac{f_1'}{p_1 f_1} = ... = \dfrac{f_n'}{p_n f_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

In the particular case of the Cobb-Douglas function:

$U(x_1,x_2,...,x_n)= x_1^{\alpha_1}...x_n^{\alpha_n}$, $\alpha_i>0$, $\sum_{i=1}^{n}\alpha_i \le 1$

we have:

$$\begin{cases} \dfrac{\alpha_1}{p_1 x_1} = ... = \dfrac{\alpha_n}{p_n x_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

Noting with λ the common value of the ratios, we get: $p_i x_i = \frac{\alpha_i}{\lambda}$, $i=\overline{1,n}$ hence from the equality of the budget: $\sum_{i=1}^{n} \frac{\alpha_i}{\lambda} = V$ that is: $\lambda = \frac{\sum_{i=1}^{n} \alpha_i}{V}$, Finally we get: $x_j = \frac{\alpha_j V}{p_j \sum_{i=1}^{n} \alpha_i}$, $j=\overline{1,n}$.

4.5. The consumer's Behavior in the Case Neutral Goods

In the case of neutral goods $B_1,...,B_m$, the utility function is:

$U(x_1,...,x_n)=f(x_{m+1},...,x_n)$

where f is of class C^2 and concave. Because in the budget hyperplane: $\sum_{i=1}^{n} p_i x_i = V$ the neutral goods consumed financial resources without to bring more utility, the optimal allocation will exclude from the analysis and the optimization problem becomes:

$$\begin{cases} \dfrac{\frac{\partial f}{\partial x_{m+1}}}{p_{m+1}} = \dots = \dfrac{\frac{\partial f}{\partial x_n}}{p_n} \\ \sum_{i=m+1}^{n} p_i x_i = V \end{cases}$$

5. Conclusion

The results obtained from the above analysis provides the optimal allocation of the demand of Marshall type, pointing out that in the case of perfectly substitutable goods proportionality coefficient the allocation depends of the proportionality of the coefficients of the utility function with goods prices.

If for perfectly complementary goods the issue is resolved completely, in the case of goods independent in the meaning of utility, the problem reduces to solving a nonlinear equation whose solution determines the actual allocation.

The last two issues of a general nature, specifically formulated the optimal conditions.

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