Considerations on the Hicks Effect for N Consumer's Goods

Cătălin Angelo Ioan

Danubius University of Galati, Department of Economics, catalin_angelo_ioan@univ-danubius.ro

Abstract: The paper analyzes the Hicksian effect of income and substitution for the case of n goods when all their prices change. We determine hereby the limits of variation of the two effects and the relationship between them.

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1 Introduction

Let consider a consumer who has the income V and is faced with the choice of n goods $B_1,...,B_n$ with initial prices $p_1,...,p_n$. Following a relocating of the market we will consider the new prices of goods $B_1,...,B_n$ as: $p'_1,..., p'_n$. Let also be an utility function $U:SC\to\mathbf{R}_+$ where SC is the space of consumption goods relative to those given.

Considering the budget zone $ZB=\{(x_1,...,x_n)\in SC \mid \sum_{i=1}^{n} p_i x_i \le V\}$ the problem of determining the consumption basket so that the utility be maximum becomes:

$$\begin{cases} \max U(x_1,...,x_n) \\ \sum_{i=1}^{n} p_i x_i \le V \\ x_1,...,x_n \in SC \end{cases}$$

In the conditions that the function U is concave, and *SC* is a convex set, it is shown ([1], [3], [4]) that the optimal solution of the problem is situated on the border area of the budget, that is it satisfies:

$$\begin{cases} \max U(x_1,...,x_n) \\ \sum_{i=1}^{n} p_i x_i = V \\ x_1,...,x_n \in SC \end{cases}$$

Applying the Lagrange multiplier method results in the end:

$$\begin{cases} \frac{U_{m,1}}{p_1} = ... = \frac{U_{m,n}}{p_n} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

with the solution:

$$\begin{cases} \overline{x}_1 = f_1(p_1,...,p_n,V) \\ ... \\ \overline{x}_n = f_n(p_1,...,p_n,V) \end{cases}$$

called Marshall demand.

Considering now the same problem in the direction of the minimization of the allocated income to meet a given level of utility, the problem is:

$$\begin{cases} \min \sum_{i=1}^{n} p_i x_i \\ U(x_1,...,x_n) \ge \overline{u} \\ x_1,...,x_n \in SC \end{cases}$$

where $\overline{u}$ is the desired utility.

Finally, it is shown that the Hicks demand satisfies:

$$\begin{cases} \frac{U_{m,1}}{p_1} = ... = \frac{U_{m,n}}{p_n} \\ U(x_1,...,x_n) = \overline{u} \end{cases}$$

with the solution:

$$\begin{cases} \tilde{x}_1 = g_1(p_1,...,p_n,\overline{u}) \\ ... \\ \tilde{x}_n = g_n(p_1,...,p_n,\overline{u}) \end{cases}$$

Now consider, in terms of changing prices that, first, the consumer will change the demand in order to preserve his original utility level. The compensated demand (Hicks type) will satisfies the problem:

$$\begin{cases} \min \sum_{i=1}^{n} p'_i x_i \\ U(x_1,...,x_n) = \overline{u} \\ x_1,...,x_n \in SC \end{cases}$$

where $\overline{u}$ is the initial level of the utility.

Let the solution:

$$\begin{cases} \tilde{x}_1 = f_1(p'_1,...,p'_n,\overline{u}) \\ ... \\ \tilde{x}_n = f_n(p'_1,...,p'_n,\overline{u}) \end{cases}$$

and V'= $\sum_{i=1}^{n} p'_i \tilde{x}_i$ - the income needed to purchase the goods basket respectively, $U(\tilde{x}_1,...,\tilde{x}_n) = \overline{u}$.

We will call the passage from the initial basket of goods $(x_1,...,x_n)$ at $(\tilde{x}_1,...,\tilde{x}_n)$ - **Hicks substitution effect** (*short - Hs*) and we have: $\Delta_{Hs,i}=\tilde{x}_i - x_i$, $i=\overline{1,n}$.

The second phase arises from the fact that if V'≠V (*V – the initial income*), the consumer will change again the demand vector (*corresponding to its actual income*), in proportion to that previously preserved its utility. In this case, the problem of the uncompensated demand is:

$$\begin{cases} \max U(x_1,...,x_n) \\ \sum_{i=1}^{n} p'_i x_i = V \\ x_1,...,x_n \in SC \end{cases}$$

with the solution:

$$\begin{cases} \tilde{\tilde{x}}_1 = g_1(p'_1,...,p'_n, V) \\ ... \\ \tilde{\tilde{x}}_n = g_n(p'_1,...,p'_n, V) \end{cases}$$

In this case, we have: $V=\sum_{i=1}^{n} p'_i \tilde{\tilde{x}}_i$, $U\left(\tilde{\tilde{x}}_1,...,\tilde{\tilde{x}}_n\right)$ – the obtained utility. We will call the transition from intermediate goods basket $(\tilde{x}_1,...,\tilde{x}_n)$ at $\left(\tilde{\tilde{x}}_1,...,\tilde{\tilde{x}}_n\right)$ - **Hicks revenue effect** (*short - Hv*) and we have: $\Delta_{Hv,i}=\tilde{\tilde{x}}_i - \tilde{x}_i$, $i=\overline{1,n}$.

The total effect of these two phases is:

$$\Delta_{H,i}=\Delta_{Hs,i}+\Delta_{Hv,i}=\tilde{x}_i - x_i + \tilde{\tilde{x}}_i - \tilde{x}_i = \tilde{\tilde{x}}_i - x_i,\ i=\overline{1,n}$$

2 Conditions for the Existence of the Cobb-Douglas Utility Function

Let a utility function of Cobb-Douglas type: $U(x_1,...,x_n)=Ax_1^{\alpha_1}...x_n^{\alpha_n}$ with A, $\alpha_i>0$, $i=\overline{1,n}$.

The conditions of existence of a utility function ([4]) imply the C^2 – differentiability and its concavity.

Computing the partial derivatives of first and second order for the function $U(x_1,...,x_n)=Ax_1^{\alpha_1}...x_n^{\alpha_n}$ we obtain:

$$\frac{\partial U}{\partial x_i}=\alpha_i A x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}=\frac{\alpha_i}{x_i}U \quad \forall i=\overline{1,n}$$

$$\frac{\partial^2 Q}{\partial x_i \partial x_j}=\alpha_i\alpha_j A x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_j^{\alpha_j-1}...x_n^{\alpha_n}=\frac{\alpha_i\alpha_j}{x_i x_j}U \quad \forall i\neq j=\overline{1,n}$$

$$\frac{\partial^2 Q}{\partial x_i^2}=\alpha_i(\alpha_i-1)A x_1^{\alpha_1}...x_i^{\alpha_i-2}...x_n^{\alpha_n}=\frac{\alpha_i(\alpha_i-1)}{x_i^2}U \quad \forall i=\overline{1,n}$$

The Hessian matrix is:

$$H_{C\text{-}D}=U\begin{pmatrix} \frac{\alpha_1(\alpha_1-1)}{x_1^2} & \frac{\alpha_1\alpha_2}{x_1x_2} & \dots & \frac{\alpha_1\alpha_n}{x_1x_n} \\ \frac{\alpha_1\alpha_2}{x_1x_2} & \frac{\alpha_2(\alpha_2-1)}{x_2^2} & \dots & \frac{\alpha_2\alpha_n}{x_2x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1\alpha_n}{x_1x_n} & \frac{\alpha_2\alpha_n}{x_2x_n} & \dots & \frac{\alpha_n(\alpha_n-1)}{x_n^2} \end{pmatrix}$$

The principal diagonal minors are therefore:

$$\Delta_k= U^k\begin{pmatrix} \frac{\alpha_1(\alpha_1-1)}{x_1^2} & \frac{\alpha_1\alpha_2}{x_1x_2} & \dots & \frac{\alpha_1\alpha_k}{x_1x_k} \\ \frac{\alpha_1\alpha_2}{x_1x_2} & \frac{\alpha_2(\alpha_2-1)}{x_2^2} & \dots & \frac{\alpha_2\alpha_k}{x_2x_k} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1\alpha_k}{x_1x_k} & \frac{\alpha_2\alpha_k}{x_2x_k} & \dots & \frac{\alpha_k(\alpha_k-1)}{x_k^2} \end{pmatrix} = U^k\frac{\prod_{i=1}^k\alpha_i}{\left(\prod_{i=1}^k x_i\right)^2}\begin{pmatrix} \alpha_1-1 & \alpha_2 & \dots & \alpha_k \\ \alpha_1 & \alpha_2-1 & \dots & \alpha_k \\ \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_k-1 \end{pmatrix} =$$

$$A^k x_1^{k\alpha_1-2}\dots x_n^{k\alpha_n-2}\prod_{i=1}^k\alpha_i\begin{pmatrix} \alpha_1-1 & \alpha_2 & \dots & \alpha_k \\ \alpha_1 & \alpha_2-1 & \dots & \alpha_k \\ \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_k-1 \end{pmatrix} = (-1)^k A^k x_1^{k\alpha_1-2}\dots x_n^{k\alpha_n-2}\prod_{i=1}^k\alpha_i\left(1-\sum_{i=1}^k\alpha_i\right).$$

For the function's concavity we must have: $\Delta_k\le 0\ \forall k=\overline{1,n}$, k=odd and $\Delta_k\ge 0\ \forall k=\overline{1,n}$, k=even.

Therefore: $1-\sum_{i=1}^k\alpha_i \ge 0$ so: $\sum_{i=1}^k\alpha_i \le 1\ \forall k=\overline{1,n}$. How $\sum_{i=1}^k\alpha_i \le \sum_{i=1}^n\alpha_i \le 1$ the only condition of concavity of the function remains: $\sum_{i=1}^n\alpha_i \le 1$, $\alpha_i>0\ \forall i=\overline{1,n}$.

3 The Analysis of the Hicks Effect for a Cobb-Douglas Utility

Let now consider a consumer who has the income V and is faced with the choice of n goods $B_1,\dots,B_n$ with initial prices $p_1, \dots,p_n$ which are further adjusted to: $p'_1,\dots, p'_n$.

We will note: $\theta_k=\frac{p'_k}{p_k}$, $k=\overline{1,n}$ - the index of the good k price change.

Let now a Cobb-Douglas utility function: $U(x_1,\dots,x_n)=Ax_1^{\alpha_1}\dots x_n^{\alpha_n}$ with A, $\alpha_i>0$, $i=\overline{1,n}$, $\sum_{i=1}^n\alpha_i \le 1$.

We will note, first: $\alpha=\sum_{i=1}^n\alpha_i$, $\sigma_k=\frac{\alpha_k}{\alpha}$, $k=\overline{1,n}$. We have, obviously: $\sum_{k=1}^n\sigma_k=1$.

Calculating the marginal utilities, we get: $U_{m,k}= \alpha_k A x_1^{\alpha_1}\dots x_k^{\alpha_k-1}\dots x_n^{\alpha_n}=\frac{\alpha_k U}{x_k}$ from where:

$$\begin{cases} \dfrac{\alpha_1}{p_1 x_1} = \ldots = \dfrac{\alpha_n}{p_n x_n} = \dfrac{\sum_{i=1}^{n}\alpha_i}{\sum_{i=1}^{n} p_i x_i} = \dfrac{\alpha}{V} \\ \sum_{i=1}^{n} p_i x_i = V \end{cases}$$

so, finally:

$$x_k = \frac{\alpha_k}{p_k \alpha} V = \frac{\sigma_k}{p_k} V, \ k=\overline{1,n}$$

If we denote: $\frac{p_k}{V}=\beta_k$ – the share of the good k price in the disposable income V, we have:

$$x_k = \frac{\sigma_k}{\beta_k}, \ k=\overline{1,n}$$

The utility corresponding to this consumption distribution is:

$$U_1(x_1,\ldots,x_n) = A\prod_{k=1}^{n} x_k^{\alpha_k} = A\frac{\prod_{k=1}^{n}\alpha_k^{\alpha_k}}{\alpha^{\alpha}\prod_{k=1}^{n}\beta_k^{\alpha_k}} = A\frac{\prod_{k=1}^{n}\sigma_k^{\alpha_k}}{\prod_{k=1}^{n}\beta_k^{\alpha_k}}$$

Let us now consider the change in assets for each B_k, $k=\overline{1,n}$, the income remaining constant. From the above relations, we obtain for: $\gamma_k=\frac{p'_k}{V}$, $k=\overline{1,n}$:

$$x_{kf} = \frac{\sigma_k}{\gamma_k}, \ k=\overline{1,n}$$

and the appropriate utility: $U_3(x_1,\ldots,x_n) = A\frac{\prod_{k=1}^{n}\sigma_k^{\alpha_k}}{\prod_{k=1}^{n}\gamma_k^{\alpha_k}}$.

Let us note also that: $\theta_i = \frac{p'_i}{p_i} = \frac{\frac{p'_i}{V}}{\frac{p_i}{V}} = \frac{\gamma_i}{\beta_i}$, $i=\overline{1,n}$.

At a price change of B_k, $k=\overline{1,n}$, for the same value of the utility $U_1(x_1,\ldots,x_n) = A\frac{\prod_{k=1}^{n}\sigma_k^{\alpha_k}}{\prod_{k=1}^{n}\beta_k^{\alpha_k}}$ we will have:

$$A\frac{\prod_{k=1}^{n}\sigma_k^{\alpha_k}}{\prod_{k=1}^{n}\beta_k^{\alpha_k}}=A\frac{\prod_{k=1}^{n}\sigma_k^{\alpha_k}}{\prod_{k=1}^{n}\delta_k^{\alpha_k}}$$

where $\delta_k=\frac{p'_k}{V'}$, V' being the new income which will ensure the utility U_1.

We therefore have:

$$\prod_{k=1}^{n}\beta_k^{\alpha_k}=\prod_{k=1}^{n}\delta_k^{\alpha_k}$$

or, in terms of income:

$$\prod_{k=1}^{n}\frac{p_k^{\alpha_k}}{V^{\alpha_k}}=\prod_{k=1}^{n}\frac{p'^{\alpha_k}_k}{V'^{\alpha_k}}\Leftrightarrow\frac{\prod_{k=1}^{n}p_k^{\alpha_k}}{V^{\alpha}}=\frac{\prod_{k=1}^{n}p'^{\alpha_k}_k}{V'^{\alpha}}$$

The new income will be:

$$V'=\left(\frac{\prod_{i=1}^{n}p'^{\alpha_i}_i}{\prod_{i=1}^{n}p_i^{\alpha_i}}\right)^{\frac{1}{\alpha}}V=\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\beta_i}\right)^{\alpha_i}\right)^{\frac{1}{\alpha}}V=V\prod_{i=1}^{n}\left(\frac{\gamma_i}{\beta_i}\right)^{\sigma_i}=V\prod_{i=1}^{n}\theta_i^{\sigma_i}$$

With this income, we have:

$$x_{k2H}=\frac{\sigma_k}{\delta_k},\ k=\overline{1,n}$$

where $\delta_k=\frac{p'_k}{V'}=\frac{p'_k}{V\prod_{i=1}^{n}\left(\frac{\gamma_i}{\beta_i}\right)^{\sigma_i}}=\frac{\gamma_k}{\prod_{i=1}^{n}\theta_i^{\sigma_i}}$.

The Hicks substitution effect is thus:

$$\Delta_{1H}x_k=x_{k2H}-x_k=\frac{\sigma_k}{\delta_k}-\frac{\sigma_k}{\beta_k}=\sigma_k\left(\frac{\prod_{i=1}^{n}\theta_i^{\sigma_i}}{\gamma_k}-\frac{1}{\beta_k}\right)=\frac{\sigma_k}{\gamma_k}\left(\prod_{i=1}^{n}\theta_i^{\sigma_i}-\theta_k\right)$$

Considering now the initial income V instead of V' we obtain:

$$\Delta_{2H}x_k=x_{kf}-x_{k2H}=\frac{\sigma_k}{\gamma_k}-\frac{\sigma_k}{\delta_k}=\frac{\sigma_k}{\gamma_k}\left(1-\prod_{i=1}^{n}\theta_i^{\sigma_i}\right)$$

which means The Hicks income effect.

Denoting for simplicity: $\Gamma=\prod_{i=1}^{n}\theta_i^{\sigma_i}$ we have therefore:

$$\Delta_{1H}x_k = x_{k2H} - x_k = \frac{\sigma_k}{\gamma_k}(\Gamma - \theta_k)$$

$$\Delta_{2H}x_k = x_{kf} - x_{k2H} = \frac{\sigma_k}{\gamma_k}(1 - \Gamma)$$

We define, in the following, the ratios:

- $\alpha_k = \frac{x_{k2H} - x_k}{x_{kf} - x_k}$ - the part of the total change in consumption due to the substitution effect;
- $\beta_k = \frac{x_{kf} - x_{k2H}}{x_{kf} - x_k}$ - the part of the total change in consumption due to the income effect;
- $r_k = \frac{\beta_k}{\alpha_k} = \frac{x_{kf} - x_{k2H}}{x_{k2H} - x_k}$ - the ratio between the income and substitution effect.

We have, obviously: $\alpha_k + \beta_k = 1$ and $r_k = \frac{1}{\alpha_k} - 1 = \frac{1}{\frac{1}{\beta_k} - 1}$.

From the above, follows:

- $\alpha_k = \frac{\Delta_{1H}x_k}{\Delta_{1H}x_k + \Delta_{2H}x_k} = \frac{\frac{\sigma_k}{\gamma_k}(\Gamma - \theta_k)}{\frac{\sigma_k}{\gamma_k}(\Gamma - \theta_k) + \frac{\sigma_k}{\gamma_k}(1 - \Gamma)} = \frac{\Gamma - \theta_k}{1 - \theta_k}$
- $\beta_k = 1 - \alpha_k = 1 - \frac{\Gamma - \theta_k}{1 - \theta_k} = \frac{1 - \Gamma}{1 - \theta_k}$
- $r_k = \frac{\beta_k}{\alpha_k} = \frac{1 - \Gamma}{\Gamma - \theta_k}$

Let the function: f: $\mathbf{R}_+^{*\,k-1} \times A \times \mathbf{R}_+^{*\,n-k} \to \mathbf{R}$ where $A = (0,1) \cup (1, \infty)$, $f(\theta_1, \ldots, \theta_n) = \frac{\prod_{i=1}^{n} \theta_i^{\sigma_i} - \theta_k}{1 - \theta_k}$.

We have:

$$\frac{\partial f}{\partial \theta_p} = \frac{\sigma_p \prod_{i=1}^{n} \theta_i^{\sigma_i}}{\theta_p (1 - \theta_k)} \quad \forall p \neq k$$

$$\frac{\partial f}{\partial \theta_k} = \frac{(\sigma_k - \sigma_k\theta_k + \theta_k)\prod_{i=1}^{n} \theta_i^{\sigma_i} - \theta_k}{\theta_k (1 - \theta_k)^2}$$

If we note: $\lambda_k=\prod_{\substack{i=1\\ i\neq k}}^{n}\theta_i^{\sigma_i}=\frac{\Gamma}{\theta_k^{\sigma_k}}$:

$$\frac{\partial f}{\partial\theta_k}=\frac{\lambda_k\sigma_k\theta_k^{\sigma_k-1}-\lambda_k\sigma_k\theta_k^{\sigma_k}+\lambda_k\theta_k^{\sigma_k}-1}{(1-\theta_k)^2}$$

Let also the function: $g(\theta_k)=\lambda_k\sigma_k\theta_k^{\sigma_k-1}-\lambda_k\sigma_k\theta_k^{\sigma_k}+\lambda_k\theta_k^{\sigma_k}-1$.

From the expression of g, we get easily: $\lim_{\theta_k\to 0}g(\theta_k)=\infty$, $g(1)=\lambda_k-1$, $\lim_{\theta_k\to\infty}g(\theta_k)=\infty$.

But since: $g'(\theta_k)=\lambda_k\sigma_k\theta_k^{\sigma_k-2}(\sigma_k-1)(1-\theta_k)$ we get that:

- $\theta_k\in(0,1)\Rightarrow g'(\theta_k)<0$ so g is strictly decreasing. In this case: $g(\theta_k)\in(\lambda_k-1,\infty)$.
- $\theta_k\in(1,\infty)\Rightarrow g'(\theta_k)>0$ so g is strictly increasing. In this case: $g(\theta_k)\in(\lambda_k-1,\infty)$.

If $\lambda_k\geq 1$ then $g(\theta_k)>0$ therefore $\frac{\partial f}{\partial\theta_k}>0$ that is f is increasing with respect to θ_k.

How $f(\theta_1,\ldots,\theta_n)=\frac{\prod_{i=1}^{n}\theta_i^{\sigma_i}-\theta_k}{1-\theta_k}=\frac{\lambda_k\theta_k^{\sigma_k}-\theta_k}{1-\theta_k}$ follows:

$$\lim_{\theta_k\to 0}f(\theta_1,\ldots,\theta_n)=0,\ \lim_{\theta_k\to 1}f(\theta_1,\ldots,\theta_n)=1-\lambda_k\sigma_k,\ \lim_{\theta_k\to\infty}f(\theta_1,\ldots,\theta_n)=1$$

Like a conclusion, in the conditions that $\prod_{\substack{i=1\\ i\neq k}}^{n}\theta_i^{\sigma_i}=\text{constant}\geq 1$, we have:

$$f(\theta_1,\ldots,\theta_n)\in(0,1-\lambda_k\sigma_k)\ \forall\theta_k\in(0,1)$$

$$f(\theta_1,\ldots,\theta_n)\in(1-\lambda_k\sigma_k,1)\ \forall\theta_k\in(1,\infty)$$

If now: $\lambda_k<1$ then as $g(1)=\lambda_k-1<0$ follows that f will change the monotony.

Let φ an arbitrary root of $g(\theta_k)=0$ that is: $\lambda_k\sigma_k\varphi^{\sigma_k-1}-\lambda_k\sigma_k\varphi^{\sigma_k}+\lambda_k\varphi^{\sigma_k}-1=0$.

It is easy to see that the equation has two roots: $\varphi_1\in(0,1)$ and $\varphi_2\in(1,\infty)$.

Therefore, we have:

- $\theta_k\in(0,\varphi_1)\Rightarrow g(\theta_k)>0\Rightarrow\frac{\partial f}{\partial\theta_k}>0\Rightarrow$ f is strictly increasing, so

$$f(\theta_1,\ldots,\theta_n)\in\left(0,\frac{\lambda_k\varphi_1^{\sigma_k}-\varphi_1}{1-\varphi_1}\right)\ \forall\theta_k\in(0,\varphi_1)$$

- $\theta_k\in(\varphi_1,\varphi_2)\Rightarrow g(\theta_k)<0\Rightarrow\frac{\partial f}{\partial\theta_k}<0\Rightarrow$ f is strictly decreasing, so

$$f(\theta_1,...,\theta_n)\in\left(\frac{\lambda_k\varphi_2^{\sigma_k}-\varphi_2}{1-\varphi_2},\frac{\lambda_k\varphi_1^{\sigma_k}-\varphi_1}{1-\varphi_1}\right)\ \forall\theta_k\in(\varphi_1,\varphi_2)$$

- $\theta_k\in(\varphi_2,\infty)\Rightarrow g(\theta_k)>0\Rightarrow\frac{\partial f}{\partial\theta_k}>0\Rightarrow f$ is strictly increasing, so

$$f(\theta_1,...,\theta_n)\in\left(\frac{\lambda_k\varphi_2^{\sigma_k}-\varphi_2}{1-\varphi_2},1\right)\ \forall\theta_k\in(\varphi_2,\infty)$$

With the simplified notations: $\lambda_k=\lambda$, $\sigma_k=\sigma$, let the equation:

$$g(\varphi)=\lambda\sigma\varphi^{\sigma-1}-\lambda\sigma\varphi^{\sigma}+\lambda\varphi^{\sigma}-1=0,\ \sigma\in(0,1),\ \varphi,\lambda\in(0,\infty)$$

We have:

$$g'(\varphi)=\lambda\sigma\varphi^{\sigma-2}(\sigma-1)(1-\varphi)$$

$$g''(\varphi)=\lambda\sigma(\sigma-1)\varphi^{\sigma-3}((\sigma-2)-(\sigma-1)\varphi)$$

If $\varphi>\frac{\sigma-2}{\sigma-1}=1-\frac{1}{\sigma-1}>1$ then $g''(\varphi)<0$.

If $\varphi<\frac{\sigma-2}{\sigma-1}=1-\frac{1}{\sigma-1}>1$ then $g''(\varphi)>0$.

After these, we have:

- $\varphi\in(0,1)\Rightarrow$ g is strictly decreasing and convex
- $\varphi\in(1,\infty)\Rightarrow$ g is strictly increasing and convex on $\left(1,\frac{\sigma-2}{\sigma-1}\right)$ and concave on $\left(\frac{\sigma-2}{\sigma-1},\infty\right)$.

We have but: $\lim_{\varphi\to0}g(\varphi)=\infty$, $g(1)=\lambda-1<0$, $\lim_{\varphi\to\infty}g(\varphi)=\infty$. On the other hand:

$$g\left(\frac{\sigma-2}{\sigma-1}\right)=2\lambda\left(1-\frac{1}{\sigma-1}\right)^{\sigma-1}-1$$

Because: $\left(1-\frac{1}{\sigma-1}\right)^{\sigma-1}=e^{\frac{\ln\left(1-\frac{1}{\sigma-1}\right)}{\frac{1}{\sigma-1}}}$ follows: $\lim_{\sigma\to1}\left(1-\frac{1}{\sigma-1}\right)^{\sigma-1}=\lim_{\sigma\to1}e^{\frac{\ln\left(1-\frac{1}{\sigma-1}\right)}{\frac{1}{\sigma-1}}}=1$. Considering the function: $h(y)=\left(1+\frac{1}{y}\right)^{-y}$ with $y>0$, we have $h(y)<1$ and $\lim_{y\to0}h(y)=1$. Therefore: $g\left(\frac{\sigma-2}{\sigma-1}\right)<2\lambda-1<1$.

For the determination of $\varphi_1\in(0,1)$ we will apply the Newton's method, taking into account that on the interval (0,1) the function g is strictly decreasing and convex. So we choose the point x_0 sufficiently close to 0 so that: $g(x_0)g''(x_0)>0$.

We get therefore:

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}=\frac{\lambda\sigma(\sigma-2)x_n^{\sigma-1}-\lambda(\sigma-1)^2x_n^{\sigma}+1}{\lambda\sigma(\sigma-1)x_n^{\sigma-2}-\lambda\sigma(\sigma-1)x_n^{\sigma-1}},\ n\geq 0$$

Finally: $\varphi_1=\lim_{n\to\infty}x_n$.

For the determination of $\varphi_2\in(1,\infty)$ we have two cases:

If $g\left(\frac{\sigma-2}{\sigma-1}\right)<0$ the root belongs to the interval $\left(\frac{\sigma-2}{\sigma-1},\infty\right)$ where g is strictly increasing and concave, so we will choose the point $x_0=\frac{\sigma-2}{\sigma-1}$ because $g(x_0)g''(x_0)>0$.

If $g\left(\frac{\sigma-2}{\sigma-1}\right)>0$ the root belongs to the interval $\left(1,\frac{\sigma-2}{\sigma-1}\right)$ where g is strictly increasing and convex, so we will choose again the point $x_0=\frac{\sigma-2}{\sigma-1}$ because $g(x_0)g''(x_0)>0$.

We therefore have:

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}=\frac{\lambda\sigma(\sigma-2)x_n^{\sigma-1}-\lambda(\sigma-1)^2x_n^{\sigma}+1}{\lambda\sigma(\sigma-1)x_n^{\sigma-2}-\lambda\sigma(\sigma-1)x_n^{\sigma-1}},\ n\geq 0$$

and $\varphi_2=\lim_{n\to\infty}x_n$.

If $g\left(\frac{\sigma-2}{\sigma-1}\right)=0$ we have, obviously that $\varphi_2=\frac{\sigma-2}{\sigma-1}$.

After the above relationships, we get, finally:

- α_k – the part of the total change in consumption due to the substitution effect belongs to:
  - $(0,1-\lambda_k\sigma_k)$ $\forall\theta_k\in(0,1)$ if $\lambda_k\geq 1$;
  - $(1-\lambda_k\sigma_k,1)$ $\forall\theta_k\in(1,\infty)$ if $\lambda_k\geq 1$;
  - $\left(0,\frac{\lambda_k\varphi_1^{\sigma_k}-\varphi_1}{1-\varphi_1}\right)$ $\forall\theta_k\in(0,\varphi_1)$ if $\lambda_k<1$;
  - $\left(\frac{\lambda_k\varphi_2^{\sigma_k}-\varphi_2}{1-\varphi_2},\frac{\lambda_k\varphi_1^{\sigma_k}-\varphi_1}{1-\varphi_1}\right)$ $\forall\theta_k\in(\varphi_1,\varphi_2)$ if $\lambda_k<1$;
  - $\left(\frac{\lambda_k\varphi_2^{\sigma_k}-\varphi_2}{1-\varphi_2},1\right)$ $\forall\theta_k\in(\varphi_2,\infty)$ if $\lambda_k<1$.
- β_k – the part of the total change in consumption due to the income effect belongs to:
  - $(\lambda_k\sigma_k,1)$ $\forall\theta_k\in(0,1)$ if $\lambda_k\geq 1$;
  - $(0,\lambda_k\sigma_k)$ $\forall\theta_k\in(1,\infty)$ if $\lambda_k\geq 1$;
  - $\left(\frac{1-\lambda_k\varphi_1^{\sigma_k}}{1-\varphi_1},1\right)$ $\forall\theta_k\in(0,\varphi_1)$ if $\lambda_k<1$;

- o $\left(\frac{1-\lambda_k\varphi_1^{\sigma_k}}{1-\varphi_1},\frac{1-\lambda_k\varphi_2^{\sigma_k}}{1-\varphi_2}\right)$ $\forall\theta_k\in(\varphi_1,\varphi_2)$ if $\lambda_k<1$;
- o $\left(0,\frac{1-\lambda_k\varphi_2^{\sigma_k}}{1-\varphi_2}\right)$ $\forall\theta_k\in(\varphi_2,\infty)$ if $\lambda_k<1$.

- r_k – the ratio between the income and substitution effect belongs to:

- o $\left(\frac{\lambda_k\sigma_k}{1-\lambda_k\sigma_k},\infty\right)$ $\forall\theta_k\in(0,1)$ if $\lambda_k\ge1$;
- o $\left(1,\frac{\lambda_k\sigma_k}{1-\lambda_k\sigma_k}\right)$ $\forall\theta_k\in(1,\infty)$ if $\lambda_k\ge1$;
- o $\left(\frac{1-\lambda_k\varphi_1^{\sigma_k}}{\lambda_k\varphi_1^{\sigma_k}-\varphi_1},\infty\right)$ $\forall\theta_k\in(0,\varphi_1)$ if $\lambda_k<1$;
- o $\left(\frac{1-\lambda_k\varphi_1^{\sigma_k}}{\lambda_k\varphi_1^{\sigma_k}-\varphi_1},\frac{1-\lambda_k\varphi_2^{\sigma_k}}{\lambda_k\varphi_2^{\sigma_k}-\varphi_2}\right)$ $\forall\theta_k\in(\varphi_1,\varphi_2)$ if $\lambda_k<1$;
- o $\left(1,\frac{1-\lambda_k\varphi_2^{\sigma_k}}{\lambda_k\varphi_2^{\sigma_k}-\varphi_2}\right)$ $\forall\theta_k\in(\varphi_2,\infty)$ if $\lambda_k<1$.

4 Conclusions

The analysis of the effect of income and substitution for the case of n goods is essential in determining the effect of price changes on consumption movement.

The present demarche, establishes the limits of variation of the two effects and the relationship between them, when price changes on all the goods and not just on two as in the classical theory.

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