Mathematical and Quantative Methods

The Demand's Behavior under the Action of Factors

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Abstract: In this paper we will study and classify the main types of goods, from the perspective of a global analysis (for n goods simultaneous). For each of the main types, we will present one concrete example resulting from special utility functions. Also we will broach the Engel's hypersurfaces which generalize the Engel's curves from the case of two goods. The problem of elasticity is also broach in order to settle essential links (for n goods) between their level and nature of the goods.

Keywords consumer; demand; utility; elasticity; Engel

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1. Introduction

The study of the demand behavior under the action of some factors such as price and disposable income is essential in determining the market structure and also the action of determining the consumption trends in the occurrence of disturbing factors (changes in price or income).

We will study and classify below the main types of goods, from the perspective of a global analysis (for n goods simultaneous). For each of the main types, we will present one concrete example resulting from special utility functions.

The next section will be devoted to the Engel's hypersurfaces which generalize the Engel's curves from the case of two goods. It will gets concrete examples, each time reducing at the well-known case of two goods.

The last part will deal with the problem of elasticity, in which we settle essential links (for n goods) between their level and nature of the goods.

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2. The Demand's Behavior under the Action of Factors

Let consider a lot of goods $B_1,...,B_n$ subject to an utility function U, $x_1,...,x_n$ – the quantities of $B_1,...,B_n$ consumed, their sale prices: $p_1,...,p_n$ and a consumer disposable income V. In what follows, we put the question to study how the demand changes under the action of specific factors.

Suppose, first, that the price of a good B_i changes with an amount dp_i in caeteris paribus assumption. Let also note dx_i – the demand variation of B_i.

We know that a **good** is called **normal** if $\frac{dx_i}{dp_i}<0$ caeteris paribus.

Following this definition, if normal goods price drops, the demand increases and vice versa, the demand decreases with the price increases.

Let now p_i^1 and p_i^2 such that $p_i^1<p_i^2$ and the demands $x_i(p_i^1)$, $x_i(p_i^2)$. From the strictly decreasing character of demand in the case of normal goods we have that: $x_i(p_i^1)>x_i(p_i^2)$. For $\lambda\in[0,1]$, let $p_i=\lambda p_i^1+(1-\lambda)p_i^2\in[p_i^1,p_i^2]$.

We therefore have: $x_i(p_i)<x_i(p_i^1)$ and $x_i(p_i)>x_i(p_i^2)$. The question is now if the demand for normal goods is convex or concave. To analyze this, consider, for a fixed price p_i, the income allocated to the purchase of $x_i(p_i)$ units of good.

We have therefore $V(p_i)=p_ix_i(p_i)$. As $V'(p_i)=x_i(p_i)+p_i x_i'(p_i)$ results that the minimizing of the income allocated will satisfy $V'(p_i)=0$ then $x_i'(p_i)=-\frac{x_i(p_i)}{p_i}<0$.

On the other hand, the minimum requirement is reduced to $V''(p_i)=2x_i'(p_i)+p_i x_i''(p_i)>0$ from where $x_i''(p_i)\geq0$. The demand function of a normal good is therefore convex. This fact can be derived also from a purely economic observation because at a linear upward trend in the price, any rational consumer will prefer to purchase a combination at extremes (in a limit of an available budget, he purchasing as long as it is accessible to the lowest price) than the entire amount at an intermediate level price.

Following these considerations we obtain that the demand function $x_i=x_i(p_i)$ being strictly decreasing is injective so, by restricting the co-domain it becomes one-to-one. The inverse function $p_i=p_i(x_i)$ is the so-called the inverse function of demand. Because the monotony of a function is preserved at the reverse, follows that the inverse function of demand is strictly decreasing for normal goods.

Let consider, for example, the case of goods whose utility is of Cobb-Douglas type: $U(x_1,x_2,...,x_n)= x_1^{\alpha_1}...x_n^{\alpha_n}$ with $\alpha_i>0$, $\sum_{i=1}^{n}\alpha_i \le 1$. In [3] we have shown that the optimal consumer solution is: $x_i=\frac{\alpha_i V}{p_i\sum_{i=1}^{n}\alpha_i}$, $i=\overline{1,n}$.

At the i-th good price adjustment with dp_i, the new optimal solution becomes: $\tilde{x}_j=x_j\ \forall j\neq i$ and $\tilde{x}_i=\frac{\alpha_i V}{(p_i+dp_i)\sum_{i=1}^{n}\alpha_i}$. The variation corresponding to the consumption of good i is therefore:

$$dx_i=\tilde{x}_i-x_i=\frac{\alpha_i V}{(p_i+dp_i)\sum_{i=1}^{n}\alpha_i}-\frac{\alpha_i V}{p_i\sum_{i=1}^{n}\alpha_i}=-\frac{\alpha_i V dp_i}{p_i(p_i+dp_i)\sum_{i=1}^{n}\alpha_i}$$

We also have: $\frac{dx_i}{dp_i}=-\frac{\alpha_i V}{p_i^2\sum_{i=1}^{n}\alpha_i}<0$. The goods are so normal.

Because $\frac{d^2x_i}{dp_i^2}=\frac{2\alpha_i V}{p_i^3\sum_{i=1}^{n}\alpha_i}>0$ it follows that the demand function is convex in relation to price.

A **Giffen good** is one for which $\frac{dx_i}{dp_i}>0$ caeteris paribus.

Following the definition, in the case of Giffen goods if the price decreases the demand decreases also and vice versa, the demand increases with the increase of the price.

Given two goods B_1 and B_2 whose utility function is $U(x_1,x_2,...,x_n)=\sqrt{x_1}+\sqrt[4]{x_2^3}$ using the method shown in [3] we get:

$$x_1=\frac{8p_2^3}{81p_1^2}\left(\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}-\frac{1}{p_1}\right),\quad x_2=-\frac{8p_2^2}{81p_1}\left(\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}-\frac{1}{p_1}\right)+\frac{V}{p_2}$$

We have now:

$$\frac{dx_1}{dp_1} = \frac{8p_2^3}{81p_1^3}\left(\frac{3}{p_1} - 2\sqrt{\frac{1}{p_1^2} + \frac{81}{4p_2^3}V} - \frac{1}{p_1^2\sqrt{\frac{1}{p_1^2} + \frac{81}{4p_2^3}V}}\right)$$

The condition that $\frac{dx_1}{dp_1}<0$ is equivalent, after laborious calculations with: $-243 < \frac{81^2 p_1^2}{p_2^3}V$ what is true. The good B_1 is therefore normally.

Analog it shows that $\frac{dx_2}{dp_2}>0$ is equivalent to $\frac{81}{4}\frac{p_1^2}{p_2^3}V^3 + V^2 - 9 > 0$. Therefore, if V is large enough, the condition is met, then B_2 is a Giffen good.

Considering another good B_j with $j\neq i$ let note dx_j – the demand variation.

A **good** B_j is said to **gross substitute** B_i if $\frac{dx_j}{dp_i}\geq 0$ caeteris paribus.

In the gross substitutes property, at an increasing of the price of B_i the demand of B_j will increase also and vice versa at a decreasing of the price of B_i the demand of B_j will decrease.

In a particular form of utility function: $U(x_1,x_2,...,x_n)= \sqrt{x_1} + \sqrt{x_2} + ... + \sqrt{x_n}$ we have from [3] that:

$$x_j = \frac{V}{16p_j^2\sum_{i=1}^{n}\frac{1}{p_i}}, \quad j=\overline{1,n}$$

Because $\frac{dx_j}{dp_i} = \frac{V}{16p_j^2p_i^2\left(\sum_{i=1}^{n}\frac{1}{p_i}\right)^2}>0$ we have that the j-th good gross substitute any good B_i, $i\neq j$.

We will say the a **good** B_j is a **gross complement** for a good B_i if $\frac{dx_j}{dp_i}\leq 0$ caeteris paribus.

The significance of gross complementarity is that if the good's B_i price increases then the demand of B_j will decrease and vice versa.

Considering again two goods B_1 and B_2 whose utility function is $U(x_1,x_2,...,x_n)= \sqrt{x_1}+\sqrt[4]{x_2^3}$, we have seen that the optimal consumption is:

$$x_1=\frac{8p_2^3}{81p_1^2}\left(\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}-\frac{1}{p_1}\right),\quad x_2=-\frac{8p_2^2}{81p_1}\left(\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}-\frac{1}{p_1}\right)+\frac{V}{p_2}$$

We have now:

$$\frac{dx_1}{dp_2}=\frac{\frac{8p_2^2}{27p_1^2}\left(\frac{p_2}{p_1^2}+\frac{81}{4p_2^2}V-\frac{p_2}{p_1}\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}\right)-81V}{27p_1^2p_2\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}}$$

The condition that $\frac{dx_1}{dp_2}\le 0$ is equivalent to:

$$\frac{p_2}{p_1^2}\le\frac{81V}{8p_2^2}\left(27p_1^2-2\right)+\frac{p_2}{p_1}\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}$$

which is true for V large enough. Therefore, the good B_1 is a gross complement for B_2.

Analogously $\frac{dx_2}{dp_1}\ge 0$ leads to $\frac{1}{p_1}+\frac{81p_1}{4p_2^3}V+1\ge\left(1+\frac{1}{p_1}\right)p_1\sqrt{\frac{1}{p_1^2}+\frac{81}{4p_2^3}V}$ which is also satisfied for V large enough.

After these remarks, we notice an interesting fact, namely that B_1 is a gross complement for B_2, but B_2 gross substitute B_1.

3. The Engel's Hypersurfaces

In the following we will consider that the prices of B_i will remain unchanged, the only modifiable being the consumer income.

A **good** B_i is called **normal in the sense of income** if: $\frac{dx_i}{dV}>0$.

Following this definition, the good B_i is normal in the sense of income, if at an increase in disposable income, the consumption of B_i increases.

An **inferior good** B_i is if $\frac{dx_i}{dV}<0$.

The good B_i is inferior, if an increase in disposable income leads to a decrease of the consumption of this good.

From the budget hyperplane equation ([3]): $\sum_{i=1}^{n} p_i x_i = V$ we obtain by differentiation (assuming constant prices): $\sum_{i=1}^{n} p_i dx_i = dV$ or: $\sum_{i=1}^{n} p_i \frac{dx_i}{dV} = 1$.

As a result of this relationship, we see that in a basket of goods, in a steady income, we can not have simultaneously only inferior goods. Indeed, if: $\frac{dx_i}{dV}<0$, $i=\overline{1,n}$ then: $1=\sum_{i=1}^{n} p_i \frac{dx_i}{dV}<0$ which is obviously a contradiction.

The change in the demand of n goods, depending on the income, generates a hypersurface in $\mathbf{R}^m$ (*where m is given by the utility function nature*) called **Engel's hypersurface**. In particular, for a good reference, we obtain the well-known Engel's curve which is the locus of change in its demand to price changes in income in terms of constancy.

In the case of perfectly substitutable goods, the utility function is ([3]):

$U(x_1,...,x_n)=a_1x_1+...+a_nx_n$, $a_i>0$, $i=\overline{1,n}$

and if $\frac{a_1}{p_1}=...=\frac{a_n}{p_n}$ all the points $(\overline{x}_1,...,\overline{x}_n)$ of the budget hyperplane $\sum_{i=1}^{n} p_i \overline{x}_i = V$ are optimal consumption basket components. At a change from the income V to V', the condition of proportionality coefficients of the utility function and prices remaining valid, the new consumption basket verifies $\sum_{i=1}^{n} p_i \overline{x}_i = V'$. The two hyperplanes being parallel, it follows that the locus of variation in demand for the goods is the Engel's hyperplane in $\mathbf{R}^{n+1}$: $\sum_{i=1}^{n} p_i x_i - V = 0$. If $\exists i\neq j=\overline{1,n}$ so that: $\frac{a_i}{p_i}\neq\frac{a_j}{p_j}$ then we consider the partition of $I=\{1,...,n\}$: $I=I_1\cup...\cup I_k$, $I_p\cap I_t=\varnothing$ such that $\forall u,v\in I_p$ we have: $\frac{a_u}{p_u}=\frac{a_v}{p_v}$ and for each $u\in I_p$ and $v\in I_t$, $p\neq t$: $\frac{a_u}{p_u}\neq\frac{a_v}{p_v}$.

Determining $\max_{p=\overline{1,k}} \lambda_p$ where $\lambda_p = \frac{a_u}{p_u}$ $\forall u \in I_p$, we obtain the locus given by: $\sum_{u \in I_p} p_u x_u = V$, $x_v=0$ $\forall v \in I-I_p$. Varying the income V, we obtained finally the Engel's hyperplane of $\mathbf{R}^{n+1}$: $\sum_{u \in I_p} p_u x_u - V = 0$, $x_v=0$ $\forall v \in I-I_p$.

In conditions that for a good B_i, the consumption of the others is constant, we see that: $x_i = \frac{V - \sum_{u \in I_p - \{i\}} p_u x_u}{p_i}$ from where $\frac{dx_i}{dV} = \frac{1}{p_i} > 0$. Therefore, perfectly substitutable goods are always normal in the sense of income.

In particular, for two perfectly substitutable goods, if $\frac{a_1}{p_1} = \frac{a_2}{p_2}$ then, in $\mathbf{R}^3$, the Engel's plane is $p_1x_1+p_2x_2-V=0$. If $\frac{a_1}{p_1} > \frac{a_2}{p_2}$ then the Engel's plane is $p_1x_1-V=0$, $x_2=0$, and if $\frac{a_1}{p_1} < \frac{a_2}{p_2}$ then the Engel's plane is $p_2x_2-V=0$, $x_1=0$.

In care of perfectly complementary goods, for which $U(x_1,...,x_n)=\min(a_1x_1,...,a_nx_n)$, $a_i>0$, $i=\overline{1,n}$, the optimal consumption basket is $\overline{x} = (\overline{x}_1,...,\overline{x}_n)$ where $\overline{x}_j = \frac{V}{a_j \sum_{i=1}^{n} \frac{p_i}{a_i}}$, $j=\overline{1,n}$. The Engel's straight line is in this case:

$$\begin{cases} x_j = \dfrac{\lambda}{a_j \sum_{i=1}^{n} \frac{p_i}{a_i}},\ j=\overline{1,n} \\ V = \lambda \end{cases},\ \lambda \in \mathbf{R}_+$$

In the particular case of two perfectly complementary goods, we have:

$$\begin{cases} x_1 = \dfrac{a_2\lambda}{p_1a_2 + p_2a_1} \\ x_2 = \dfrac{a_1\lambda}{p_1a_2 + p_2a_1},\ \lambda \in \mathbf{R}_+ \\ V = \lambda \end{cases}$$

Noting $\frac{\lambda}{p_1a_2 + p_2a_1}$ =μ we can simply write:

$$\begin{cases} x_1 = a_2\mu \\ x_2 = a_1\mu \quad , \mu \in \mathbf{R}_+ \\ V = (p_1a_2 + p_2a_1)\mu \end{cases}$$

As above, because $x_j = \frac{V}{a_j\sum_{i=1}^{n}\frac{p_i}{a_i}}$ we have: $\frac{dx_j}{dV} = \frac{1}{a_j\sum_{i=1}^{n}\frac{p_i}{a_i}}$ >0. Therefore, the complementary goods are always perfectly normal in the meaning of income.

In the case of independent goods in the meaning of utility, let consider the above example of two goods B_1 and B_2 for which the utility function is $U(x_1,x_2,...,x_n)= \sqrt{x_1} + \sqrt[4]{x_2^3}$. Since the optimal consumption is: $x_1 = \frac{8p_2^3}{81p_1^2}\left(\sqrt{\frac{1}{p_1^2} + \frac{81}{4p_2^3}V} - \frac{1}{p_1}\right)$, $x_2 = -\frac{8p_2^2}{81p_1}\left(\sqrt{\frac{1}{p_1^2} + \frac{81}{4p_2^3}V} - \frac{1}{p_1}\right) + \frac{V}{p_2}$ we have: $\frac{dx_1}{dV} = \frac{1}{p_1^2\sqrt{\frac{1}{p_1^2} + \frac{81}{4p_2^3}V}}$ >0 therefore x_1 is normal in the meaning of income. Also $\frac{dx_2}{dV} = -\frac{1}{p_1p_2\sqrt{\frac{1}{p_1^2} + \frac{81}{4p_2^3}V}} + \frac{1}{p_2}$ >0 and hence x_2 is normal in the meaning of income.

4. The Elasticity of Demand

Another onset of the demand change relative to the prices, refers to relative variations of the goods request in relation to their price or income allocated. The concept of elasticity measures precisely the situation demand.

We will define **the elasticity of demand in relation to the product price at the time t_2, relative to the initial time t_1** as:

$$\varepsilon_{x_i,p_i} = \frac{\frac{x_i(t_2) - x_i(t_1)}{x_i(t_1)}}{\frac{p_i(t_2) - p_i(t_1)}{p_i(t_1)}} = \frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}} = \frac{\delta x_i}{\delta p_i}$$

where we note for simplicity x_i and p_i – the demand level and the price level respectively at t_1, δx_i and δp_i being the relative variation of x_i and p_i respectively at the moment t_1.

We will define **the elasticity of demand in relation to the product price at the time t_1, relative to the initial time t_2** as:

$$\varepsilon_{f,x_i,p_i} = \frac{\dfrac{x_i(t_1) - x_i(t_2)}{x_i(t_2)}}{\dfrac{p_i(t_1) - p_i(t_2)}{p_i(t_2)}} = \frac{\dfrac{\Delta x_i}{x_i}}{\dfrac{\Delta p_i}{p_i}} = \frac{\delta x_i(1+\delta p_i)}{\delta p_i(1+\delta x_i)}$$

where we note for simplicity x_i and p_i – the demand level and the price level respectively at t_2, δx_i and δp_i being the relative variation of x_i and p_i respectively at the moment t_1.

For a more accurate measurement of elasticity, it can be used **the arc elasticity of demand** in relation to the product price, relative to the times t_1 and t_2:

$$\varepsilon_{a,x_i,p_i} = \frac{\dfrac{x_i(t_2) - x_i(t_1)}{\dfrac{x_i(t_1) + x_i(t_2)}{2}}}{\dfrac{p_i(t_2) - p_i(t_1)}{\dfrac{p_i(t_1) + p_i(t_2)}{2}}} = \frac{\delta x_i(2+\delta p_i)}{\delta p_i(2+\delta x_i)}$$

where δx_i and δp_i are the relative variation of x_i and p_i respectively, at the time t_1.

One can easily check that:

$$\varepsilon_{a,x_i,p_i} \text{-1} = \frac{2}{\dfrac{1}{\varepsilon_{x_i,p_i} - 1} + \dfrac{1}{\varepsilon_{f,x_i,p_i} - 1}}$$

so the deviation from 1 of the arc elasticity is the harmonic average of deviations from 1 of the elasticities in relation to initial and final “moments”.

In the differentiable case, we define **the point elasticity of demand in relation to the product price** as:

$$\varepsilon_{x_i,p_i} = \frac{\dfrac{dx_i}{x_i}}{\dfrac{dp_i}{p_i}} = \frac{dx_i}{dp_i} \cdot \frac{p_i}{x_i}$$

From the above, it follows that the normal goods are those for which $\varepsilon_{x_i,p_i}<0$ and Giffen goods for which: $\varepsilon_{x_i,p_i}>0$.

We have now the well-know classification:

- $\left|\varepsilon_{x_i,p_i}\right|=0$ – the demand is perfectly inelastic;
- $\left|\varepsilon_{x_i,p_i}\right|\in(0,1)$ – the demand is inelastic;
- $\left|\varepsilon_{x_i,p_i}\right|=1$ – an unitary elastic demand;
- $\left|\varepsilon_{x_i,p_i}\right|\in(1,\infty)$ – the demand is elastic;
- $\left|\varepsilon_{x_i,p_i}\right|=\infty$ - the demand is perfectly elastic.

Let us note, also, that the elasticity being a point-wise concept, it can vary depending on the product price and therefore its character may change over time.

So we have:

$$\frac{d\varepsilon_{x_i,p_i}}{dp_i}=\frac{d}{dp_i}\left(\frac{dx_i}{dp_i}\cdot\frac{p_i}{x_i}\right)=\frac{1}{x_i^2}\left(\frac{d^2x_i}{dp_i^2}+x_i\frac{dx_i}{dp_i}\right)$$

Therefore, on the demand curve where $\frac{d^2x_i}{dp_i^2}+x_i\frac{dx_i}{dp_i}\geq 0$ the elasticity is increasing like function of the price, and where $\frac{d^2x_i}{dp_i^2}+x_i\frac{dx_i}{dp_i}\leq 0$ it is decreasing relative to the price.

In particular, for a linear function of demand for a normal good: $x_i=a-bp_i$, $a,b>0$, we have: $\frac{d^2x_i}{dp_i^2}+x_i\frac{dx_i}{dp_i}=-bx_i<0$ so the elasticity of a linear demand function for a normal good is strictly decreasing. On the other hand, for $x_i=a-bp_i$, $a,b>0$, we have: $\varepsilon_{x_i,p_i}=\frac{dx_i}{dp_i}\cdot\frac{p_i}{x_i}=\frac{-bp_i}{a-bp_i}=\frac{bp_i}{bp_i-a}$. Because $\lim_{p_i\to 0}\varepsilon_{x_i,p_i}=0$, $\lim_{p_i\to\frac{a}{b}}\varepsilon_{x_i,p_i}=-\infty$, $\varepsilon_{x_i,\frac{a}{2b}}=-1$ follows that the linear demand function for a normal good is elastic for $p_i\in\left(\frac{a}{2b},\frac{a}{b}\right)$, is unitary elastic for $p_i=\frac{a}{2b}$ and inelastic for $p_i\in\left(0,\frac{a}{2b}\right)$.

Returning to our general discussion, the income obtained from the sale of x_i units is $V=p_ix_i$. Differentiating this equality, we get:

$$dV=x_idp_i+p_idx_i= p_ix_i\left(\frac{dp_i}{p_i}+\frac{dx_i}{x_i}\right)= p_ix_i\left(\frac{dp_i}{p_i}+\frac{dp_i}{p_i}\varepsilon_{x_i,p_i}\right)= x_i\left(1+\varepsilon_{x_i,p_i}\right)dp_i$$

Therefore, at an increase in the price of the product ($dp_i>0$), the income will increase ($dV>0$) if and only if $\varepsilon_{x_i,p_i}>-1$ which happens in the inelastic demand case or for Giffen goods, and at a reduction in the price of the product ($dp_i<0$), the income will increase ($dV>0$) if and only if $\varepsilon_{x_i,p_i}<-1$ which happens in the case of elastic demand for normal goods. In the case of unitary elasticity of normal goods ($\varepsilon_{x_i,p_i}=-1$) we have $dV=0$ so we do not suffer any revenue earned change. In this case, the effect of price changes (up or down) has no consequence in terms of total receipts from the sale of the product.

Another type is the **cross-elasticity**. We define thus:

- **The cross elasticity of demand for the i-th good in relation to the price of the j-th good at time t_2, relative to the initially time t_1**, as:

$$\varepsilon_{x_i,p_j}=\frac{\frac{x_i(t_2)-x_i(t_1)}{x_i(t_1)}}{\frac{p_j(t_2)-p_j(t_1)}{p_j(t_1)}}=\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_j}{p_j}}=\frac{\delta x_i}{\delta p_j}$$

- **The cross elasticity of demand for the i-th good in relation to the price of the j-th good at time t_1, relative to the final time t_2**, as:

$$\varepsilon_{f,x_i,p_j}=\frac{\frac{x_i(t_1)-x_i(t_2)}{x_i(t_2)}}{\frac{p_j(t_1)-p_j(t_2)}{p_j(t_2)}}=\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_j}{p_j}}=\frac{\delta x_i(1+\delta p_j)}{\delta p_j(1+\delta x_i)}$$

- **The arc cross elasticity of demand for the i-th good in relation to the price of the j-th good**, as:

$$\varepsilon_{a,x_i,p_j} = \frac{\dfrac{x_i(t_2) - x_i(t_1)}{\dfrac{x_i(t_1) + x_i(t_2)}{2}}}{\dfrac{p_j(t_2) - p_j(t_1)}{\dfrac{p_j(t_1) + p_j(t_2)}{2}}} = \frac{\delta x_i (2 + \delta p_j)}{\delta p_j (2 + \delta x_i)}$$

- **The point cross elasticity of demand for the i-th good in relation to the price of the j-th good**, as:

$$\varepsilon_{x_i,p_j} = \frac{\dfrac{dx_i}{x_i}}{\dfrac{dp_j}{p_j}} = \frac{dx_i}{dp_j} \cdot \frac{p_j}{x_i}$$

The cross elasticity of demand relative to the price of another product means the percentage change in demand for the first good at a percentage change of the second good price.

From the above, we have that the i-th good is a gross substitute for the good j if $\varepsilon_{x_i,p_j} > 0$ and the i-th good is gross complement for good j if $\varepsilon_{x_i,p_j} \le 0$.

The income corresponding to x_i units of good at the price p_i is: $V = p_i x_i + p_j x_j$. Differentiating this equality, we get:

$$dV = x_i dp_i + p_i dx_i + x_j dp_j + p_j dx_j = p_i x_i \left(\frac{dp_i}{p_i} + \frac{dx_i}{x_i} \right) + p_j x_j \left(\frac{dp_j}{p_j} + \frac{dx_j}{x_j} \right) =$$

$$p_i x_i \left(\frac{dp_i}{p_i} + \frac{dp_j}{p_j} \varepsilon_{x_i,p_j} \right) + p_j x_j \left(\frac{dp_j}{p_j} + \frac{dp_j}{p_j} \varepsilon_{x_j,p_j} \right) =$$

$$x_i dp_i + x_j dp_j + \frac{p_i}{p_j} x_i \varepsilon_{x_i,p_j} dp_j + x_j \varepsilon_{x_j,p_j} dp_j .$$

Assuming constant p_i we finnaly get:

$$dV = x_j dp_j + \frac{p_i}{p_j} x_i \varepsilon_{x_i,p_j} dp_j + x_j \varepsilon_{x_j,p_j} dp_j = \left(x_j + \frac{p_i}{p_j} x_i \varepsilon_{x_i,p_j} + x_j \varepsilon_{x_j,p_j} \right) dp_j =$$
$$\left(p_j x_j \left(1 + \varepsilon_{x_j,p_j} \right) + p_i x_i \varepsilon_{x_i,p_j} \right) p_j dp_j .$$

An increase of the revenue earned (dV>0) to an increase in the j-th good price (dp_j>0) will be happened if and only if $p_j x_j (1+\varepsilon_{x_j,p_j}) + p_i x_i \varepsilon_{x_i,p_j}$ >0 and at a decreasing (dp_j<0) if and only if: $p_j x_j (1+\varepsilon_{x_j,p_j}) + p_i x_i \varepsilon_{x_i,p_j}$ <0.

Thus, for normal goods with inelastic demand or Giffen we have: $p_j x_j (1+\varepsilon_{x_j,p_j})$ >0, so their gross substitute (ε_{x_i,p_j} >0) will lead to an increase in income when the price increase. For the same goods j, the necessary condition (but not sufficient) for increased income to a reduction of the price is that the i-th good be a gross complement to j.

For normal goods with elastic demand we have $p_j x_j (1+\varepsilon_{x_j,p_j})$<0, then the necessary condition (but not sufficient) to increase revenue at an increasing of the price that the i-th good to be a gross substitute for j, and if the price is reduced the gross complementarity condition is sufficient.

A final type of elasticity is based on income. We define thus:

- **the elasticity of demand with respect to the income at time t_2, relative to the initial time t_1** as:

$$\varepsilon_{x_i,V} = \frac{\dfrac{x_i(t_2)-x_i(t_1)}{x_i(t_1)}}{\dfrac{V(t_2)-V(t_1)}{V(t_1)}} = \frac{\dfrac{\Delta x_i}{x_i}}{\dfrac{\Delta V}{V}} = \frac{\delta x_i}{\delta V}$$

- **the elasticity of demand with respect to the income at time t_1, relative to the final time t_2** as:

$$\varepsilon_{f,x_i,V} = \frac{\dfrac{x_i(t_1)-x_i(t_2)}{x_i(t_2)}}{\dfrac{V(t_1)-V(t_2)}{V(t_2)}} = \frac{\dfrac{\Delta x_i}{x_i}}{\dfrac{\Delta V}{V}} = \frac{\delta x_i(1+\delta V)}{\delta V(1+\delta x_i)}$$

- **the arc elasticity of demand with respect to the income** as:

$$\varepsilon_{a,x_i,V} = \frac{\dfrac{x_i(t_2)-x_i(t_1)}{\frac{x_i(t_1)+x_i(t_2)}{2}}}{\dfrac{V(t_2)-V(t_1)}{\frac{V(t_1)+V(t_2)}{2}}} = \frac{\delta x_i(2+\delta V)}{\delta V(2+\delta x_i)}$$

- **the point elasticity of demand with respect to the income** as:

$$\varepsilon_{x_i,V} = \frac{\frac{dx_i}{x_i}}{\frac{dV}{V}} = \frac{dx_i}{dV} \cdot \frac{V}{x_i}$$

The elasticity of demand with respect to the income means the percentage change in demand for product at a percentage change of consumer disposable income.

From the above, we have that i-th good is normal in the meaning of income if $\varepsilon_{x_i,V}$ >0 and the good is inferior if $\varepsilon_{x_i,V}$ <0.

The products for which $\varepsilon_{x_i,V}$ >1 are called superior goods (or luxury), and those for which $\varepsilon_{x_i,V}$ <1 – necessity goods.

5. Conclusion

The analysis of the behavior and nature of n goods in relationship with each other is of fundamental importance. The classical analysis consider only two goods, considering a partitioning in a good and the rest.

Several problems arise in this case: what happens when mixed nature of the goods in that basket is left? How the cross-elasticities can be define when the basket of goods contains different prices?

We hope that this article answers at some of these problems by treating goods and obtaining global interesting results, particularly considering special utility functions.

6. References

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