A GENERAL TYPE OF ALMOST CONTACT MANIFOLDS

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Abstract: *Among almost contact manifolds Sasakian manifolds, Kenmotsu manifolds (called also "a certain class of almost contact manifolds") and cosymplectic manifolds have been studied by many authors.*

The purpose of this paper is to obtain a class of almost contact manifolds which will generalize the above manifolds.

The paper generalizes the RK-manifolds introduced by Lieven Vanhecke. I give some results concerning the submanifolds of these spaces, the behaviour of these submanifolds at conformal, projective and concircular transformations. Also I obtain a similar result with those on RK-manifolds but in a form a little weaker when they satisfy the axiom of 2p+1-coholomorphic spheres.

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1. Introduction

Among almost contact manifolds Sasakian manifolds, Kenmotsu manifolds (*called also "a certain class of almost contact manifolds"*) and cosymplectic manifolds have been studied by many authors. In [1], [2], [3] we find the principal results about these manifolds.

The purpose of this paper is to obtain a class of almost contact manifolds which will generalize the above manifolds.

After some general results, we have obtained the Riemann-Christoffel tensor in the case of constant φ-sectional curvature. In the last paragraph we study a subclass of this general type which is richer in information.

2. Preliminaries

We call an **almost contact metric manifold**, one denoted by M^{2n+1} for which:

(1) $\varphi^2X=-X+\eta(X)\xi$

(2) $\eta(\xi)=1$

(3) $\varphi\xi=0$

(4) $\eta(\varphi X)=0$

(5) $g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y)\ \forall X,Y\in \mathcal{X}(M)$

where φ is a (1,1)-type tensor field, η a 1-form, ξ a vector field (named the characteristic vector field) and g is the associated Riemannian metric on M.

The 2-fundamental form is:

(6) $\phi(X,Y)=g(X,\varphi Y)\ \forall X,Y\in \mathcal{X}(M)$

On an almost contact manifold we define the tensor:

(7) $N^1(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+\varphi^2[X,Y]+2d\eta(X,Y)\xi\ \forall X,Y\in \mathcal{X}(M)$

A manifold with an almost contact metric structure and $N^1=0$ is called **normal manifold**.

An almost contact manifold with $\phi=d\eta$ is called a **contact manifold**. A normal contact manifold is a **Sasakian manifold**.

If on an almost contact manifold we have: $(\nabla_X\varphi)Y=g(X,Y)\xi-\eta(Y)X$ $\forall X,Y\in \mathcal{X}(M)$ the manifold is Sasakian. We also have $\nabla_X\xi=-\varphi X\ \forall X\in \mathcal{X}(M)$.

An almost contact manifold is a **Kenmotsu manifold** if $(\nabla_X\varphi)Y=-g(X,\varphi Y)\xi-\eta(Y)\varphi X$, $\nabla_X\xi=X-\eta(X)\xi\ \forall X,Y\in \mathcal{X}(M)$.

A **cosymplectic manifold** is a normal manifold with ϕ and η closed. On a cosymplectic manifold we have: $\nabla_X\varphi=0$, $\nabla_X\xi=0\ \forall X\in \mathcal{X}(M)$.

For every $p\in M$ and $X\in T_pM$, X orthogonal on ξ we define the φ-sectional curvature like $K(X,\varphi X)$ where K is the sectional curvature.

3. A general type of almost contact manifolds

Definition An almost contact manifold M^{2n+1} is called **a general type of almost contact manifold** (short gt-manifold) if there are a (1, 1)-type tensor field $\Psi:\mathcal{X}(M)\to\mathcal{X}(M)$ and a function $\beta\in \mathcal{F}(M)$ which satisfy the following conditions:

(8) $(\nabla_X\varphi)Y=g(\Psi X,Y)\xi-\eta(Y)\Psi X\ \forall X,Y\in \mathcal{X}(M)$

(9) $\nabla_X\xi=-\Psi\varphi X\ \forall X\in \mathcal{X}(M)$

(10) $g(\Psi X,X)=\beta\ \forall X\perp\xi,\ g(X,X)=1$

(11) $\nabla_\xi\Psi=0$

In what follows for the simplification we write:

(12) $\eta(\Psi\xi)=\alpha$

Let in (8) $Y=\xi$. We obtain:

(13) $-\varphi\nabla_X\xi=\eta(\Psi X)\xi-\Psi X\ \forall X\in \mathcal{X}(M)$

Applying φ in (13) we obtain:

(14) $\varphi\Psi=\Psi\varphi$

From (9),(13) we have:

(15) $\eta(\Psi X)\xi=\eta(X)\Psi\xi\ \forall X\in \mathcal{X}(M)$

For $X=\xi$ in (15) and using (12) we have:

(16) $\Psi\xi=\alpha\xi$

and

(17) $\eta(\Psi X)=\alpha\eta(X)\ \forall X\in \mathcal{X}(M)$

From (17) we obtain that the contact distribution $D=\{X\mid\eta(X)=0\}$ is invariant through Ψ.

From (9) we obtain that

(18) $\nabla_\xi\xi=0$

In consequence we have the following:

Theorem 1 In a gt-manifold the integral curves of ξ are geodesics.

Using (8),(16) we have also:

(19) $\nabla_\xi\varphi=0$

Now if in (10) X is not unitary we have $g(\Psi X,X)=\beta g(X,X)\ \forall X\perp\xi$ and puting $X=Y-\eta(Y)\xi$ we obtain:

(20) $g(\Psi Y,Y)=\beta(Y,Y)+(\alpha-\beta)\eta^2(Y)\ \forall Y\in \mathcal{X}(M)$

Reciprocally, from (20) we obtain (10).

Lemma 2 On an almost contact manifold M^{2n+1} which satisfy (8),(9) we have that (10) is equivalent with $d\eta=\beta\phi$.

Proof We have seen that (10) is equivalent with (20). Let suppose that (20) are valid. Polarizing, we obtain:

(21) $g(\Psi X,Y)+g(\Psi Y,X)=2\beta g(X,Y)+2(\alpha-\beta)\eta(X)\eta(Y)\ \forall X,Y\in \mathcal{X}(M)$

We have also $(\nabla_X\eta)Y=\nabla_X g(Y,\xi)-g(\nabla_X Y,\xi)=g(Y,\nabla_X\xi)=g(\Psi X,\varphi Y)$ and

(22) $2d\eta(X,Y)=(\nabla_X\eta)Y-(\nabla_Y\eta)X=g(\Psi X,\varphi Y)-g(\Psi Y,\varphi X)\ \forall X,Y\in \mathcal{X}(M)$

Writing (21) for $Y \to \varphi Y$ we obtain:

(23) $2\beta\phi(X,Y)=g(\Psi X,\varphi Y)-g(\Psi Y,\varphi X)\ \forall X,Y\in \mathcal{X}(M)$

From (22),(23) we have that:

(24) $d\eta=\beta\phi$

Suppose now that (24) are valid. Going back, we obtain (23) and for $X\to\varphi Y$ we obtain (20). Q. E. D.

From (20) we obtain also a formula which we need later:

(25) $\mathrm{tr}\,\Psi=2n\beta+\alpha$

where $\mathrm{tr}\,\Psi$ is the trace of the operator Ψ.

From (6),(8) we have:

(26) $3d\phi(X,Y,Z)=X\phi(Y,Z)-Y\phi(X,Z)+Z\phi(X,Y)-\phi([X,Y],Z)+\phi([X,Z],Y)-\phi([Y,Z],X)=g(Y,(\nabla_X\varphi)Z)-g(X,(\nabla_Y\varphi)Z)+g(X,(\nabla_Z\varphi)Z)=\eta(X)[g(\Psi Z,Y)-g(\Psi Y,Z)]+\eta(Y)[g(\Psi X,Z)-g(\Psi Z,X)]+\eta(Z)[g(\Psi Y,X)-g(\Psi X,Y)]\ \forall X,Y,Z\in \mathcal{X}(M)$

From (24) we have:

Theorem 3 A gt-manifold with $\beta=0$ has η closed.

From (26) we obtain:

Theorem 4 A gt-manifold with Ψ a symmetric operator has ϕ closed.

From (7),(8),(24) we obtain:

(27) $N^1(X,Y)=0\ \forall X,Y\in \mathcal{X}(M)$

therefore we have:

Theorem 5 A gt-manifold is a normal manifold.

4. Examples

1. For $\Psi=I$ and $\beta=1$ we obtain Sasakian manifolds

2. For $\Psi=\varphi$ and $\beta=0$ we have Kenmotsu manifolds

3. For $\Psi=0$ and $\beta=0$ we have cosymplectic manifolds

5. Curvature properties

Now we have:

(28) $R(X,Y)\xi=\nabla_X\nabla_Y\xi-\nabla_Y\nabla_Z\xi-\nabla_{[X,Y]}\xi=\alpha\eta(Y)\Psi X-\alpha\eta(X)\Psi Y+\varphi((\nabla_Y\Psi)X)-\varphi((\nabla_X\Psi)Y)$

Using (9), (16) we have:

(29) $(\nabla_X\Psi)\xi=X(\alpha)\xi-\alpha\varphi\Psi X+\varphi\Psi^2X$

For $Y=\xi$ in (28) and using (11),(16),(29) we obtain:

(30) $R(X,\xi)\xi=\Psi^2X-\alpha^2\eta(X)\xi$

From (30) we have:

(31) $K(X,\xi)=g(R(X,\xi)\xi,X)=g(\Psi^2X,X)\ \forall X\perp\xi,\ g(X,X)=1$

On the other hand, from (21) we obtain:

(32) $g(\Psi^2X,X)=-g(\Psi X,\Psi Y)+2\beta g(\Psi X,Y)-2\alpha(\alpha-\beta)\eta(X)\eta(Y)$

Using now (31),(32) we obtain finally:

(33) $K(X,\xi)=2\beta^2-g(\Psi X,\Psi X)\ \forall X\perp\xi,\ g(X,X)=1$

Theorem 6 A gt-manifold has $K(X,\xi)\leq 2\beta^2$ where $X\perp\xi$, $g(X,X)=1$

Corollary 7 A gt-manifold with $\beta=0$ has $K(X,\xi)\leq 0$.

We now define a (0,4)-tensor field $A:\mathcal{X}(M)^4\to\mathcal{F}(M)$:

(34) $A(X,Y,Z,V)=g(\varphi X,\Psi Z)g(\varphi Y,\Psi V)-g(X,\Psi Z)g(Y,\Psi V)\ \forall X,Y,Z,V\in\mathcal{X}(M)$

We obtain immediately:

(35) $A(X,Y,Z,V)=A(Y,X,V,Z)$

(36) $A(\varphi X,\varphi Y,Z,V)=A(X,Y,\varphi Z,\varphi V)=-A(X,Y,Z,V)$

$A(\varphi X,\varphi Y,\varphi Z,\varphi V)=A(X,Y,Z,V)$

$A(\varphi X,Y,Z,V)=A(X,\varphi Y,Z,V)$

$A(X,Y,\varphi Z,V)=A(X,Y,Z,\varphi V)\ \forall X,Y,Z,V\perp\xi$

We also define $B:\mathcal{X}(M)^4\to\mathcal{F}(M)$ a (0,4)-tensor field:

(37) $B(X,Y,Z,V)=A(X,Y,Z,V)-A(X,Y,V,Z)$

We have from (35),(36),(37) that:

(38) $B(X,Y,Z,V)=B(Y,X,V,Z)=-B(Y,X,Z,V)=-B(X,Y,V,Z)$

$B(\varphi X,Y,Z,\varphi V)=B(X,Y,Z,V) \ \forall X,Y,Z,V\perp\xi$

Using now (8),(9),(34)-(38) we can prove that:

(39) $R(\varphi X,\varphi Y,\varphi Z,\varphi V)=R(X,Y,Z,V)+B(X,Y,Z,V)-B(V,Z,Y,X)$

$R(X,Y,\varphi Z,\varphi V)=R(X,Y,Z,V)+B(V,Z,Y,X)$

$R(\varphi X,\varphi Y,Z,V)=R(X,Y,V,Z)+B(X,Y,Z,V)$

$R(X,\varphi Y,Z,\varphi V)+R(\varphi X,Y,Z,\varphi V)=B(X,Y,V,Z) \ \forall X,Y,Z,V\perp\xi$

Let suppose now that $K(X,\varphi X)=K=$constant. We have:

Theorem 8 If a gt-manifold has constant φ-sectional curvature then:

$$\begin{aligned}(40)\ 4R(X,Y,Z,V)=\ &2B(X,Y,V,Z)+B(X,V,Y,Z)+B(X,Z,V,Y)+\\ &4g((\nabla_Y\Psi)X-(\nabla_X\Psi)Y,\eta(Z)\varphi Y-\eta(V)\varphi Z)+\\ &4g((\nabla_V\Psi)Z-(\nabla_Z\Psi)V,\eta(X)\varphi Y-\eta(Y)\varphi X)+\\ &\eta(V)\eta(Y)((3\alpha-8\beta)g(\Psi X,Z)+(2\alpha\beta-K)g(X,Z)+4g(\Psi X,\Psi Z))-\\ &\eta(Y)\eta(Z)((3\alpha-8\beta)g(\Psi X,V)+(2\alpha\beta-K)g(X,V)+4g(\Psi X,\Psi V))+\\ &\eta(X)\eta(Z)((\alpha-8\beta)g(\Psi Y,V)+(4\alpha\beta-K)g(Y,V)+4g(\Psi Y,\Psi V))-\\ &\eta(X)\eta(V)((\alpha-8\beta)g(\Psi Y,Z)+(4\alpha\beta-K)g(Y,Z)+4g(\Psi Y,\Psi Z))+\\ &\alpha\eta(X)\eta(Y)[2g(\Psi Z,V)-2\beta g(Z,V)+2(\alpha-\beta)\eta(Z)\eta(V)]+\\ &K[g(X,Z)g(Y,V)-g(X,V)g(Y,Z)+\phi(X,Z)\phi(Y,V)-\\ &\phi(X,V)\phi(Y,Z)+2\phi(X,Y)\phi(Z,V)] \ \forall X,Y,Z,V\in \mathcal{X}(M)\end{aligned}$$

with the above notations and $R(X,Y,Z,V)=g(R(X,Y)V,Z)$.

Proof From the hypothesis, we have:

(41) $R(X,\varphi X,X,\varphi X)=Kg(X,X)^2 \ \forall X\perp\xi$

For $X\to X+Y$ in (41) then $X\to X-Y$ in (41) and adding:

(42)
$2R(X,\varphi X,Y,\varphi Y)+2R(X,\varphi Y,Y,\varphi X)+R(Y,\varphi X,Y,\varphi X)+R(X,\varphi Y,X,\varphi Y)=4Kg(X,Y)^2+2Kg(X,X)g(Y,Y)\ \forall X,Y\perp\xi$

If in (42) we put $X\to X+\varphi Z$ then in what we obtained $Y\to Y+\varphi V$ and using (39):

(43)
$2R(X,Z,Y,V)+2R(X,V,Z,Y)+R(\varphi X,Y,\varphi V,Z)=2K[g(X,Y)g(Z,V)+\phi(X,V)\phi(Y,Z)+\phi(X,Z)\phi(Y,V)+B(X,Z,V,Y)+B(V,X,Y,Z)+B(Y,\varphi X,\varphi V,Z)\ \forall X,Y,Z,V\perp\xi$

If in ((43) we change Y with Z and subtract from (43) we have:

(44) $4R(X,Y,V,Z)=2B(X,V,Z,Y)+B(X,Z,V,Y)+B(X,Y,Z,V)+K[g(X,Y)g(Z,V)-g(X,Z)g(Y,V)+2\phi(X,V)\phi(Y,Z)+\phi(X,Z)\phi(Y,V)-\phi(X,Y)\phi(Z,V)]\ \forall X,Y,Z,V\perp\xi$

If in (44) we replace X with $X-2\eta(X)\xi$, Y with $Y-\eta(Y)\xi$, Z with $Z-\eta(Z)\xi$ and V with $V-\eta(V)\xi$ we obtain (40). Q.E.D.

If we return now at examples, we obtain the well-known expressions.

The calculus of the Ricci tensor and the scalar of curvature using (25) and (40) is immediate.

About Ricci tensor, let note that on a gt-manifold we have from (30) that:

(45) $Ric(\xi,\xi)=tr\Psi^2-\alpha^2$

6. A special general type of almost contact manifolds

Definition We call **special general type of an almost contact manifold** (short special gt-manifold) a gt-manifold M^{2n+1} which has in addition:

(46) $(\nabla_X\Psi)Y=(\nabla_Y\Psi)X\ \forall X,Y\in \mathcal{X}(M)$

From section 4 we have that Sasakian manifolds and cosymplectic manifolds are special gt-manifolds.

Using (11),(29),(46) we have for $X=\xi$ that

(47) $Y(\alpha)\xi-\alpha\varphi\Psi Y+\varphi\Psi^2Y=0$

From (4),(47) we have:

(48) $Y(\alpha)=0$ therefore α is constant.

(49) $\Psi^2Y-\alpha\Psi Y\in Span(\xi)$

From (49) for $Y=\xi$ we obtain that:

(50) $\Psi^2 Y=\alpha\Psi Y$

From (31) we obtain that:

(51) $K(X,\xi)=\alpha\beta\ \forall X\perp\xi,\ g(X,X)=1$

Using (32),(50) we have:

(52) $g(\Psi X,\Psi Y)=(2\beta-\alpha)g(\Psi X,Y)-2\alpha(\alpha-\beta)\eta(X)\eta(Y)\ \forall X,Y\in \mathcal{X}(M)$

If we have now $\alpha=2\beta$ from (52) where $X=Y=\xi$ we obtain that $\alpha=\beta=0$ and reciprocally if $\alpha=\beta=0$ we have that $\alpha=2\beta$.

If $\beta=0$ from (52) where $X=Y=\xi$ we obtain $\alpha=0$ and again from (52) we have $\Psi X=0$. From section 4, 3 we have that the manifold is cosymplectic.

Theorem 9 A special gt-manifold is cosymplectic, if and only if $\beta=0$.

Suppose now that the manifold is not cosymplectic. Interchanging X and Y in (52) and subtract from it:

(53) $(2\beta-\alpha)[g(\Psi X,Y)-g(\Psi Y,X)]=0\ \forall X,Y\in \mathcal{X}(M)$

From the hypothesis we have that $2\beta\neq\alpha$ then

(54) $g(\Psi X,Y)=g(\Psi Y,X)\ \forall X,Y\in \mathcal{X}(M)$

therefore Ψ is a symmetric operator.

Using the facts that a cosymplectic manifold has ϕ closed and the theorem 4, we conclude:

Theorem 10 A special gt-manifold has ϕ closed.

We can now reformulate the theorem 8:

Theorem 11 If a special gt-manifold which is not cosymplectic has constant φ-sectional curvature then:

(55) $4R(X,Y,Z,V)=g(\varphi X,\Psi V)g(\varphi Y,\Psi Z)+g(\varphi X,\Psi Z)g(\varphi V,\Psi Y)-2g(\varphi X,\Psi Y)g(\varphi Z,\Psi V)+$

$3g(X,\Psi Z)g(Y,\Psi V)-3g(X,\Psi V)g(Y,\Psi Z)+$

$\eta(V)\eta(Y)((2\alpha\beta-K)g(X,Z)+\alpha g(\Psi X,Z))-$

$\eta(Y)\eta(Z)((2\alpha\beta-K)g(X,V)+\alpha g(\Psi X,V))+$

$\eta(X)\eta(Z)((4\alpha\beta-K)g(Y,V)-3\alpha g(\Psi Y,V))-$

$\eta(X)\eta(V)((4\alpha\beta-K)g(Y,Z)-3\alpha g(\Psi Y,Z))+$

$\alpha\eta(X)\eta(Y)[2g(\Psi Z,V)-2\beta g(Z,V)+2(\alpha-\beta)\eta(Z)\eta(V)]+$

$K[g(X,Z)g(Y,V)-g(X,V)g(Y,Z)+\phi(X,Z)\phi(Y,V)-$

$\phi(X,V)\phi(Y,Z)+2\phi(X,Y)\phi(Z,V)]\ \forall X,Y,Z,V\in \mathcal{X}(M)$

where R(X,Y,Z,V)=g(R(X,Y)V,Z) and K is the constant φ-sectional curvature.

From (55) we obtain also:

(56) $2Ric(X,Y)=\eta(X)\eta(Y)(7\alpha^2+(n-6)\alpha\beta-K(n+1))+g(X,Y)(\alpha\beta+K(n+1))+$

$g(\varphi X,Y)(\alpha+3\beta(n-1))\ \forall X,Y\in \mathcal{X}(M)$

Ric being the Ricci tensor on M^{2n+1}.

(57) $S=4\alpha^2+(3n-4)\alpha\beta+3n(n-1)\beta^2$

where S is the scalar of curvature.

From (57) we obtain immediately:

Theorem 12 A special gt-manifold, not cosymplectic, having constant φ-sectional curvature and of dimension greater than 3 has positive scalar of curvature.

Almost Hermitian manifolds with J-invariant sectional curvature

Let (M,g) a differentiable manifold with the metric tensor g. M is named **almost Hermitian** if there exists an endomorphism $J:\mathcal{X}(M)\to\mathcal{X}(M)$ of the Lie algebra of tensor fields $\mathcal{X}(M)$ such that $J^2=-I$ and g is J-invariant that is $g(JX,JY)=g(X,Y)\ \forall X,Y\in \mathcal{X}(M)$.

In [4] L. Vanhecke defines **RK-manifolds** like manifolds almost hermitian with J-invariant curvature Riemann tensor, that is $R(JX,JY,JZ,JV)=R(X,Y,Z,V)$ $\forall X,Y,Z,V\in \mathcal{X}(M)$.

In [3] are defined **para-Kähler manifolds** like almost hermitian manifolds with $R(X,Y,JZ,JV)=R(X,Y,Z,V)\ \forall X,Y,Z,V\in \mathcal{X}(M)$.

A **Kähler manifold** is an almost Hermitian manifold for which the 2-fundamental form is closed, where $\Phi(X,Y)=g(JX,Y)\ \forall X,Y\in \mathcal{X}(M)$ and the Nijenhuis tensor corresponding to J vanishes. In a Kähler manifold we have ([1]): $R(X,JY,Z,V)=R(Y,JX,Z,V)\ \forall X,Y,Z,V\in \mathcal{X}(M)$.

We have, in consequence, that Kähler manifolds are para-Kähler which their turn are RK-manifolds. Let note the sectional curvature by the 2-plane (X,Y) in any point of the manifold with k(X,Y) and $K(X,Y)=k(X,Y)[g(X,X)g(Y,Y)-g(X,Y)^2]$. We note also H(X)=k(X,JX) the holomorphic sectional curvature corresponding to X. It is proved in [5] that on a RK-manifold we have k(X,Y)=k(JX,JY), k(X,JY)=k(JX,Y), S(X,Y)=S(JX,JY), S(X,JY)+S(JX,Y)=0 $\forall X,Y\in \mathcal{X}(M)$ where S is the Ricci tensor.

In this paper I shall enlarge the RK-manifolds class and I shall study some properties of these manifolds.

2. Almost RK-manifolds

Definition 1 An **almost RK-manifold** (short **RKA-manifold**) is an almost Hermitian manifold for which $K(X,Y)=K(JX,JY)$ $\forall X,Y\in \mathcal{X}(M)$.

Remarks An RK-manifold is an RKA-manifold. Manifolds with constant curvature are also RKA-manifolds.

From the definition follows immediately that:

(1) $R(X,Y,V,Z)+R(X,Z,V,Y)=R(JX,JY,JV,JZ)+R(JX,JZ,JV,JY)$ $\forall X,Y,Z,V\in \mathcal{X}(M)$

If we take an orthonormal basis in M: $X_1,...,X_n$ and put $Y=Z=X_i$ and summing for i, we obtain; $S(X,V)=S(JX,JV)$ $\forall X,V\in \mathcal{X}(M)$. In consequence, the property of the Ricci tensor to be invariant at the action of J remains valid in RKA-manifolds.

Let now study the behaviour of RKA-manifolds at the time when they admit some special submanifolds.

Let $(M,g)\subset(\overline{M},\overline{g})$ a submanifold of an almost Hermitian manifold $(\overline{M},\overline{g})$. The Gauss equation is:

(2) $$\overline{R}(X,Y,Z,V)=R(X,Y,Z,V)-\overline{g}(h(X,Z),h(Y,V))+\overline{g}(h(X,V),h(Y,Z))$$ $\forall X,Y,Z,V\in \mathcal{X}(M)$

Definition 2 A **submanifold** $(M,g)\subset(\overline{M},\overline{g})$ is called **totally cuasi-umbilical** if the second fundamental form h is:

$$h(X,Y)=g(X,Y)H+[\omega(X)\omega(Y)+\omega(JX)\omega(JY)]A \quad \forall X,Y\in \mathcal{X}(M)$$

where H is the mean curvature vector and $A\in \mathcal{X}(M)^{\perp}$, ω being a 1-form on M.

In particular, if ω=0 we obtain **totally umbilical submanifolds** and if, in addition H=0, we have **totally geodesic submanifolds**.

For totally cuasi-umbilical submanifolds, we have:

(3) $\overline{K}(X,Y)=K(X,Y)+\overline{g}(H,H)[g^2(X,Y)-g(X,X)g(Y,Y)]+$
$\overline{g}(H,A)[2\omega(X)\omega(Y)g(X,Y)+2\omega(JX)\omega(JY)g(X,Y)-g(X,X)(\omega^2(Y)+\omega^2(JY))-$
$g(Y,Y)(\omega^2(X)+\omega^2(JX))]-\overline{g}(A,A)[\omega(X)\omega(JY)-\omega(Y)\omega(JX)]^2\ \forall X,Y\in \mathcal{X}(M)$

Writing (3) for JX and JY and subtract the two relations, we obtain:

(4) $\overline{K}(JX,JY)-\overline{K}(X,Y)=K(JX,JY)-K(X,Y)\ \forall X,Y\in \mathcal{X}(M)$

where we have noted with bar all the quantities on M.

In consequence, we have:

Theorem 1 A totally cuasi-umbilical submanifold of an RKA-manifold is an RKA-manifold.

Corollary 1 A totally umbilical submanifold of an RKA-manifold is an RKA-manifold.

Corollary 2 A totally geodesic submanifold of an RKA-manifold is an RKA-manifold.

The conformal curvature tensor of a manifold is:

(5) $C(X,Y,Z,V)=r(X,Y,Z,V)+g(X,V)L(Y,Z)+g(Y,Z)L(X,V)-g(X,Z)L(Y,V)-$
$g(Y,V)L(X,Z)\ \forall X,Y,Z,V\in \mathcal{X}(M)$

where $L(X,Y)=\frac{1}{n-2}\left(S(X,Y)-\frac{\rho}{2(n-1)}g(X,Y)\right)$, ρ being the scalar of curvature.

Immediately, we obtain that:

(6) $C(X,Y,X,Y)-C(JX,JY,JX,JY)=K(X,Y)-K(JX,JY)\ \forall X,Y\in \mathcal{X}(M)$

From (6) follows:

Theorem 2 If an RKA-manifold is conformable with another manifold the second is also RKA-manifold.

In the same manner, considering the Weyl projective tensor: $P(X,Y)Z=R(X,Y)Z+\frac{1}{n-1}(S(X,Z)Y-S(Y,Z)X)$ and the Yano concircular tensor $K(X,Y)Z=R(X,Y)Z-\frac{\rho}{n(n-1)}(g(Y,Z)X-g(X,Z)Y)$ where n=dim M, we obtain:

Theorem 3 At projective transformations RKA-manifolds applied on RKA-manifolds.

Theorem 4 At concircular transformations RKA-manifolds applied on RKA-manifolds.

3. RKA-manifolds with punctual constant type

In what follows are necessary some definitions.

Definition 3 Let $p\in M$. A subspace N_p of T_pM is called **holomorphic subspace** if $J(N_p)\subset N_p$ and **antiholomorphic** if $J(N_p)\subset N_p^{\perp}$.

Definition 4 A 2p+1-dimensional subspace is called **2p+1-coholomorphic plane** if it contains a 2p-holomorphic plane.

It shows in [5] that a 2p+1-coholomorphic plane contains a p+1-antiholomorphic plane and $1\le p\le q-1$ where $\dim M=2q$.

Definition 5 An almost Hermitian manifold has **constant type** in $p\in M$ if for any $X\in T_pM$ we have: $\lambda(X,Y)=\lambda(X,Z)$ where (X,Y), (X,Z) are antiholomorphic planes, $g(Y,Y)=g(Z,Z)$ and $\lambda(X,Y)=R(X,Y,X,Y)-R(X,Y,JX,JY)$. If the **manifold** has constant type in every point $p\in M$ it is called **with punctual constant type**.

Definition 6 An almost hermitian manifold M satisfies the axiom of (2p+1)-coholomorphic spheres if for any $m\in M$ and any 2p+1-coholomorphic plane N_m of T_pM it exists a 2p+1-dimensional totally umbilical submanifold S in order to $m\in S$ and $T_mS=N_m$ with p fixed integer and $2\le p\le q-1$, $\dim M=2q$.

In the same manner like in [4] we shall prove the following:

Theorem 5 Let M an RKA-manifold with punctual constant type. If M satisfy the axiom of 2p+1-coholomorphic spheres for some p and if $\dim M\ge 6$ then the holomorphic sectional curvature depends only from the point.

Proof Let $m\in M$ We consider two orthonormal vectors X,Y in T_pM in order to (X,Y) is an antiholomorphic plane. We take now a 2p+1-coholomorphic plane N_m which contains X,Y,JX and JY is normal to N_m. From the axiom of 2p+1-coholomorphic spheres, it exists a 2p+1-totally umbilical submanifold S in order to $m\in S$ and $T_mS=N_m$. Let now the Codazzi equation for a totally umbilical submanifold:

(7) $(R(X,Y)Z)^{\perp}=g(Y,Z)D_XH-g(X,Z)D_YH\ \forall X,Y,Z\in \mathcal{X}(M)$

where D is the connection of the normal fibre bundle of S in M.

If in (7) we consider X,JX,Y we obtain $(R(X,JX)Y)^{\perp}=0$. But JY is normal to N_m therefore:

(8) $R(X,JX,Y,JY)=0\ \forall X,Y\in T_mM$ with (X,Y) an antiholomorphic plane.

(X+Y,JX-JY) is obvious an antiholomorphic plane then, using (1),(8) follows:

(9) $K(X+Y,JX-JY)=H(X)+H(Y)+2K(X,JY)+2K(X,Y)-2\lambda(X,Y)$

Also, from (8) we have:

(10) $K(X,Y)+K(X,JY)=\lambda(X,Y)+\lambda(X,JY)$

We take in (10) X+Y and JX-JY instead of X and Y:

(11) $K(X+Y,JX-JY)+K(X+Y,X-Y)=\lambda(X+Y,JX-JY)+\lambda(X+Y,X-Y)$

After elementary computations, we have:

(12) $K(X+Y,X-Y)=4K(X,Y)$

(13) $\lambda(X+Y,JX-JY)=4\lambda(X,JY)$

(14) $\lambda(X+Y,X-Y)=4\lambda(X,Y)$

Using (12),(13),(14) in (11) we obtain:

(15) $K(X+Y,JX-JY)=-4K(X,Y)+4\lambda(X,JY)+4\lambda(X,Y)$

On the other hand we have:

(16) $K(X,JY)=\lambda(X,Y)+\lambda(X,JY)-K(X,Y)$

Using now (15),(16) in (9) we obtain:

(17) $K(X,Y)=\frac{1}{2}\lambda(X,JY)+\lambda(X,Y)-\frac{1}{4}\left(H(X)+H(Y)\right)$

If we put in (17) JY instead of Y we have:

(18) $K(X,JY)=\frac{1}{2}\lambda(X,Y)+\lambda(X,JY)-\frac{1}{4}\left(H(X)+H(Y)\right)$

From (17) and (18) follows:

(19) $\lambda(X,Y)+\lambda(X,JY)=H(X)+H(Y)$

If M has constant punctual type, let note him with α, we obtain:

(20) $H(X)+H(Y)=2\alpha$.

But dim $M\geq 6$ then $H(X)=\alpha$. The theorem is completely proved.

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