General Economics

The Production Functions from the Point of View of 3- Dimensional Geometry

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Abstract: In this paper we shall make an analysis of production functions from the space point of view. We shall obtain some interesting results like that all the points of the surface are parabolic, the total curvature is always null, the conditions when a production function is minimal and finally we give the equations of the geodesics on the surface i.e. the curves of minimal length between two points.

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1. Introduction

In the theory of production functions, usual all computations and phenomenon are studied on projections of the surface, or for a constant level of production. A complete analysis can be made only at the entire surface.

In the economical analysis, the production functions had a long and interesting history.

A production function is defined like $P:\mathbf{R}_+\times\mathbf{R}_+\rightarrow\mathbf{R}_+$, $P=P(K,L)$ where P is the production, K - the capital and L – the labour such that:

(1) $P(0,0)=0$;

(2) P is differentiable of order 2 in any interior point of the production set;

(3) P is a homogenous function of degree 1, that is $P(rK,rL)=rP(K,L)$ $\forall r\in\mathbf{R}$;

(4) $\frac{\partial P}{\partial K}\geq 0$, $\frac{\partial P}{\partial L}\geq 0$;

(5) $\frac{\partial^2 P}{\partial K^2}\leq 0$, $\frac{\partial^2 P}{\partial L^2}\leq 0$.

From Euler's formula for homogenous functions we have:

$$(6)\quad \frac{\partial P}{\partial L} = \frac{P}{L} - \frac{K}{L}\frac{\partial P}{\partial K}$$

By derivation with L and after with K in (6) we obtain:

$$\frac{\partial^2 P}{\partial L^2} = \frac{\frac{\partial P}{\partial L}L - P}{L^2} + \frac{\chi}{L}\frac{\partial P}{\partial K} - \chi\frac{\partial^2 P}{\partial L \partial K} = -\chi\frac{\partial^2 P}{\partial L \partial K}$$

$$\frac{\partial^2 P}{\partial L \partial K} = \frac{1}{L}\frac{\partial P}{\partial K} - \frac{1}{L}\frac{\partial P}{\partial K} - \chi\frac{\partial^2 P}{\partial K^2} = -\chi\frac{\partial^2 P}{\partial K^2}$$

therefore:

$$(7)\quad \frac{\partial^2 P}{\partial L^2} = -\chi\frac{\partial^2 P}{\partial L \partial K}$$

$$(8)\quad \frac{\partial^2 P}{\partial K^2} = -\frac{1}{\chi}\frac{\partial^2 P}{\partial L \partial K}$$

$$(9)\quad \frac{\partial^2 P}{\partial L^2} = \chi^2\frac{\partial^2 P}{\partial K^2}$$

2. Some notions of the space differential geometry

The graph representation of a production function is a surface.
Let:

$$(10)\quad p = \frac{\partial P}{\partial L},\ q = \frac{\partial P}{\partial K},\ r = \frac{\partial^2 P}{\partial L^2},\ s = \frac{\partial^2 P}{\partial L \partial K},\ t = \frac{\partial^2 P}{\partial K^2}.$$

For a constant value of one parameter we obtain a curve on the surface. For example: $P=P(K,L_0)$ or $P=P(K_0,L)$ are both curves on the production surface. They are obtained from the intersection of the plane $L=L_0$ or $K=K_0$ with the surface $P=P(K,L)$.

The curvature of a curve is, from an elementary point of view, the degree of deviation of the curve relative to a straight line.

In the study of the surfaces, two quadratic forms are very useful.

The first fundamental quadratic form of the surface is:

$$(11)\quad g = g_{11}dL^2 + 2g_{12}dLdK + g_{22}dK^2$$

where: $g_{11}=1+p^2$, $g_{12}=pq$, $g_{22}=1+q^2$.

The area element is $d\sigma = \sqrt{g_{11}g_{22} - g_{12}^2}\, dKdL = \sqrt{\Delta}\, dKdL$ and the surface area A when $(K,L) \in R$ (a region in the plane K-O-L) is $A = \iint_R d\sigma dKdL$ where $\Delta = g_{11}g_{22} - g_{12}^2$.

The second fundamental form of the surface is:

(12) $h=h_{11}dL^2+2\,h_{12}dLdK+h_{22}dK^2$

where: $h_{11}=\frac{r}{\sqrt{1+p^2+q^2}}$, $h_{12}=\frac{s}{\sqrt{1+p^2+q^2}}$, $h_{22}=\frac{t}{\sqrt{1+p^2+q^2}}$.

Considering the quantity $\delta=h_{11}h_{22}-h_{12}^2$ we have that:

- If $\delta>0$ in each point of the surface, we will say that it is elliptical. Such surfaces are the hyperboloid with two sheets, the elliptical paraboloid and the ellipsoid.
- If $\delta<0$ in each point of the surface, we will say that it is hyperbolic. Such surfaces are the hyperboloid with one sheet and the hyperbolic paraboloid.
- If $\delta=0$ in each point of the surface, we will say that it is parabolic. Such surfaces are the cone surfaces and the cylinder surfaces.

Considering a surface S and an arbitrary curve through a point P of the surface who has the tangent vector v in P, let the plane π determined by the vector v and the normal N in P at S. The intersection of π with S is a curve C_n named normal section of S. Its curvature is called normal curvature.

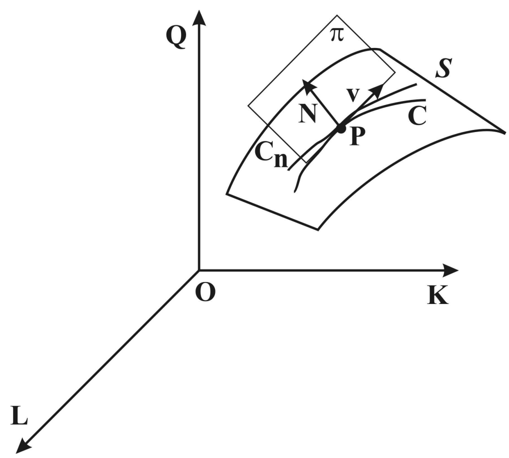

Figure-1: The normal section of a curve

If we have a direction $m=\frac{dL}{dK}$ in the tangent plane of the surface in an arbitrary point P we have that the normal curvature is given by:

(13) $k(m)=\frac{h_{11}m^2+2h_{12}m+h_{22}}{g_{11}m^2+2g_{12}m+g_{22}}$

The extreme values k_1 and k_2 of the function k(m) call the principal curvatures of the surface in that point. They satisfy also the equation:

(14) $(g_{11}g_{22}-g_{12}^2)k^2-(g_{11}h_{22}-2g_{12}h_{12}+g_{22}h_{11})k+(h_{11}h_{22}-h_{12}^2)=0$

The values of m, who give the extremes, call principal directions in that point.

They also satisfy the equation:

(15) $(g_{11}s-g_{12}r)m^2+(g_{11}t-g_{22}r)m+(g_{12}t-g_{22}s)=0$

The curve $\frac{dL}{dK}=m$ (where m is one of the principal directions) is called line of curvature on the surface. On such a curve we have the maximum or minimum variation of the value of Q in a neighbourhood of P.

The quantity $K=k_1k_2$ is named the total curvature in the considered point and $H=\frac{k_1+k_2}{2}$ is named the mean curvature of the surface in that point.

We have therefore:

(16) $K=\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{\delta}{\Delta}$ and $H=\frac{g_{11}h_{22}-2g_{12}h_{12}+g_{22}h_{11}}{g_{11}g_{22}-g_{12}^2}$

A surface with K=constant call surface with constant total curvature and if H=0 call minimal surface.

If we consider now in the tangent plane π at the surface in a point P a direction m, if $h_{11}m^2+2h_{12}m+h_{22}=0$ we will say that m is an asymptotic direction, and the equation: $h_{11}\left(\frac{dL}{dK}\right)^2+2h_{12}\frac{dL}{dK}+h_{22}=0$ gives the asymptotic curves of the surface in the point P.

3. The space differential geometry for production functions

From (6), (10) we have that:

(17) $g_{11}=1+\left(\frac{\partial P}{\partial L}\right)^2=1+\frac{1}{L^2}(P-Kq)^2$

(18) $g_{12}=\frac{\partial P}{\partial L}\frac{\partial P}{\partial K}=\frac{q}{L}(P-Kq)$

(19) $g_{22}=1+\left(\frac{\partial P}{\partial K}\right)^2=1+q^2$

(20) $\Delta=g_{11}g_{22}-g_{12}^2=1+q^2+\frac{1}{L^2}(P-Kq)^2$

We have also: $1+p^2+q^2=1+\left(\frac{\partial P}{\partial L}\right)^2+\left(\frac{\partial P}{\partial K}\right)^2=\Delta$ and:

(21) $h_{11}=\frac{1}{\sqrt{\Delta}}\left(\frac{K}{L}\right)^2 t$

(22) $h_{12}=-\frac{1}{\sqrt{\Delta}}\frac{K}{L}t$

(23) $h_{22}=\frac{1}{\sqrt{\Delta}}t$

(24) $\delta=h_{11}h_{22}-h_{12}^2=0$

From (24) we have that all the points of the surface are parabolic.

The principal curvatures satisfy the equation:

(25) $\sqrt{\Delta}^3 L^2k^2-t(P^2+L^2+K^2)k=0$

therefore: $k_1=0$, $k_2=\frac{t(P^2+K^2+L^2)}{\sqrt{\Delta}^3L^2}<0$.

The values of m corresponding to k_1 and k_2 satisfy the equation:

(26) $(E\mu-F\lambda)m^2+(E\nu-G\lambda)m+(F\nu-G\mu)=0$

If $t\neq0$ then

$$-K\left[L^2+P(P-Kq)\right]m^2+L\left[L^2+P^2-2qKP-K^2\right]m+L^2\left[qP+K\right]=0$$

We have now:

(27) $K=\frac{\lambda\nu-\mu^2}{EG-F^2}=0$

therefore the surface is with null constant total curvature and:

(28) $H=\frac{t(P^2+L^2+K^2)}{L^2\sqrt{\Delta}^3}$.

In order to have a minimal surface we must have: $t=0$ therefore $\frac{\partial^2P}{\partial K^2}=0$ i.e. $\frac{\partial P}{\partial K}=f(L)$ and after: $P=f(L)K+g(L)$ where f,g are differentiable functions of order two.

The asymptotic directions are, if $t\neq0$:

(29) $\left(\frac{K}{L}m-1\right)^2=0$

therefore: m=$\frac{L}{K}$. But m=$\frac{dL}{dK}$ gives that K=CL with C=constant.

With notations x^1=L, x^2=K, let define now the Christoffel symbols of first order:

$$(30)\quad |ij,k| = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

and of second order:

$$(31)\quad \begin{vmatrix} i \\ jk \end{vmatrix} = g^{i1}|jk,1| + g^{i2}|jk,2|$$

where $g^{11}=\frac{1}{\Delta}G$, $g^{12}=-\frac{1}{\Delta}F$, $g^{22}=\frac{1}{\Delta}E$ are the components of the inverse matrix of $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$.

We have now:

$$(32)\quad |11,1| = \frac{1}{2}\frac{\partial g_{11}}{\partial L},\ |11,2| = \frac{\partial g_{12}}{\partial L} - \frac{1}{2}\frac{\partial g_{11}}{\partial K},\ |12,1| = \frac{1}{2}\frac{\partial g_{11}}{\partial K},\ |12,2| = \frac{1}{2}\frac{\partial g_{22}}{\partial L},$$

$$|22,1| = \frac{\partial g_{12}}{\partial K} - \frac{1}{2}\frac{\partial g_{22}}{\partial L},\ |22,2| = \frac{1}{2}\frac{\partial g_{22}}{\partial K}$$

$$(33)\quad \begin{vmatrix} 1 \\ 11 \end{vmatrix} = g^{11}|11,1| + g^{12}|11,2| = \frac{1}{\Delta}\left[\frac{1}{2}g_{22}\frac{\partial g_{11}}{\partial L} - g_{12}\left(\frac{\partial g_{12}}{\partial L} - \frac{1}{2}\frac{\partial g_{11}}{\partial K}\right)\right],$$

$$\begin{vmatrix} 2 \\ 11 \end{vmatrix} = g^{21}|11,1| + g^{22}|11,2| = \frac{1}{\Delta}\left[-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial L} + g_{11}\left(\frac{\partial g_{12}}{\partial L} - \frac{1}{2}\frac{\partial g_{11}}{\partial K}\right)\right],$$

$$\begin{vmatrix} 1 \\ 12 \end{vmatrix} = g^{11}|12,1| + g^{12}|12,2| = \frac{1}{\Delta}\left[\frac{1}{2}g_{22}\frac{\partial g_{11}}{\partial K} - \frac{1}{2}g_{12}\frac{\partial g_{22}}{\partial L}\right],$$

$$\begin{vmatrix} 2 \\ 12 \end{vmatrix} = g^{21}|12,1| + g^{22}|12,2| = \frac{1}{\Delta}\left[-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial K} + \frac{1}{2}g_{11}\frac{\partial g_{22}}{\partial L}\right],$$

$$\begin{vmatrix} 1 \\ 22 \end{vmatrix} = g^{11}|22,1| + g^{12}|22,2| = \frac{1}{\Delta}\left[g_{22}\left(\frac{\partial g_{12}}{\partial K} - \frac{1}{2}\frac{\partial g_{22}}{\partial L}\right) - g_{12}\frac{1}{2}\frac{\partial g_{22}}{\partial K}\right],$$

$$\begin{vmatrix} 2 \\ 22 \end{vmatrix} = g^{21}|22,1| + g^{22}|22,2| = \frac{1}{\Delta}\left[-g_{12}\left(\frac{\partial g_{12}}{\partial K} - \frac{1}{2}\frac{\partial g_{22}}{\partial L}\right) + \frac{1}{2}g_{11}\frac{\partial g_{22}}{\partial K}\right]$$

From (6)-(10) we can write:

$$(34)\quad p=\frac{P-Kq}{L},\ s=-\frac{K}{L}t,\ r=\left(\frac{K}{L}\right)^2 t$$

We have from (17)-(19):

$$(35)\quad \frac{\partial g_{11}}{\partial K} = -\frac{2Kt(P-Kq)}{L^2},\ \frac{\partial g_{11}}{\partial L} = \frac{2K^2t(P-Kq)}{L^3}$$

$$(36)\quad \frac{\partial g_{12}}{\partial K} = \frac{t(P-2Kq)}{L},\ \frac{\partial g_{12}}{\partial L} = -\frac{Kt}{L^2}(P-2Kq)$$

$$(37)\quad \frac{\partial g_{22}}{\partial K} = 2qt,\ \frac{\partial g_{22}}{\partial L} = -\frac{K}{L}2qt$$

From (33)-(37) we obtain:

$$(38)\quad \left|\begin{matrix}1\\11\end{matrix}\right| = \frac{K^2t(P-Kq)}{\Delta^2L^3},\ \left|\begin{matrix}2\\11\end{matrix}\right| = \frac{K^2tq}{\Delta^2L^2},\ \left|\begin{matrix}1\\12\end{matrix}\right| = -\frac{Kt(P-Kq)}{\Delta^2L^2},$$

$$\left|\begin{matrix}2\\12\end{matrix}\right| = -\frac{Kqt}{\Delta^2L},\ \left|\begin{matrix}1\\22\end{matrix}\right| = \frac{(P-Kq)t}{\Delta^2L},\ \left|\begin{matrix}2\\22\end{matrix}\right| = \frac{tq}{\Delta^2}$$

A geodesic is in common language the shortest curve between two points. The equation of a geodesic is:

$$(39)\quad \frac{d^2x^i}{ds^2} + \left|\begin{matrix}i\\jk\end{matrix}\right| \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

that is:

$$(40)\quad \frac{d^2L}{ds^2} + \left|\begin{matrix}1\\11\end{matrix}\right|\left(\frac{dL}{ds}\right)^2 + 2\left|\begin{matrix}1\\12\end{matrix}\right|\frac{dL}{ds}\frac{dK}{ds} + \left|\begin{matrix}1\\22\end{matrix}\right|\left(\frac{dK}{ds}\right)^2 = 0$$

$$(41)\quad \frac{d^2K}{ds^2} + \left|\begin{matrix}2\\11\end{matrix}\right|\left(\frac{dL}{ds}\right)^2 + 2\left|\begin{matrix}2\\12\end{matrix}\right|\frac{dL}{ds}\frac{dK}{ds} + \left|\begin{matrix}2\\22\end{matrix}\right|\left(\frac{dK}{ds}\right)^2 = 0$$

After a long computation, we have:

$$(42)\quad \Delta L^3\frac{d^2L}{ds^2} + t(P-Kq)\left(K\frac{dL}{ds} - L\frac{dK}{ds}\right)^2 = 0$$

$$(43)\quad \Delta L^2\frac{d^2K}{ds^2} + tq\left(K\frac{dL}{ds} - L\frac{dK}{ds}\right)^2 = 0$$

Because P=P(K(s),L(s)) we have:

$$\frac{dP}{ds} = \frac{\partial P}{\partial K}\frac{dK}{ds} + \frac{\partial P}{\partial L}\frac{dL}{ds} = q\frac{dK}{ds} + \frac{P-Kq}{L}\frac{dL}{ds} = \frac{1}{L}\left[q\left(L\frac{dK}{ds} - K\frac{dL}{ds}\right) + P\frac{dL}{ds}\right]$$

therefore:

$$(44)\quad q = \frac{P\frac{dL}{ds} - L\frac{dP}{ds}}{K\frac{dL}{ds} - L\frac{dK}{ds}}$$

and also:

$$(45)\ \text{P-Kq}=L\frac{K\frac{dP}{ds}-P\frac{dK}{ds}}{K\frac{dL}{ds}-L\frac{dK}{ds}}$$

$$(46)\ \Delta=1+q^2+\frac{1}{L^2}(P-Kq)^2=$$

$$\frac{\left(K\frac{dL}{ds}-L\frac{dK}{ds}\right)^2+\left(P\frac{dL}{ds}-L\frac{dP}{ds}\right)^2+\left(K\frac{dP}{ds}-P\frac{dK}{ds}\right)^2}{\left(K\frac{dL}{ds}-L\frac{dK}{ds}\right)^2}$$

If we note now:

$$(47)\ A=K\frac{dL}{ds}-L\frac{dK}{ds}$$

$$(48)\ B=P\frac{dL}{ds}-L\frac{dP}{ds}$$

$$(49)\ C=K\frac{dP}{ds}-P\frac{dK}{ds}$$

the equations (42), (43) become (again after a long calculus):

$$(50)\ \left[A^2+B^2\right]\frac{d^2L}{ds^2}+BC\frac{d^2K}{ds^2}-AC\frac{d^2P}{ds^2}=0$$

$$(51)\ BC\frac{d^2L}{ds^2}+\left[A^2+C^2\right]\frac{d^2K}{ds^2}-AB\frac{d^2P}{ds^2}=0$$

from where:

$$\frac{\frac{d^2L}{ds^2}}{\begin{vmatrix} BC & -AC \\ A^2+C^2 & -AB \end{vmatrix}}=\frac{\frac{d^2K}{ds^2}}{\begin{vmatrix} -AC & A^2+B^2 \\ -AB & BC \end{vmatrix}}=\frac{\frac{d^2P}{ds^2}}{\begin{vmatrix} A^2+B^2 & BC \\ BC & A^2+C^2 \end{vmatrix}}$$

or simply:

$$(52)\ \frac{\frac{d^2L}{ds^2}}{C(A^2+C^2-B^2)}=\frac{\frac{d^2K}{ds^2}}{B(A^2+B^2-C^2)}=\frac{\frac{d^2P}{ds^2}}{A(A^2+B^2+C^2)}$$

The equations of geodesics are: L=L(s), K=K(s) where s is the element of arc on the curves.

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