On The Nature of Level Hyper Surfaces in the Economic Theory

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Abstract: In this paper, we will determine the conditions where a level hypersurface (curve) will preserve the character of the original function. The applications are in the theory of the utility and in the theory of production functions.

Keywords: utility; production; convexity; concavity

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1 Introduction

In most of the works in the economics, the authors noticed an implicit assumption or sometimes graphically demonstrated by methods more or less "common sense" relative to the nature of convexity or concavity of a level curve corresponding to an economical concept, be it utility or production function.

If in the classical theories that concern two quantities (goods, factors of production) the nature of the level curves can be determined from purely economic considerations, in the general theory of n goods or factors of production ([5]), this is not at all obvious.

For these reasons, we will broach the issue of level hypersurfaces nature relatively to the basic function, obtaining a result that will demonstrate that when appropriate marginal indicators are positive, the nature will change (from concavity to convexity and vice versa) and if they are negative, the character is preserved.

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2 Main Theorem

For the beginning

The implicit functions theorem (Goursat)

Let a function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D - open set, $n\geq 2$, $f\in C^1(D)$, $(x_1,\dots,x_{n-1},y)\to f(x_1,\dots,x_{n-1},y)$ and $c=(a_1,\dots,a_{n-1},b)\in D$ such that $f(c)=0$. If $\frac{\partial f}{\partial y}(c)\neq 0$ then $\exists W=U\times V\in \mathcal{V}(c)$ such that $U\subset\mathbf{R}^{n-1}, V\subset\mathbf{R}$ and $\varphi:U\to V$, $\varphi\in C^1(U)$, $b=\varphi(a_1,\dots,a_{n-1}), f(x_1,\dots,x_{n-1},\varphi(x_1,\dots,x_{n-1}))=0$ $\forall(x_1,\dots,x_{n-1})\in U$, $\frac{\partial\varphi}{\partial x_i}=-\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial y}}$, $i=\overline{1,n-1}$.

Also, if $f\in C^s(D)$, $s\geq 1$ then $\varphi\in C^s(U)$.

Let therefore $f:D\subset\mathbf{R}^n\to\mathbf{R}$, $\chi\in\mathbf{R}$ such that $f(x_1,\dots,x_n)=\chi$ and suppose that (after a possible renumbering) $\frac{\partial f}{\partial x_n}\neq 0$. Considering the function $g=f-\chi$, the implicit function theorem shows that $\exists\varphi:U\to V$ such that: $x_n=\varphi(x_1,\dots,x_{n-1})$ and $f(x_1,\dots,x_{n-1},\varphi(x_1,\dots,x_{n-1}))=\chi$. In addition: $\frac{\partial\varphi}{\partial x_i}=-\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_n}}$, $i=\overline{1,n-1}$.

We have now:

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j}=\frac{-\frac{\partial^2 f}{\partial x_i\partial x_j}\left(\frac{\partial f}{\partial x_n}\right)^2+\frac{\partial^2 f}{\partial x_i\partial x_n}\frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_n}+\frac{\partial^2 f}{\partial x_n\partial x_j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_n}-\frac{\partial^2 f}{\partial x_n^2}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}}{\left(\frac{\partial f}{\partial x_n}\right)^3},$$

$$i,j=\overline{1,n-1}$$

We put now the issue of determining the second differential of φ respecting to f. We have thus:

$$\left(\frac{\partial f}{\partial x_n}\right)^3 d^2\varphi=\left(\frac{\partial f}{\partial x_n}\right)^3\sum_{i,j=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_j}dx_i dx_j=$$

$$\sum_{i,j=1}^{n-1}\left(-\frac{\partial^2 f}{\partial x_i\partial x_j}\left(\frac{\partial f}{\partial x_n}\right)^2+\frac{\partial^2 f}{\partial x_i\partial x_n}\frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_n}+\frac{\partial^2 f}{\partial x_n\partial x_j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_n}-\frac{\partial^2 f}{\partial x_n^2}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right)dx_i dx_j$$

$=$

$$-\left(\frac{\partial f}{\partial x_n}\right)^2\sum_{i,j=1}^{n-1}\frac{\partial^2 f}{\partial x_i\partial x_j}dx_i dx_j+2\frac{\partial f}{\partial x_n}\sum_{i,j=1}^{n-1}\frac{\partial^2 f}{\partial x_i\partial x_n}\frac{\partial f}{\partial x_j}dx_i dx_j-\frac{\partial^2 f}{\partial x_n^2}\sum_{i,j=1}^{n-1}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}dx_i dx_j$$

$=$

$$-\left(\frac{\partial f}{\partial x_n}\right)^2 d^2f+2\left(\frac{\partial f}{\partial x_n}\right)^2\sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial x_i\partial x_n}dx_i dx_n+2\frac{\partial f}{\partial x_n}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_n}\frac{\partial f}{\partial x_j}dx_i dx_j-2\frac{\partial f}{\partial x_n}\sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial x_i\partial x_n}\frac{\partial f}{\partial x_n}dx_i dx_n-$$

$$2\frac{\partial f}{\partial x_n}\sum_{j=1}^{n-1}\frac{\partial^2 f}{\partial x_n^2}\frac{\partial f}{\partial x_j}dx_n dx_j-\frac{\partial^2 f}{\partial x_n^2}\sum_{i,j=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}dx_i dx_j+2\frac{\partial^2 f}{\partial x_n^2}\sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_n}dx_i dx_n=$$

$$-\left(\frac{\partial f}{\partial x_n}\right)^2 d^2f+2\left(\frac{\partial f}{\partial x_n}\right)^2\sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial x_i\partial x_n}dx_i dx_n+2\frac{\partial f}{\partial x_n}df\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_n}dx_i-2\frac{\partial f}{\partial x_n}\sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial x_i\partial x_n}\frac{\partial f}{\partial x_n}dx_i dx_n-$$

$$2\frac{\partial f}{\partial x_n}\frac{\partial^2 f}{\partial x_n^2}df dx_n+2\left(\frac{\partial f}{\partial x_n}\right)^2\frac{\partial^2 f}{\partial x_n^2}dx_n^2-\frac{\partial^2 f}{\partial x_n^2}(df)^2+2\frac{\partial^2 f}{\partial x_n^2}\frac{\partial f}{\partial x_n}df dx_n-2\frac{\partial^2 f}{\partial x_n^2}\left(\frac{\partial f}{\partial x_n}\right)^2 dx_n^2=$$

$$-\left(\frac{\partial f}{\partial x_n}\right)^2 d^2f+2\frac{\partial f}{\partial x_n}df\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_n}dx_i-\frac{\partial^2 f}{\partial x_n^2}(df)^2$$

Because df=0 we get:

$$d^2\varphi=\sum_{i,j=1}^{n-1}\frac{\partial^2\varphi}{\partial x_i\partial x_j}dx_i dx_j=-\frac{1}{\frac{\partial f}{\partial x_n}}d^2f$$

We can formulate the following theorem:

Theorem

Let $f:D\subset\mathbf{R}^n\to\mathbf{R}$, $\chi\in\mathbf{R}$ such that $f(x_1,...,x_n)=\chi$ and $\exists k=\overline{1,n}:\ \frac{\partial f}{\partial x_k}\neq 0$. Considering $\varphi:U\to V$, $U\subset\mathbf{R}^{n-1}$, $V\subset\mathbf{R}$ such that $x_k=\varphi(x_1,...,x_{k-1},x_{k+1},...,x_n)$ and $f(x_1,...,x_{k-1},\varphi(x_1,...,x_{k-1},x_{k+1},...,x_n), x_{k+1},...,x_n)=\chi$ then:

- if $\frac{\partial f}{\partial x_k}>0$ then f is convex (concave) if and only if φ is concave (convex);

- if $\frac{\partial f}{\partial x_k}<0$ then f is convex (concave) if and only if φ is convex (concave).

3 Application for the Utility Function

For n fixed assets, let the consumer space: $SC=\{(x_1,...,x_n) \mid x_i\geq 0,\ i=\overline{1,n}\}\subset\mathbf{R}^n$ where $x\in SC$, $x=(x_1,...,x_n)$ is a consumption basket or basket of goods. We also consider an arbitrary norm $\|\cdot\|$ defined on $\mathbf{R}^n$.

We define also on *SC*, the relationship of indifference SC noted, below, with: ~.

If two baskets x and y are related: x~y, this means that any combination of goods x and y it is indifferent to the consumer. Also, we will note $x \nsim y$ the fact that x is not indifferent to y.

We impose to the indifference relationship the following axioms:

I.1. $\forall x\in SC \Rightarrow x\sim x$ (reflexivity);

I.2. $\forall x,y\in SC$, $x\sim y\Rightarrow y\sim x$ (symmetry);

I.3. $\forall x,y,z\in SC$, $x\sim y$, $y\sim z\Rightarrow x\sim z$ (transitivity);

I.4. $\forall x,y\in SC$, $x\sim y$, $\|x\|<\|y\| \Rightarrow \exists z\in SC$ such that $x\sim z$ and $\|x\|<\|z\|<\|y\|$ (the axiom of continuity);

I.5. $\forall x\in SC\Rightarrow\exists u\sim x$ such that $\|u\|\leq\|v\|$ $\forall v\sim x$ (the condition of lower bounds of the indifference classes).

Considering (*SC*,~), let $x\in SC$. The equivalence class of x: $[x]=\{y\in SC \mid y\sim x\}$ will be the total consumption baskets indifferent in relation to x. We will call [x] – the indifference class of x.

For $x\in SC$, we will call, in the assumption of continuity axiom, the indifference class of x as the indifference hypersurface or for n=2 – the indifference curve.

Relative to the axiom I.5, we call u the minimal basket of goods in the meaning of the norm relative to the indifference class of $x\in SC$ and we will note m(x).

Now we define preference relationship on classes marked, below, with $\succeq$ through the following axioms:

P.1. $\forall[x]\in SC/\sim\Rightarrow[x]\succeq[x]$ (reflexivity);

P.2. $\forall[x],[y]\in SC/\sim$, $[x]\succeq[y]$, $[y]\succeq[x]\Rightarrow[x]=[y]$ (skew-symmetry);

P.3. $\forall[x],[y],[z]\in SC/\sim$, $[x]\succeq[y]$, $[y]\succeq[z]\Rightarrow[x]\succeq[z]$ (transitivity);

P.4. $\forall x,y\in SC\Rightarrow[x]\succeq[y]$ or $[y]\succeq[x]$ (the total ordering);

P.5. $\forall x\in SC\Rightarrow\exists y\in SC$ such that $y\not\sim x$ and $[y]\succeq[x]$;

P.6. $[x]\succeq[y]$ if and only if $\|m(x)\|\geq\|m(y)\|$ (the compatibility with the existence of minimum baskets);

P.7. $\forall x,y\in SC$, $x>y\Rightarrow[x]\succeq[y]$ and $x\not\sim y$ (the compatibility with strict inequality relation).

We now define the utility function as:

$$U:SC\to\mathbf{R}_+,\ (x_1,\ldots,x_n)\to U(x_1,\ldots,x_n)\in\mathbf{R}_+\ \forall(x_1,\ldots,x_n)\in SC$$

satisfying the following axioms:

U.1. $\forall x,y\in SC/\sim$: $[x]=[y]\Leftrightarrow U([x])=U([y])$;

U.2. $\forall x,y\in SC/\sim$: $[x]\succeq[y]\Leftrightarrow U([x])\geq U([y])$;

U.3. $U(0)=0$

We require to the utility function the additional conditions:

U.4. The utility function is concave;

U.5. The utility function is of class C^2 on the inside of *SC*.

Considering $\chi>0$, the graph corresponding to the equation solutions $U(x)=a$ is called curve (in $\mathbf{R}^2$) or isoutility hypersurface (in $\mathbf{R}^n$).

We define also the marginal utility relative to a good "k": $U_{m,k}=\frac{\partial U}{\partial x_k}$. The economic theory says that for a rational consumer, according to the constancy of consumption of other goods, the marginal utility must be positive, otherwise the consumer recording a decrease of total utility which implicitly would lead to economic nonsense.

So the question arises about the isoutility hypersurface (curve) nature in relation to the good "k".

From the theorem, it follows, however, that if the isoutility hypersurface is represented explicitly by $x_k=\varphi(x_1,...,x_{k-1},x_{k+1},...,x_n)$ then it is opposed to the corresponding character of the utility function.

Therefore, as U is a concave function, it follows that all of the isoutility hypersurfaces will be convex.

4 Application for the Production Function

We define on $\mathbf{R}^n$ the production space for n fixed resources as $SP=\{(x_1,...,x_n) \mid x_i\geq 0, i=\overline{1,n}\}$ where $x\in SP$, $x=(x_1,...,x_n)$ is a ordered set of resources.

Because in a production process, depending on the nature of technology applied, but also its specificity, not any amount of resources possible, we will restrict the production space to a subset $D_p\subset SP$ called field of production.

We will call production function an application $Q:D_p\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+$ $\forall(x_1,...,x_n)\in D_p$ satisfying the following axioms:

FP1. D_p is convex;

FP2. $Q(0)=0$;

FP3. $Q\in C^2(D_p)$;

FP4. Q is monotonically increasing in each variable;

FP5. Q is concave.

Considering a production function $Q:D_p\to\mathbf{R}_+$ and $\overline{Q}\in\mathbf{R}_+$ **-** fixed, the set of inputs which generate $\overline{Q}$ is called isocuant.

As above, we assume that the marginal productivity of factors of production relative to x_k: $\eta_{x_k}=\dfrac{\partial Q}{\partial x_k}$ is positive, representing the trend of variation of production to changes in factor x_k.

The condition is absolutely normal in the theory, because no economic agent will not supplement the factors of production (labor, capital etc.) if this will lead to a decrease in effective results.

From theorem, therefore follows that for a isoquant explicitly represented by $x_k=\varphi(x_1,...,x_{k-1}, x_{k+1},...,x_n)$, its character is opposed to the feature production. Therefore, as Q is a concave function, it follows that isoquants will be convex.

5 Conclusions

Following the above analysis, we detach a series of conclusions important in the theoretical and practical approaches. On the one hand, the above considerations cover, we hope, a present gap in most economic works, where the character of the level hypersurfaces (curves) is granted. If in the classical theory involving consideration of only two quantities (goods, factors of production) the nature of level curves might be somehow determined, in the general theory of n goods or factors of production this is not at all obvious.

On the other hand, we will offer a "justification" from a purely mathematical nature of the necessity of positiveness marginal indicators, in light of the fact that in the negativity area the character of the level hypersurfaces (curves) will change, the whole theory of minimize income or costs being overturn.

6 References

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