A GENERAL TYPE OF ALMOST CONTACT MANIFOLDS

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Abstract: Among almost contact manifolds Sasakian manifolds, Kenmotsu manifolds (called also “a certain class of almost contact manifolds”) and cosymplectic manifolds have been studied by many authors.

The purpose of this paper is to obtain a class of almost contact manifolds which will generalize the above manifolds.

The paper generalizes the RK-manifolds introduced by Lieven Vanhecke. I give some results concerning the submanifolds of these spaces, the behaviour of these submanifolds at conformal, projective and concircular transformations. Also I obtain a similar result with those on RK-manifolds but in a form a little weaker when they satisfy the axiom of 2p+1-coholomorphic spheres.

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1. Introduction

Among almost contact manifolds Sasakian manifolds, Kenmotsu manifolds (called also “a certain class of almost contact manifolds”) and cosymplectic manifolds have been studied by many authors. In [1], [2], [3] we find the principal results about these manifolds.

The purpose of this paper is to obtain a class of almost contact manifolds which will generalize the above manifolds.

After some general results, we have obtained the Riemann-Christoffel tensor in the case of constant \( \varphi \)-sectional curvature. In the last paragraph we study a subclass of this general type which is richer in information.
2. Preliminaries

We call an almost contact metric manifold, one denoted by \( M^{2n+1} \) for which:
(1) \( \phi^2 X = -X + \eta(X)\xi \)
(2) \( \eta(\xi) = 1 \)
(3) \( \phi \xi = 0 \)
(4) \( \eta(\phi X) = 0 \)
(5) \( g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \forall X, Y \in X(M) \)

where \( \phi \) is a (1,1)-type tensor field, \( \eta \) a 1-form, \( \xi \) a vector field (named the characteristic vector field) and \( g \) is the associated Riemannian metric on \( M \).

The 2-fundamental form is:

(6) \( \phi(X, Y) = g(X, \phi Y) \forall X, Y \in X(M) \)

On an almost contact manifold we define the tensor:

(7) \( N^i(X, Y) = [\phi X, \phi Y] - \phi[X, Y] - \phi^2[X, Y] + 2d\eta(X, Y)\xi \forall X, Y \in X(M) \)

A manifold with an almost contact metric structure and \( N^i = 0 \) is called normal manifold.

An almost contact manifold with \( \phi = d\eta \) is called a contact manifold. A normal contact manifold is a Sasakian manifold.

If on an almost contact manifold we have: \( (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \forall X, Y \in X(M) \) the manifold is Sasakian. We also have \( \nabla_X \xi = -\phi X \forall X \in X(M) \).

An almost contact manifold is a Kenmotsu manifold if \( (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \nabla_X \xi = -\eta(X)\xi \forall X, Y \in X(M) \).

A cosymplectic manifold is a normal manifold with \( \phi \) and \( \eta \) closed. On a cosymplectic manifold we have: \( \nabla_X \phi = 0, \nabla_X \xi = 0 \forall X \in X(M) \).

For every \( p \in M \) and \( X \in T_p M \), \( X \) orthogonal on \( \xi \) we define the \( \phi \)-sectional curvature like \( K(X, \phi X) \) where \( K \) is the sectional curvature.

3. A general type of almost contact manifolds

Definition An almost contact manifold \( M^{2n+1} \) is called a general type of almost contact manifold (short gt-manifold) if there are a (1,1)-type tensor field \( \Psi : X(M) \rightarrow X(M) \) and a function \( \beta \in F(M) \) which satisfy the following conditions:
(8) \( (\nabla_X \phi)Y = g(\Psi X, Y)\xi - \eta(Y)\Psi X \forall X, Y \in X(M) \)
(9) \( \nabla_X \xi = -\Psi \phi X \forall X \in X(M) \)
(10) \( g(\Psi X, X) = \beta \forall X \perp \xi, g(X, X) = 1 \)
In what follows for the simplification we write:

(12) \( \eta(\Psi\xi) = \alpha \)

Let in (8) \( Y = \xi \). We obtain:

(13) \( -\varphi \nabla_X \xi = \eta(\Psi X) \xi - \Psi X \quad \forall X \in X(M) \)

Applying \( \varphi \) in (13) we obtain:

(14) \( \varphi \Psi = \Psi \varphi \)

From (9), (13) we have:

(15) \( \eta(\Psi X) \xi = \eta(X) \Psi \xi \quad \forall X \in X(M) \)

For \( X = \xi \) in (15) and using (12) we have:

(16) \( \Psi \xi = \alpha \xi \)

and

(17) \( \eta(\Psi X) = \alpha \eta(X) \quad \forall X \in X(M) \)

From (17) we obtain that the contact distribution \( D = \{ X \mid \eta(X) = 0 \} \) is invariant through \( \Psi \).

From (9) we obtain that

(18) \( \nabla_\xi \xi = 0 \)

In consequence we have the following:

**Theorem 1** In a gt-manifold the integral curves of \( \xi \) are geodesics.

Using (8), (16) we have also:

(19) \( \nabla_\xi \varphi = 0 \)

Now if in (10) \( X \) is not unitary we have \( g(\Psi X, X) = \beta g(X, X) \quad \forall X \perp \xi \) and putting \( X = Y - \eta(Y) \xi \) we obtain:

(20) \( g(\Psi Y, Y) = \beta(Y, Y) + (\alpha - \beta) \eta^2(Y) \quad \forall Y \in X(M) \)

Reciprocally, from (20) we obtain (10).

**Lemma 2** On an almost contact manifold \( M^{2n+1} \) which satisfy (8), (9) we have that (10) is equivalent with \( d\eta = \beta \phi \).

**Proof** We have seen that (10) is equivalent with (20). Let suppose that (20) are valid. Polarizing, we obtain:

(21) \( g(\Psi X, Y) + g(\Psi Y, X) = 2\beta g(X, Y) + 2(\alpha - \beta) \eta(X) \eta(Y) \quad \forall X, Y \in X(M) \)

We have also \((\nabla_X \eta) Y = \nabla_X g(Y, \xi) - g(\nabla_X Y, \xi) = g(Y, \nabla_X \xi) = g(\Psi X, \varphi Y)\) and

(22) \( 2d\eta(X, Y) = (\nabla_X \eta) Y - (\nabla_Y \eta) X = g(\Psi X, \varphi Y) - g(\Psi Y, \varphi X) \quad \forall X, Y \in X(M) \)
Writing (21) for $Y \rightarrow \phi Y$ we obtain:

(23) $2\beta \phi (X,Y)=g(\Psi X,\phi Y)-g(\Psi Y,\phi X) \ \forall X,Y \in X(M)$

From (22),(23) we have that:

(24) $d\eta = \beta \phi$

Suppose now that (24) are valid. Going back, we obtain (23) and for $X \rightarrow \phi Y$ we obtain (20). Q. E. D.

From (20) we obtain also a formula which we need later:

(25) $\text{tr } \Psi = 2n\beta + \alpha$

where $\text{tr } \Psi$ is the trace of the operator $\Psi$.

From (6),(8) we have:

(26) $3d\phi (X,Y,Z)=X\phi (Y,Z)-Y\phi (X,Z)+Z\phi (X,Y)-\phi ([X,Y],Z)+\phi ([X,Z],Y)-\phi ([Y,Z],X)=g(Y,(\nabla_X \phi)Z)-g(X,(\nabla_Y \phi)Z)+g(X,(\nabla_Z \phi)Z)=\eta (X)[g(\Psi Z,Y)-g(\Psi Y,Z)]+\eta (Y)[g(\Psi X,Z)-g(\Psi Z,X)]+\eta (Z)[g(\Psi Y,X)-g(\Psi X,Y)] \ \forall X,Y,Z \in X(M)$

From (24) we have:

**Theorem 3** A gt-manifold with $\beta = 0$ has $\eta$ closed.

From (26) we obtain:

**Theorem 4** A gt-manifold with $\Psi$ a symmetric operator has $\phi$ closed.

From (7),(8),(24) we obtain:

(27) $N^i(X,Y)=0 \ \forall X,Y \in X(M)$

therefore we have:

**Theorem 5** A gt-manifold is a normal manifold.

4. Examples

1. For $\Psi = I$ and $\beta = 1$ we obtain Sasakian manifolds
2. For $\Psi = \phi$ and $\beta = 0$ we have Kenmotsu manifolds
3. For $\Psi = 0$ and $\beta = 0$ we have cosymplectic manifolds
5. Curvature properties

Now we have:

\[(28) \ R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_Z \xi - \nabla_{[X,Y]} \xi = \alpha \eta(Y) \Psi X - \alpha \eta(X) \Psi Y + \phi(\nabla_Y \Psi)X - \phi(\nabla_X \Psi)Y \]

Using (9), (16) we have:

\[(29) \ (\nabla_X \Psi)\xi = X(\alpha)\xi - \alpha \phi \Psi X + \phi \Psi^2 X \]

For \(Y = \xi\) in (28) and using (11), (16), (29) we obtain:

\[(30) \ R(X,\xi)\xi = \Psi^2 X - \alpha^2 \eta(X)\xi \]

From (30) we have:

\[(31) \ K(X,\xi) = g(R(X,\xi)\xi, X) = g(\Psi^2 X, X) \quad \forall X \perp \xi, \ g(X, X) = 1 \]

On the other hand, from (21) we obtain:

\[(32) \ g(\Psi^2 X, X) = -g(\Psi X, \Psi Y) + 2\beta g(\Psi X, Y) - 2\alpha (\alpha - \beta) \eta(X) \eta(Y) \]

Using now (31), (32) we obtain finally:

\[(33) \ K(X,\xi) = 2\beta^2 g(\Psi X, \Psi X) \quad \forall X \perp \xi, \ g(X, X) = 1 \]

**Theorem 6** A gt-manifold has \(K(X,\xi) \leq 2\beta^2\) where \(X \perp \xi, \ g(X, X) = 1\)

**Corollary 7** A gt-manifold with \(\beta = 0\) has \(K(X,\xi) \leq 0\).

We now define a \((0,4)\)-tensor field \(A: \mathfrak{X}(M)^4 \to \mathcal{F}(M)\):

\[(34) \ A(X,Y,Z,V) = g(\phi X, \Psi Z) g(\phi Y, \Psi V) - g(X, \Psi Z) g(Y, \Psi V) \quad \forall X,Y,Z,V \in \mathfrak{X}(M) \]

We obtain immediately:

\[(35) \ A(X,Y,Z,V) = A(Y,X,V,Z) \]

\[(36) \ A(\phi X, \phi Y, Z, V) = A(X,Y,\phi Z, \phi V) = -A(X,Y,Z,V) \]

\[A(\phi X, \phi Y, \phi Z, \phi V) = A(X,Y,Z,V) \]

\[A(\phi X, Y, Z, V) = A(X, \phi Y, Z, V) \]

\[A(X,Y,\phi Z, V) = A(X,Y,Z,\phi V) \quad \forall X,Y,Z,V \perp \xi \]

We also define \(B: \mathfrak{X}(M)^4 \to \mathcal{F}(M)\) a \((0,4)\)-tensor field:
(37) $B(X,Y,Z,V) = A(X,Y,Z,V) - A(X,Y,V,Z)$

We have from (35),(36),(37) that:

(38) $B(X,Y,Z,V) = B(Y,X,V,Z) = -B(Y,X,Z,V) = -B(X,Y,V,Z)$

$B(\phi X,Y,Z,\phi V) = B(X,Y,Z,V) \forall X,Y,Z,V \perp \xi$

Using now (8),(9),(34)-(38) we can prove that:

(39) $R(\phi X,\phi Y,\phi Z,\phi V) = R(X,Y,Z,V) + B(X,Y,V,Z) - B(V,Z,Y,X)$

$R(X,Y,\phi Z,\phi V) = R(X,Y,Z,V) + B(V,Z,Y,X)$

$R(\phi X,\phi Y,Z,V) = R(X,Y,V,Z) + B(X,Y,Z,V)$

$R(X,\phi Y,Z,\phi V) + R(\phi X,Y,Z,\phi V) = B(X,Y,V,Z) \forall X,Y,Z,V \perp \xi$

Let suppose now that $K(\phi X) = K = \text{constant}$. We have:

**Theorem 8** If a gt-manifold has constant $\phi$-sectional curvature then:

(40) $4R(X,Y,Z,V) = 2B(X,Y,V,Z) + B(X,V,Y,Z) + B(X,Z,V,Y) + 4g((\nabla Y \Psi X - (\nabla X \Psi)Y, \eta(Z)\phi Y - \eta(V)\phi Z) + 4g((\nabla v \Psi)Z - (\nabla Z \Psi)Y, \eta(X)\phi Y - \eta(Y)\phi X) + \eta(V)\eta(Y)((3\alpha - 8\beta)g(\Psi X,Z) + (2\alpha\beta - K)g(X,Z) + 4g(\Psi X,\Psi Z)) - \eta(Y)\eta(Z)((3\alpha - 8\beta)g(\Psi X,V) + (2\alpha\beta - K)g(X,V) + 4g(\Psi X,\Psi V)) + \eta(X)\eta(Z)((\alpha - 8\beta)g(\Psi Y,V) + (4\alpha\beta - K)g(Y,V) + 4g(\Psi Y,\Psi V)) - \eta(X)\eta(V)((\alpha - 8\beta)g(\Psi Y,Z) + (4\alpha\beta - K)g(Y,Z) + 4g(\Psi Y,\Psi Z)) + \alpha\eta(X)\eta(Y)[2g(\Psi Z,V) - 2\beta g(Z,V) + 2(\alpha - \beta)\eta(Z)\eta(V)] + K[g(X,Z)g(Y,V) - g(X,V)g(Y,Z) + \phi(X,Z)\phi(Y,V) - \phi(X,V)\phi(Y,Z) + 2\phi(X,Y)\phi(Z,V)] \forall X,Y,Z,V \in X(M)$

with the above notations and $R(X,Y,Z,V) = g(R(X,Y)V,Z)$.

**Proof** From the hypothesis, we have:

(41) $R(X,\phi X,X,\phi X) = Kg(X,X)^2 \forall X \perp \xi$

For $X \rightarrow X + Y$ in (41) then $X \rightarrow X - Y$ in (41) and adding:
(42) 
\[ 2R(X,\phi X,Y,\phi Y)+2R(X,\phi Y,Y,\phi X)+R(Y,\phi X,Y,\phi X)+R(X,\phi Y,X,\phi Y)=4Kg(X,Y)^2+2Kg(X,X)g(Y,Y) \forall X,Y \perp \xi \]

If in (42) we put \( X \rightarrow X+\phi Z \) then in what we obtained \( Y \rightarrow Y+\phi V \) and using (39):

(43) 
\[ 2R(X,Z,Y,V)+2R(X,V,Z,Y)+R(\phi X,Y,\phi V,Z)=2K[g(X,Y)g(Z,V)+\phi(X,V)\phi(Y,Z)+\phi(X,Z)\phi(Y,V)+B(X,Z,V,Y)+B(Y,\phi X,\phi V,Z) \forall X,Y,Z,V \perp \xi \]

If in ((43) we change \( Y \) with \( Z \) and subtract from (43) we have:

(44) 
\[ 4R(X,Y,V,Z)=2B(X,V,Z,Y)+B(X,Z,V,Y)+B(Y,X,Z,V)+K[g(X,Y)g(Z,V)-g(X,Z)g(Y,V)+2\phi(X,V)\phi(Y,Z)+\phi(X,Z)\phi(Y,V)-\phi(X,Y)\phi(Z,V)] \forall X,Y,Z,V \perp \xi \]

If in (44) we replace \( X \) with \( X-2\eta(\xi) \), \( Y \) with \( Y-\eta(Y)\xi \), \( Z \) with \( Z-\eta(Z)\xi \) and \( V \) with \( V-\eta(V)\xi \) we obtain (40). Q.E.D.

If we return now at examples, we obtain the well-known expressions.

The calculus of the Ricci tensor and the scalar of curvature using (25) and (40) is immediate.

About Ricci tensor, let note that on a gt-manifold we have from (30) that:

(45) \[ \text{Ric}(\xi,\xi)=\text{tr}\Psi^2-\alpha^2 \]

6. A special general type of almost contact manifolds

Definition We call special general type of an almost contact manifold (short special gt-manifold) a gt-manifold \( M^{2n+1} \) which has in addition:

(46) \( (\nabla_X\Psi)Y=(\nabla_Y\Psi)X \forall X,Y \in X(M) \)

From section 4 we have that Sasakian manifolds and cosymplectic manifolds are special gt-manifolds.

Using (11),(29),(46) we have for \( X=\xi \) that

(47) \[ Y(\alpha)\xi-\alpha\phi\Psi Y+\phi\Psi^2Y=0 \]

From (4),(47) we have:

(48) \[ Y(\alpha)=0 \] therefore \( \alpha \) is constant.

(49) \[ \Psi^2Y-\alpha\Psi Y \in \text{Span}(\xi) \]
From (49) for $Y=\xi$ we obtain that:

(50) $\Psi^2Y=\alpha\Psi Y$

From (31) we obtain that:

(51) $K(X,\xi)=\alpha\beta$ $\forall X\perp \xi$, $g(X,X)=1$

Using (32),(50) we have:

(52) $g(\Psi X,\Psi Y)=(2\beta-\alpha)g(\Psi X,Y)-2\alpha(\alpha-\beta)\eta(X)\eta(Y)$ $\forall X,Y \in X(M)$

If we have now $\alpha=2\beta$ from (52) where $X=Y=\xi$ we obtain that $\alpha=\beta=0$ and reciprocally if $\alpha=\beta=0$ we have that $\alpha=2\beta$.

If $\beta=0$ from (52) where $X=Y=\xi$ we obtain $\alpha=0$ and again from (52) we have $\Psi X=0$. From section 4, 3 we have that the manifold is cosymplectic.

**Theorem 9** A special gt-manifold is cosymplectic, if and only if $\beta=0$.

Suppose now that the manifold is not cosymplectic. Interchanging $X$ and $Y$ in (52) and subtract from it:

(53) $(2\beta-\alpha)[g(\Psi X,Y)-g(\Psi Y,X)]=0$ $\forall X,Y \in X(M)$

From the hypothesis we have that $2\beta\neq\alpha$ then

(54) $g(\Psi X,Y)=g(\Psi Y,X)$ $\forall X,Y \in X(M)$

therefore $\Psi$ is a symmetric operator.

Using the facts that a cosymplectic manifold has $\phi$ closed and the theorem 4, we conclude:

**Theorem 10** A special gt-manifold has $\phi$ closed.

We can now reformulate the theorem 8:

**Theorem 11** If a special gt-manifold which is not cosymplectic has constant $\varphi$-sectional curvature then:

(55) $4R(X,Y,Z,V)=g(\varphi X,\Psi V)g(\varphi Y,\Psi Z)+g(\varphi X,\Psi Z)g(\varphi V,\Psi Y)-2g(\varphi X,\Psi Y)g(\varphi Z,\Psi V)+$

$3g(X,\Psi Z)g(Y,\Psi V)-3g(X,\Psi V)g(Y,\Psi Z)+$

$\eta(V)\eta(Y)((2\alpha\beta-K)g(X,Z)+\alpha g(\Psi X,Z))-...$
\[ \eta(Y)\eta(Z)((2\alpha\beta-K)g(X,V)+\alpha g(\Psi X,V))+ \\
\eta(X)\eta(Z)((4\alpha\beta-K)g(Y,V)-3\alpha g(\Psi Y,V))- \\
\eta(X)\eta(V)((4\alpha\beta-K)g(Y,Z)-3\alpha g(\Psi Y,Z))+ \\
\alpha\eta(X)\eta(Y)[2g(\Psi Z,V)-2\beta g(Z,V)+2(\alpha-\beta)\eta(Z)\eta(V)]+ \\
K[g(X,Z)g(Y,V)-g(X,V)g(Y,Z)+\phi(X,Z)\phi(Y,V)- \\
\phi(X,V)\phi(Y,Z)+2\phi(X,Y)\phi(Z,V)] \forall X,Y,Z,V \in X(M) \]

where \( R(X,Y,Z,V)=g(R(X,Y)V,Z) \) and \( K \) is the constant \( \varphi \)-sectional curvature.

From (55) we obtain also:

(56) \[ 2\text{Ric}(X,Y)=\eta(X)\eta(Y)(7\alpha^2+(n-6)\alpha\beta-K(n+1))+g(X,Y)(\alpha\beta+K(n+1))+ \\
g(\phi X,Y)(\alpha+3\beta(n-1)) \forall X,Y \in X(M) \]

Ric being the Ricci tensor on \( M^{2n+1} \).

(57) \[ S=4\alpha^2+(3n-4)\alpha\beta+3n(n-1)\beta^2 \]

where \( S \) is the scalar of curvature.

From (57) we obtain immediately:

**Theorem 12** A special gt-manifold, not cosymplectic, having constant \( \varphi \)-sectional curvature and of dimension greater than 3 has positive scalar of curvature.

**Almost Hermitian manifolds with J-invariant sectional curvature**

Let \((M,g)\) a differentiable manifold with the metric tensor \( g \). \( M \) is named **almost Hermitian** if there exists an endomorphism \( J: X(M) \rightarrow X(M) \) of the Lie algebra of tensor fields \( X(M) \) such that \( J^2=-I \) and \( g \) is \( J \)-invariant that is \( g(JX,JY)=g(X,Y) \ \forall X,Y \in X(M) \).

In [4] L. Vanhecke defines **RK-manifolds** like manifolds almost hermitian with \( J \)-invariant curvature Riemann tensor, that is \( R(JX,JY,JZ,JV)=R(X,Y,Z,V) \ \forall X,Y,Z,V \in X(M) \).

In [3] are defined **para-Kähler manifolds** like almost hermitian manifolds with \( R(X,Y,JZ,JV)=R(X,Y,Z,V) \ \forall X,Y,Z,V \in X(M) \).

A **Kähler manifold** is an almost Hermitian manifold for which the 2-fundamental form is closed, where \( \Phi(X,Y)=g(JX,Y) \ \forall X,Y \in X(M) \) and the Nijenhuis tensor corresponding to \( J \) vanishes. In a Kähler manifold we have ([11]): \( R(X,JY,Z,V)=R(Y,JX,Z,V) \ \forall X,Y,Z,V \in X(M) \).
We have, in consequence, that Kähler manifolds are para-Kähler which their turn are RK-manifolds. Let note the sectional curvature by the 2-plane \((X,Y)\) in any point of the manifold with \(k(X,Y) = k(X,Y)[g(X,X)g(Y,Y) - g(X,Y)^2]\). We note also \(H(X) = k(X,JX)\) the holomorphic sectional curvature corresponding to \(X\). It is proved in [5] that on a RK-manifold we have 
\[k(X,Y) = k(JX,JY),\]
\[k(X,JY) = k(JX,Y),\]
\[S(X,Y) = S(JX,JY),\]
\[S(X,JY) + S(JX,Y) = 0\]
\[\forall X,Y \in X(M)\] where \(S\) is the Ricci tensor.

In this paper I shall enlarge the RK-manifolds class and I shall study some properties of these manifolds.

2. Almost RK-manifolds

Definition 1 An almost RK-manifold (short RKA-manifold) is an almost Hermitian manifold for which 
\[k(X,Y) = k(JX,JY)\]
\[\forall X,Y \in X(M).\]

Remarks An RK-manifold is an RKA-manifold. Manifolds with constant curvature are also RKA-manifolds.

From the definition follows immediately that:

(1) \[R(X,Y,V,Z) + R(X,Z,V,Y) = R(JX,JY,JV,JZ) + R(JX,JZ,JV,JY)\]

\[\forall X,Y,V,Z \in X(M)\]

If we take an orthonormal basis in \(M\): \(X_1, ..., X_n\) and put \(Y = Z = X_i\) and summing for \(i\), we obtain;
\[S(X,V) = S(JX,JV)\]
\[\forall X,V \in X(M).\] In consequence, the property of the Ricci tensor to be invariant at the action of \(J\) remains valid in RKA-manifolds.

Let now study the behaviour of RKA-manifolds at the time when they admit some special submanifolds.

Let \((M,g) \subseteq (\overline{M}, \overline{g})\) a submanifold of an almost Hermitian manifold \((\overline{M}, \overline{g})\). The Gauss equation is:

(2) \[
\overline{R}(X,Y,Z,V) = R(X,Y,Z,V) - \overline{g}(h(X,Z), h(Y,V)) + \overline{g}(h(X,V), h(Y,Z))
\]
\[\forall X,Y,Z,V \in X(M)\]

Definition 2 A submanifold \((M,g) \subseteq (\overline{M}, \overline{g})\) is called totally quasi-umbilical if the second fundamental form \(h\) is:

\[h(X,Y) = g(X,Y)H + [\omega(X)\omega(Y) + \omega(JX)\omega(JY)]A\]
\[\forall X,Y \in X(M)\]

where \(H\) is the mean curvature vector and \(A \in X(M)\), \(\omega\) being a 1-form on \(M\).

In particular, if \(\omega = 0\) we obtain totally umbilical submanifolds and if, in addition \(H = 0\), we have totally geodesic submanifolds.

For totally quasi-umbilical submanifolds, we have:
Writing (3) for $JX$ and $JY$ and subtract the two relations, we obtain:

\[(4) \quad K(JX,JY) - K(X,Y) = K(JX,JY) - K(X,Y) \forall X,Y \in X(M)\]

where we have noted with bar all the quantities on $M$.

In consequence, we have:

**Theorem 1** A totally quasi-umbilical submanifold of an RKA-manifold is an RKA-manifold.

**Corollary 1** A totally umbilical submanifold of an RKA-manifold is an RKA-manifold.

**Corollary 2** A totally geodesic submanifold of an RKA-manifold is an RKA-manifold.

The conformal curvature tensor of a manifold is:

\[(5) \quad C(X,Y,Z,V) = r(X,Y,Z,V) + g(X,V)L(Y,Z) + g(Y,Z)L(X,V) - g(X,Z)L(Y,V) - g(Y,V)L(X,Z) \forall X,Y,Z,V \in X(M)\]

where $L(X,Y) = \left( -\varrho \right)_{Y,X}$. $\varrho$ being the scalar of curvature.

Immediately, we obtain that:

\[(6) \quad C(X,Y,X,Y) - C(JX,JY,JX,JY) = K(X,Y) - K(JX,JY) \forall X,Y \in X(M)\]

From (6) follows:

**Theorem 2** If an RKA-manifold is conformable with another manifold the second is also RKA-manifold.

In the same manner, considering the Weyl projective tensor:

\[P(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}(S(X,Z)Y - S(Y,Z)X)\]

and the Yano concircular tensor

\[K(X,Y)Z = R(X,Y)Z - \frac{\varrho}{n(n-1)}(g(Y,Z)X - g(X,Z)Y)\]

where $n = \text{dim } M$, we obtain:

**Theorem 3** At projective transformations RKA-manifolds applied on RKA-manifolds.

**Theorem 4** At concircular transformations RKA-manifolds applied on RKA-manifolds.
3. RKA-manifolds with punctual constant type

In what follows are necessary some definitions.

**Definition 3** Let \( p \in M \). A subspace \( N_p \) of \( T_pM \) is called **holomorphic subspace** if \( J(N_p) \subseteq N_p \) and **antiholomorphic** if \( J(N_p) \subseteq N_p^\perp \).

**Definition 4** A \( 2p+1 \)-dimensional subspace is called **\( 2p+1 \)-coholomorphic plane** if it contains a \( 2p \)-holomorphic plane.

It shows in [5] that a \( 2p+1 \)-coholomorphic plane contains a \( p+1 \)-antiholomorphic plane and \( 1 \leq p \leq q-1 \) where \( \dim M = 2q \).

**Definition 5** An almost Hermitian manifold has **constant type** in \( p \in M \) if for any \( X \in T_pM \) we have: \( \lambda(X,Y) = \lambda(X,Z) \) where \( (X,Y), (X,Z) \) are antiholomorphic planes, \( g(Y,Y) = g(Z,Z) \) and \( \lambda(X,Y) = R(X,Y,X,Y) - R(X,Y,JX,JY) \). If the manifold has constant type in every point \( p \in M \) it is called **with punctual constant type**.

**Definition 6** An almost hermitian manifold \( M \) satisfies the axiom of \( (2p+1) \)-coholomorphic spheres if for any \( m \in M \) and any \( 2p+1 \)-coholomorphic plane \( N_m \) of \( T_pM \) it exists a \( 2p+1 \)-dimensional totally umbilical submanifold \( S \) in order to \( m \in S \) and \( T_mS = N_m \) with \( p \) fixed integer and \( 2 \leq p \leq q-1 \), \( \dim M = 2q \).

In the same manner like in [4] we shall prove the following:

**Theorem 5** Let \( M \) an RKA-manifold with punctual constant type. If \( M \) satisfy the axiom of \( 2p+1 \)-coholomorphic spheres for some \( p \) and if \( \dim M \geq 6 \) then the holomorphic sectional curvature depends only from the point.

**Proof** Let \( m \in M \) We consider two orthonormal vectors \( X,Y \) in \( T_pM \) in order to \( (X,Y) \) is an antiholomorphic plane. We take now a \( 2p+1 \)-coholomorphic plane \( N_m \) which contains \( X,Y,JX \) and \( JY \) is normal to \( N_m \). From the axiom of \( 2p+1 \)-coholomorphic spheres, it exists a \( 2p+1 \)-totally umbilical submanifold \( S \) in order to \( m \in S \) and \( T_mS = N_m \). Let now the Codazzi equation for a totally umbilical submanifold:

\[
(7) \ (R(X,Y)Z)^\perp = g(Y,Z)D_XH - g(X,Z)D_YH \ \forall X,Y,Z \in X(M)
\]

where \( D \) is the connection of the normal fibre bundle of \( S \) in \( M \).

If in (7) we consider \( X,JX,Y \) we obtain \( (R(X,JX,Y))^\perp = 0 \). But \( JY \) is normal to \( N_m \) therefore:

\[
(8) \ R(X,JX,Y,JY) = 0 \ \forall X,Y \in T_mM \text{ with } (X,Y) \text{ an antiholomorphic plane}.
\]

\( X+JX-JY \) is obvious an antiholomorphic plane then, using (1),(8) follows:

\[
(9) \ K(X+Y,JX-JY) = H(X) + H(Y) + 2K(X,JY) + 2K(X,Y) - 2\lambda(X,Y)
\]
Also, from (8) we have:

(10) \( K(X,Y) + K(X,JY) = \lambda(X,Y) + \lambda(X,JY) \)

We take in (10) \( X+Y \) and \( JX-JY \) instead of \( X \) and \( Y \):

(11) \( K(X+Y,JX-JY) + K(X+Y,X-Y) = \lambda(X+Y, JX-JY) + \lambda(X+Y, X-Y) \)

After elementary computations, we have:

(12) \( K(X+Y,X-Y) = 4K(X,Y) \)
(13) \( \lambda(X+Y,JX-JY) = 4\lambda(X,JY) \)
(14) \( \lambda(X+Y,X-Y) = 4\lambda(X,Y) \)

Using (12), (13), (14) in (11) we obtain:

(15) \( K(X+Y,JX-JY) = -4K(X,Y) + 4\lambda(X,JY) + 4\lambda(X,Y) \)

On the other hand we have:

(16) \( K(X,JY) = \lambda(X,Y) + \lambda(X,JY) - K(X,Y) \)

Using now (15), (16) in (9) we obtain:

(17) \( K(X,Y) = \frac{1}{2} \lambda(X,JY) + \lambda(X,Y) - \frac{1}{4}(H(X) + H(Y)) \)

If we put in (17) \( JY \) instead of \( Y \) we have:

(18) \( K(X,JY) = \frac{1}{2} \lambda(X,Y) + \lambda(X,JY) - \frac{1}{4}(H(X) + H(Y)) \)

From (17) and (18) follows:

(19) \( \lambda(X,Y) + \lambda(X,JY) = H(X) + H(Y) \)

If \( M \) has constant punctual type, let note him with \( \alpha \), we obtain:

(20) \( H(X) + H(Y) = 2\alpha \).

But \( \dim M \geq 6 \) then \( H(X) = \alpha \). The theorem is completely proved.

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5. Kobayashi, M., *Contact generic submanifolds of a certain class of almost contact manifolds*, Tensor N.S., vol. 48 (1989);