An adjustment of the Electre method for the case of intervals

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Abstract. The Electre method aims as objective the choosing of the best variant in the conditions of existence of some decision criteria. An interesting problem arises when different variants associated to the values are not constant but are in intervals. The paper treats how to determine the optimal variant for two types of action and two decision criteria.

Keywords: Electre, coefficients of importance, decision

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1 Introduction

The Electre method is mainly aimed at determining the best choice of action in the condition of existence of some decision criteria. Many problems arising in this method but also extensions of this have generated a series of subsequent development (Mousseau, 2001), (Buchanan, 2007), (Almeida, 2008).

The method consists of a number of action's variants $V_1, V_2, ..., V_n$ whose choice is faced with a decision maker. Let also, be a number of m criteria $C_1, C_2,..., C_m$ which have each an importance coefficient (*usually determined subjectively*) $k_1, k_2,..., k_m$. To each pair (V_i,C_j) we assign a numerical value v_{ij} (*if it is a qualitative appreciation we will convert it in a number of hierarchy*). To determine the optimal action, the coefficients of importance will be normalize by the relationship: $\nu_j=\frac{k_j}{\sum_{p=1}^{m} k_p}$, $j=\overline{1,m}$ getting: $\sum_{j=1}^{m} \nu_j=1$.

The next step is to determine the nature of the method (maximizing or minimizing) harmonizing the data in that, in the case of a criterion of contrary nature to the problem, the corresponding range changing its sign.

Also, at least two corresponding values of a criterion must be different, otherwise the selection criterion becoming insignificant.

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Because the different criterions use various units, we will determine the utilities U_{ij} that correspond to the pairs (V_i, C_j) as follows: for the problem of maximizing: $U_{ij}=\dfrac{v_{ij}-\min\limits_{k=1,\dots,n} v_{kj}}{\max\limits_{k=1,\dots,n} v_{kj}-\min\limits_{k=1,\dots,n} v_{kj}}$ and for the minimization: $U_{ij}=\dfrac{\max\limits_{k=1,\dots,n} v_{kj}-v_{ij}}{\max\limits_{k=1,\dots,n} v_{kj}-\min\limits_{k=1,\dots,n} v_{kj}}$ and after we will construct their table. Let noted also that the utilities are quantities located always between the interval: [0,1].

Finally we compute the concordance indicators according to the following:

$$c(V_i,V_j)=\sum_{\substack{p=1,\dots,m\\ U_{ip}\ge U_{jp}}} \nu_p = 1-\sum_{\substack{p=1,\dots,m\\ U_{ip}<U_{jp}}} \nu_p$$

and the discordance indicators:

$$d(V_i,V_j)=\max_{p=1,\dots,m}(U_{jp}-U_{ip},0)$$

In order to determine the best choice, we establish two values p and q such that $p,q\in(0,1)$ and $p+q=1$ to measure the limits of concordance and discordance. We will say that a variant V_i is preferred to a variant V_j if:

$$\begin{cases} c(V_i,V_j)\ge p \\ d(V_i,V_j)\le q \end{cases}$$

Taking into account that q=1-p we have therefore and noting $c'(V_i,V_j)=\sum\limits_{\substack{p=1,\dots,m\\ U_{ip}<U_{jp}}} \nu_p$ named the contra-concordance, we have that: $c(V_i,V_j)=1-c'(V_i,V_j)$

so we get:

$$\begin{cases} 1-c'(V_i,V_j)\ge p \\ 1-d(V_i,V_j)\ge p \end{cases}$$

therefore:

$$1-p\ge \max\left(c'(V_i,V_j),d(V_i,V_j)\right)$$

A variant V_i will satisfy the optimal condition if for a given p:

$$1-p\ge \max\left(c'(V_i,V_j),d(V_i,V_j)\right)\forall j=\overline{1,n}$$

We thus determine p by condition:

$$1-p=\min_{i=\overline{1,n}}\left(\max_{j=\overline{1,n}} c'(V_i,V_j),\max_{j=\overline{1,n}} d(V_i,V_j)\right)$$

the optimal variant (variants) corresponding to this value.

2 The Electre method for intervals

Let consider the problem of maximization:

Criterion Variant	C_1 k_1	C_2 k_2
V_1	[a,b]	$[\alpha,\beta]$
V_2	[c,d]	$[\gamma,\delta]$

where: $k_1+k_2=1$.

Because: $[a,b]=\{(1-s)a+sb \mid s\in[0,1]\}$ and analogously for the other intervals, we have:

Criterion Variant	C_1 k_1	C_2 k_2
V_1	(1-s)a+sb	$(1-p)\alpha+p\beta$
V_2	(1-t)c+td	$(1-q)\gamma+q\delta$

with $s,t,p,q\in[0,1]$.

Let $A=\max(b,d)$, $B=\min(a,c)$, $C=\max(\beta,\delta)$, $D=\min(\alpha,\gamma)$.

We have $[a,b]\cup[c,d]=[B,A]$ and $[\alpha,\beta]\cup[\gamma,\delta]=[D,C]$ respectively.

We define the appropriate utilities of v_{ij} like the ratio of the difference between v_{ij} and the minimum of the interval appropriate for the criterion and the length of the union-interval.

We have therefore the utilities:

Criterion Variant	C_1 k_1	C_2 k_2
V_1	$U_1=\frac{(1-s)a+sb-B}{A-B}$	$U_3=\frac{(1-p)\alpha+p\beta-D}{C-D}$
V_2	$U_2=\frac{(1-t)c+td-B}{A-B}$	$U_4=\frac{(1-q)\gamma+q\delta-D}{C-D}$

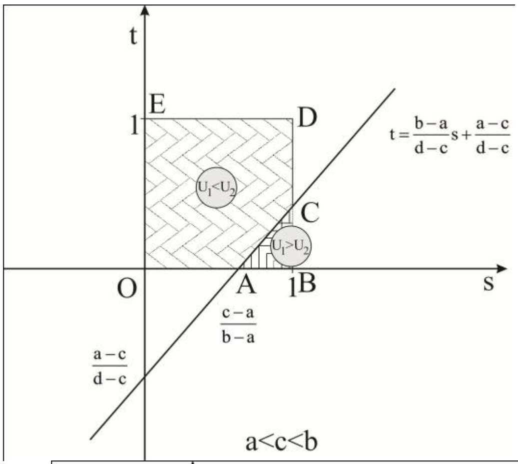
The condition that $U_1<U_2$ becomes: $\frac{(1-s)a+sb-B}{A-B}<\frac{(1-t)c+td-B}{A-B}\Leftrightarrow(1-s)a+sb<(1-t)c+td\Leftrightarrow$

$(b-a)s-(d-c)t+a-c<0$.

With the straight line: $(b-a)s-(d-c)t+a-c=0$ the upper condition is reduced to the determination of the appropriate half-plane. Because the coordinates of the origin satisfy the inequality: $a-c<0$, follows with $s,t\in[0,1]$, the next cases:

- $a<c<b$ (*fig.1*) all the points inside the pentagon OACDE satisfy $U_1<U_2$, and the points inside the triangle ABC satisfy $U_1>U_2$;

- $a<b<c$ (*fig.2*) all the points inside the square OABC satisfy $U_1<U_2$;
- $c<a<d$ (*fig.3*) the points inside the triangle CDE satisfay $U_1<U_2$, and the points inside the pentagon OABCE satisfy $U_1>U_2$;
- $c<d<a$ (*fig.4*) all the points inside the square OABC satisfy $U_1>U_2$.



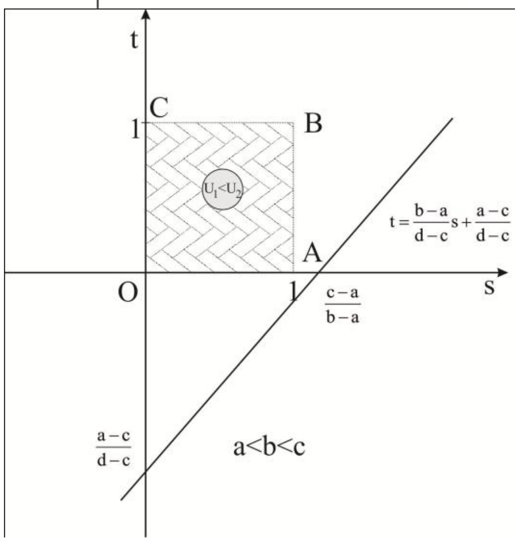
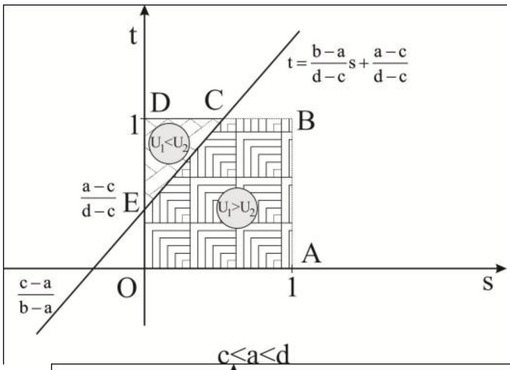


Fig.1 Fig.2



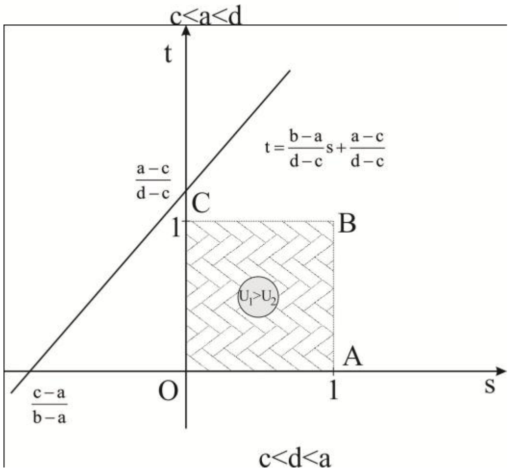


Fig.3 Fig.4

In a very similar manner we analyze the comparison between U_3 and U_4 whose geometric areas we will note similar but with character ‘.

Before continue, let make a remark that:

$$V=U_2-U_1=-\frac{(b-a)s-(d-c)t+a-c}{A-B}$$

$$W=U_4-U_3=-\frac{(\beta-\alpha)p-(\delta-\gamma)q+\alpha-\gamma}{C-D}$$

Finally we have the following cases:

1. $a<c<b$, $\alpha<\gamma<\beta$

   a. $(s,t)\in$OACDE (*fig.1*), $(p,q)\in$O’A’C’D’E’ (*fig.1*)

The table of contra-concordance and discordance indicators becomes:

	V_1	**V_2**	$\max\limits_{j=1,n} c'(V_i,V_j)$	$\max\limits_{j=1,n} d(V_i,V_j)$	max=1-p

V_1	0	0	1	max(V,W)	1	max(V,W)	1
V_2	0	0	0	0	0	0	0

The optimal variant coresponds to the minimum between the two values, that is V_2 with p=1.

b. $(s,t)\in OACDE$ (*fig.1*), $(p,q)\in A'B'C'$ (*fig.1*)

	V_1		**V_2**		$\max\limits_{j=1,n} c'(V_i,V_j)$	$\max\limits_{j=1,n} d(V_i,V_j)$	max=1-p
V_1	0	0	k_1	V	k_1	V	$\max(k_1,V)$
V_2	k_2	W	0	0	k_2	W	$\max(k_2,W)$

The optimal variant coresponds to the minimum between $\max(k_1,V)$ and $\max(k_2,W)$.

c. $(s,t)\in ABC$ (*fig.1*), $(p,q)\in O'A'C'D'E'$ (*fig.1*)

	V_1		**V_2**		$\max\limits_{j=1,n} c'(V_i,V_j)$	$\max\limits_{j=1,n} d(V_i,V_j)$	max=1-p
V_1	0	0	k_2	W	k_2	W	$\max(k_2,W)$
V_2	k_1	V	0	0	k_1	V	$\max(k_1,V)$

The optimal variant coresponds to the minimum between $\max(k_2,W)$ and $\max(k_1,V)$.

d. $(s,t)\in ABC$ (*fig.1*), $(p,q)\in A'B'C'$ (*fig.1*)

	V_1		**V_2**		$\max\limits_{j=1,n} c'(V_i,V_j)$	$\max\limits_{j=1,n} d(V_i,V_j)$	max=1-p
V_1	0	0	0	0	0	0	0
V_2	1	max(V,W)	0	0	1	max(V,W)	1

The optimal variant coresponds to the minimum between the two values, that is V_1 with p=1.

2. $a<c<b$, $\alpha<\beta<\gamma$

a. $(s,t)\in OACDE$ (*fig.1*)

	V_1		**V_2**		$\max\limits_{j=1,n} c'(V_i,V_j)$	$\max\limits_{j=1,n} d(V_i,V_j)$	max=1-p
V_1	0	0	1	max(V,W)	1	max(V,W)	1
V_2	0	0	0	0	0	0	0

The optimal variant coresponds to the minimum between the two values, that is V_2 with p=1.

b. (s,t)∈ABC (*fig.1*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	k_2	W	k_2	W	$\max(k_2,W)$
V_2	k_1	V	0	0	k_1	V	$\max(k_1,V)$

The optimal variant coresponds to the minimum between $\max(k_2,W)$ and $\max(k_1,V)$.

3. a<c<b, γ<α<δ

a. (s,t)∈OACDE (*fig.1*), (p,q)∈C'D'E' (*fig.3*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	1	max(V,W)	1	max(V,W)	1
V_2	0	0	0	0	0	0	0

The optimal variant coresponds to the minimum between the two values, that is V_2 with p=1.

b. (s,t)∈OACDE (*fig.1*), (p,q)∈O'A'B'C'E' (*fig.3*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	k_1	V	k_1	V	$\max(k_1,V)$
V_2	k_2	W	0	0	k_2	W	$\max(k_2,W)$

The optimal variant coresponds to the minimum between $\max(k_2,W)$ and $\max(k_1,V)$.

c. (s,t)∈ABC (*fig.1*), (p,q)∈ C'D'E' (*fig.3*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	k_2	W	k_2	W	$\max(k_2,W)$
V_2	k_1	V	0	0	k_1	V	$\max(k_1,V)$

The optimal variant coresponds to the minimum between $\max(k_2,W)$ and $\max(k_1,V)$.

d. (s,t)∈ABC (*fig.1*), (p,q)∈O'A'B'C'E' (*fig.3*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	0	0	0	0	0
V_2	1	max(V,W)	0	0	1	max(V,W)	1

The optimal variant coresponds to the minimum between the two values, that is V_1 with p=1.

4. a<c<b, $\gamma<\delta<\alpha$

a. $(s,t)\in$OACDE (*fig.1*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	k_1	V	k_1	V	$\max(k_1,V)$
V_2	k_2	W	0	0	k_2	W	$\max(k_2,W)$

The optimal variant coresponds to the minimum between $\max(k_2,W)$ and $\max(k_1,V)$.

b. $(s,t)\in$ABC (*fig.1*)

	V_1		V_2		$\max_{j=1,n} c'(V_i, V_j)$	$\max_{j=1,n} d(V_i, V_j)$	max=1-p
V_1	0	0	0	0	0	0	0
V_2	1	max(V,W)	0	0	1	max(V,W)	1

The optimal variant coresponds to the minimum between the two values, that is V_2 with p=1.

In the same manner there are treated the cases:

5. a<b<c, $\alpha<\gamma<\beta$
6. a<b<c, $\alpha<\beta<\gamma$
7. a<b<c, $\gamma<\alpha<\delta$
8. a<b<c, $\gamma<\delta<\alpha$
9. c<a<d, $\alpha<\gamma<\beta$
10. c<a<d, $\alpha<\beta<\gamma$
11. c<a<d, $\gamma<\alpha<\delta$
12. c<a<d, $\gamma<\delta<\alpha$
13. c<d<a, $\alpha<\gamma<\beta$

14. $c<d<a$, $\alpha<\beta<\gamma$
15. $c<d<a$, $\gamma<\alpha<\delta$
16. $c<d<a$, $\gamma<\delta<\alpha$

3 Conclusion

The above method of choice the optimal variant can be applied in the case when we cannot know exactly the values corresponding to each variant and/or criterion.

The difficulties of this algorithm consist in the fact that the approach of the variables require a geometrically analysis of the position of different intervals.

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