The Consumer's Budget from a Geometrically Point of View

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Abstract. This paper treats the consumer's budget problem for arbitrary n goods. The analysis is based mainly on the comparison of volumes of Rn budget zones that allow for interesting conclusions on the effect of fees and taxes in relation to purchasing power.

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1 The Consumer's Budget

Let consider a lot of goods $B_1,...,B_n$, SC - their space consumption and their sale prices: $p_1,...,p_n$. For a basket of goods $(x_1,...,x_n)\in SC$, a consumer must pay: $p_1x_1+...+p_nx_n=\sum_{i=1}^{n}p_ix_i$ u.m.

If the consumer has the acquisition of income V for n goods (in various amounts), it generates a **budget constraint** that is limiting the possibilities of purchasing to the set

$$ZB=\{(x_1,...,x_n)\in SC \mid \sum_{i=1}^{n}p_ix_i \le V\}$$

called **the budget zone**.

Considering **the budget hyperplane** H: $\sum_{i=1}^{n}p_ix_i=V$ it divides the space $\mathbf{R}^n$ into two regions:

$$H_1=\{(x_1,...,x_n)\in \mathbf{R}^n \mid \sum_{i=1}^{n}p_ix_i \le V\},\ H_2=\{(x_1,...,x_n)\in \mathbf{R}^n \mid \sum_{i=1}^{n}p_ix_i \ge V\}$$

called closed half-spaces and whose intersection is exactly H. It is known that if a point in $\mathbf{R}^n$ satisfies one of above the inequalities, then all points on the same side of the hyperplane (*i.e. those points for which the segment determined by them not intersects the hyperplane*) satisfy the same inequality.

Therefore, considering the point 0, we see that $\sum_{i=1}^{n}p_i\cdot 0\le V$, therefore the hyperplane H_1 is that which contains the origin.

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Returning to our analysis, we see therefore that the budget zone ZB is the part of the half-space determined by the budget hyperplane, which contains the origin and has all positive coordinates.

Let us note also that the budget hyperplane intersections with coordinate axes are the points $A_i\left(0,...,\frac{V}{p_i},...,0\right)$, $i=\overline{1,n}$.

We will call **volume budget**, the volume of the budget zone in $\mathbf{R}^n$.

Thus, in $\mathbf{R}^2$ the volume budget is the area of the budget zone equal to $\frac{V^2}{2p_1p_2}$, in $\mathbf{R}^3$ is the volume of the triangular prism, determined by the budget hyperplane: $\frac{V^3}{6p_1p_2p_3}$ and, in general, in $\mathbf{R}^n$ is: $\frac{V^n}{n!p_1p_2...p_n}$. Sometimes, the volume of the budget will allow us to compare the budget zones from a numerical point of view.

Considering now a consumer basket $(x_1,...,x_n)$ on the budget hyperplane, therefore $\sum_{j=1}^{n} p_j x_j=V$, let consider a variation of consumption of each good B_j equal with dx_j, $j=\overline{1,n}$. To remain on the budget hyperplane (i.e. the consumer allocate the same amount for the purchase of goods) we should have: $\sum_{j=1}^{n} p_j\left(x_j+dx_j\right)=V$ therefore $\sum_{j=1}^{n} p_j x_j+\sum_{j=1}^{n} p_j dx_j=V$. Because $\sum_{j=1}^{n} p_j x_j=V$ we get $\sum_{j=1}^{n} p_j dx_j=0$. For a fixed good B_i, we obtain: $\sum_{j=1}^{n} p_j\frac{dx_j}{dx_i}=0$. Noting $y_j=\frac{dx_j}{dx_i}$, we get **the substitution hyperplane** between the i-th good and the others:

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left(-\frac{p_j}{p_i}\right)y_j=1$$

In particular, because for an initially input $\overline{x}$ we have the partial substitution marginal rate: $RMS(i,j,\overline{x})=\frac{dx_j}{dx_i}=y_j$, we get:

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left(-\frac{p_j}{p_i}\right)RMS(i,j,\overline{x})=1$$

For two goods, we have the well-known results: $RMS(1,2,\overline{x})=\frac{dx_2}{dx_1}=-\frac{p_1}{p_2}$ and, analogously: $RMS(2,1,\overline{x})=\frac{dx_1}{dx_2}=-\frac{p_2}{p_1}$.

Geometrically speaking, in $\mathbf{R}^{n-1}$ the vectors:

$$RMS_i=\left(RMS(i,1,\bar{x}),\ldots,RMS(i,i-1,\bar{x}),1,RMS(i,i+1,\bar{x}),\ldots,RMS(i,n,\bar{x})\right)$$

and

$$P_i=\left(-\frac{p_1}{p_i},\ldots,-\frac{p_{i-1}}{p_i},-1,-\frac{p_{i+1}}{p_i},\ldots,-\frac{p_n}{p_i}\right)$$

are orthogonal.

Coming back, we put the issue of determining a consumer basket, within budget restrictions, to be minimal in the sense of norm.

Let therefore $x=(x_1,\ldots,x_n)\in SC$ such that $\sum_{j=1}^{n} p_j x_j = V$. The Euclidean norm of x is: $\|x\|=\sqrt{x_1^2+\ldots+x_n^2}$. Let consider the straight line orthogonal on the budget hyperplane passing through the origin:

$$\frac{x_1}{p_1}=\ldots=\frac{x_n}{p_n}$$

Noting $x_j=p_j\lambda$, $j=\overline{1,n}$ and replacing in the hyperplane equation, we get: $\lambda\sum_{j=1}^{n} p_j^2 = V$ where: $\lambda=\dfrac{V}{\sum_{j=1}^{n} p_j^2}$. We got so that point M of intersection of the straight line orthogonal to the hyperplane has the coordinates: $\dfrac{V}{\sum_{j=1}^{n} p_j^2}\left(p_1,\ldots,p_n\right)$.

The norm of the vector u=OM is thus: $\|u\|=\dfrac{V}{\sqrt{\sum_{j=1}^{n} p_j^2}}$. From the Cauchy-Schwarz inequality: $\sum_{j=1}^{n} p_j x_j \le \sqrt{\sum_{j=1}^{n} p_j^2}\sqrt{\sum_{j=1}^{n} x_j^2}$ we have that: $V\le \dfrac{V}{\|u\|}\|x\|$ therefore: $\|x\|\ge\|u\|$.

After these considerations we have that the consumption basket: $\dfrac{V}{\sum_{j=1}^{n} p_j^2}\left(p_1,\ldots,p_n\right)$ meet the budget restrictions and it is minimal in the sense of the norm.

2. The Budget Changes

Considering again a lot of goods $B_1,...,B_n$, SC – their space of consumption, sale prices: $p_1,...,p_n$ and V – the consumer income, suppose first that (*after a possible renumbering*) that the goods prices of $B_1,...,B_k$ will change, becoming $p_1^{'},...,p_k^{'}$.

If the consumer income remains constant V, then the budget hyperplane becomes

$$H': \sum_{i=1}^{k} p_i^{'} x_i + \sum_{i=k+1}^{n} p_i x_i = V$$

Two situations are now interesting:

- if $p_i^{'} < p_i$, $i=\overline{1,k}$ then: $\forall x=(x_1,...,x_n)\in ZB$ implies: $\sum_{i=1}^{n} p_i x_i \le V$. We have now: $\sum_{i=1}^{k} p_i^{'} x_i + \sum_{i=k+1}^{n} p_i x_i < \sum_{i=1}^{k} p_i x_i + \sum_{i=k+1}^{n} p_i x_i = \sum_{i=1}^{n} p_i x_i \le V$ therefore $x\in ZB'$ or, in other words: $ZB\subset ZB'$. Therefore, at a decreasing of the prices of some goods, the budget zone increases in the meaning of inclusion. Moreover, the volume of budget zone becomes: $\frac{V^n}{n!p_1^{'}\cdots p_k^{'}p_{k+1}\cdots p_n} > \frac{V^n}{n!p_1\cdots p_k p_{k+1}\cdots p_n}$;
- if $p_i^{'} > p_i$, $i=\overline{1,k}$ then: $\forall x=(x_1,...,x_n)\in ZB'$ implies: $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{k} p_i x_i + \sum_{i=k+1}^{n} p_i x_i < \sum_{i=1}^{k} p_i^{'} x_i + \sum_{i=k+1}^{n} p_i x_i \le V$ therefore $x\in ZB$ or, in other words: $ZB'\subset ZB$. Therefore, at an increasing of the prices of some goods, the budget zone decreases in the meaning of inclusion. Moreover, the volume of budget zone becomes: $\frac{V^n}{n!p_1^{'}\cdots p_k^{'}p_{k+1}\cdots p_n} < \frac{V^n}{n!p_1\cdots p_k p_{k+1}\cdots p_n}$.

In particular, for two goods B_1 and B_2, any good price change of B_1 (*for example*) leads to the right budget: $p_1^{'}x_1 + p_2 x_2 = V$. Its intersections with the axes are the points: $A\left(0, \frac{V}{p_2}\right)$ and $B\left(\frac{V}{p_1^{'}}, 0\right)$.

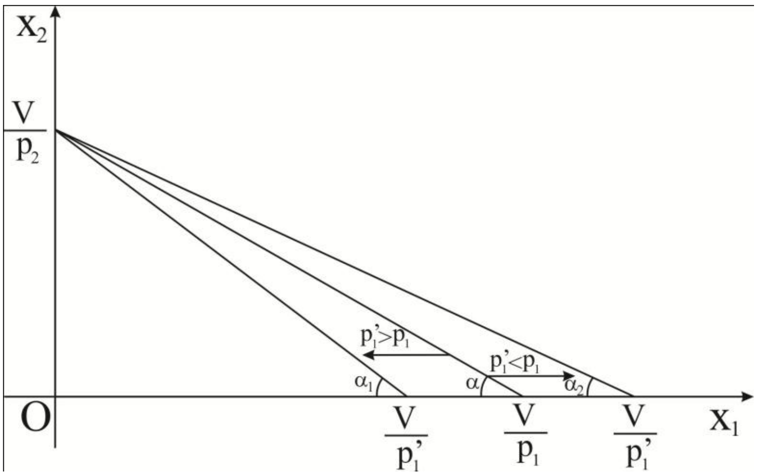

Figure 1

The displacement of the budget straight line to a price change

From figure 1, we easily see, that the downward to the origin (no slope, which is measured from the Ox_1 axis to the straight line in the trigonometric sense) of the straight line of the budget becomes bigger with increasing the good's B_1 price, i.e.: tg $\alpha_1 = \frac{p_1^{'}}{p_2} > \frac{p_1}{p_2}$ =tg α and becomes lower at a price reduction of B_1: tg $\alpha_2 = \frac{p_1^{'}}{p_2} < \frac{p_1}{p_2}$ =tg α.

Considering the two hyperplanes H: $\sum_{i=1}^{n} p_i x_i = V$ and H': $\sum_{i=1}^{k} p_i^{'} x_i + \sum_{i=k+1}^{n} p_i x_i = V$, the normals to these (orthogonal lines on them) have parameters:

$$N = (p_1, ..., p_k, p_{k+1}, ..., p_n),\ N' = (p_1^{'}, ..., p_k^{'}, p_{k+1}, ..., p_n).$$

The angle α of the two hyperplanes, defined as the angle of their normals is given by:

$$\cos\alpha = \frac{\sum_{i=1}^{k} p_i p_i^{'} + \sum_{i=k+1}^{n} p_i^2}{\sqrt{\sum_{i=1}^{n} p_i^2}\sqrt{\sum_{i=1}^{k} p_i^{'2} + \sum_{i=k+1}^{n} p_i^2}}$$

In particular, at a price changes with a multiplicative constant $\lambda > 0$, we have: $p_i^{'} = \lambda p_i$, $i = \overline{1,n}$ and the budget hyperplane: H': $\sum_{i=1}^{n} \lambda p_i x_i = V$ or $\sum_{i=1}^{n} p_i x_i = \frac{V}{\lambda}$. Both hyperplanes are therefore parallel, because $\cos\alpha = \frac{\sum_{i=1}^{n} \lambda p_i^2}{\sqrt{\sum_{i=1}^{n} p_i^2}\sqrt{\sum_{i=1}^{n} \lambda^2 p_i^2}} = \frac{\sum_{i=1}^{n} \lambda p_i^2}{\lambda \sum_{i=1}^{n} p_i^2} = 1$ or

α=0. If λ>1 then $p_i^{'} > p_i$, i=$\overline{1,n}$ and the budget zone is reduced in the meaning of inclusion and if λ<1 then $p_i^{'} < p_i$, i=$\overline{1,n}$ and the budget zone is increase in the meaning of inclusion.

If, in parallel with a prices change with a multiplicative constant λ>0 we have also a multiplication of the income with the same constant λ, then the budget hyperplane H': $\sum_{i=1}^{n}\lambda p_i x_i$ =λV becomes (*after simplification of* λ): H': $\sum_{i=1}^{n} p_i x_i$ =V that is precisely H. In this case, the budget hyperplane and its corresponding zone remain unchanged. Defining **the purchasing power** as the number of products or services that can be purchased with a currency unit, we see in this case that it remains constant at the same multiplication factor incomes and prices. This phenomenon is the so-called "the money illusion".

Another phenomenon that brings significant changes to the budget hyperplane is **the tax on consumption** of goods or services. These are sums of money paid by the consumer (*in the present supposed to be the only payer*) to the Government on the quantities of goods or services purchased.

The taxes on consumption of goods or services are essentially two: taxes and the VAT amount.

The Tax is the tax amount paid for each unit purchased, regardless of the good price.

Thus, if on $B_1,...,B_k$ we apply taxes $q_1,...,q_k$>0 then, if the consumer's income remains constant V, the budget hyperplane becomes H': $\sum_{i=1}^{k} p_i x_i + \sum_{i=1}^{k} q_i x_i + \sum_{i=k+1}^{n} p_i x_i$ =V. Noting $p_i^{'} = p_i + q_i$, i=$\overline{1,k}$ we get: H': $\sum_{i=1}^{k} p_i^{'} x_i + \sum_{i=k+1}^{n} p_i x_i$ =V. Therefore, in terms of consumer, the tax amount appears as increased price of the good. From the above, follows that the budget zone is reduced.

VAT is the tax paid for each unit of the good's price.

Thus, if to $B_1,...,B_k$ it applies VATs: $r_1,...,r_k$>0 then, if the consumer's income remains constant V, the budget hyperplane becomes: H': $\sum_{i=1}^{k} (1+r_i) p_i x_i + \sum_{i=k+1}^{n} p_i x_i$ =V. Noting $p_i^{'} = (1+r_i) p_i$, i=$\overline{1,k}$ we will obtain: H': $\sum_{i=1}^{k} p_i^{'} x_i + \sum_{i=k+1}^{n} p_i x_i$ =V. Again, from the consumer's point of view, the VAT appears like a higher price for the goods. The budget zone will decreases.

One question arises now: if from the consumer's point of view, the two taxes appear as increased prices, what differentiate their?

The volume of the budget zone after TAX is: $V_c = \frac{V^n}{n!(p_1+q_1)\dots(p_k+q_k)p_{k+1}\dots p_n}$

and after VAT: $V_v = \frac{V^n}{n!(1+r_1)\dots(1+r_k)p_1\dots p_k p_{k+1}\dots p_n}$. We have therefore $V_c<V_v$ if and only if:

$$(p_1+q_1)\dots(p_k+q_k) > (1+r_1)\dots(1+r_k)p_1\dots p_k$$

We have therefore:

- if $(p_1+q_1)\dots(p_k+q_k) > (1+r_1)\dots(1+r_k)p_1\dots p_k$ then the consumption tax is more disadvantageous for the consumer, diminishing the consumer's zone, VAT becoming preferential;
- if $(p_1+q_1)\dots(p_k+q_k) < (1+r_1)\dots(1+r_k)p_1\dots p_k$ then VAT becomes more disadvantageous for the consumer, the consumption tax becoming preferential;
- if $(p_1+q_1)\dots(p_k+q_k) = (1+r_1)\dots(1+r_k)p_1\dots p_k$ both taxes have the same effect.

In particular, for two goods, there are the following situations:

- if only the good B_1 (B_2 common analog) is subject to the two taxes, then $p_1+q_1>(1+r_1)p_1 \Leftrightarrow q_1>r_1p_1$ implies that the tax value is preferred; $p_1+q_1<(1+r_1)p_1 \Leftrightarrow q_1<r_1p_1$ implies that the consumption tax is preferable, and $q_1=r_1p_1$ – the indifference of the two taxes;
- if both goods are subject to additional taxation, then: $(p_1+q_1)(p_2+q_2)>(1+r_1)(1+r_2)p_1p_2 \Leftrightarrow p_1q_2+p_2q_1+q_1q_2>(r_1+r_2+r_1r_2)$ implies the preference for the value tax, the contrary inequality involving the consumption tax preference;
- if both goods are subject to additional taxation identical, then for $q_1=q_2=q$ and $r_1=r_2=r$ we have: $(p_1+q)(p_2+q)>(1+r)^2p_1p_2 \Leftrightarrow q^2+q(p_1+p_2)>(r^2+2r)p_1p_2$ and implies the preference for value tax or in the contrary for the consumption tax.

Another way to change the purchasing power comes from taxes on income. While tax is a compulsory payment to be made by citizens or businesses to the state, taxes are payments made to the state budget where citizens or businesses are the beneficiaries of certain services. Income taxes are of two types: taxes in absolute or relative value tax.

The absolute tax is a payment of a fixed amount of income. Thus, if V is the consumer's income, after a tax T it will remain with a disposable income of V-T u.m. The budget hyperplane is in this case: H: $\sum_{i=1}^{n} p_i x_i$ =V-T and the volume of the budget: $V_{iva} = \frac{(V-T)^n}{n!p_1\dots p_n}$.

Tax in relative value is a percentage of income payment. Thus, if V is the consumer's income, after tax rV, where $r\in(0,1)$ is the percentage of tax, he will remain with a disposable income of V-rV=(1-r)V u.m. The budget hyperplane is in this case: H:= $\sum_{i=1}^{n} p_i x_i$ =(1-r)V and the volume of the budget: $V_{ivr}=\frac{(1-r)^n V^n}{n!p_1 \ldots p_n}$.

In both cases, we see that $V_{iva}<V$ and $V_{ivr}<V$, so the purchasing power diminishes.

A naturally question arises: which of the forms of taxation is more advantageous for the consumer?

For $V_{iva}<V_{ivr}$ we must have: $(V-T)^n<(1-r)^nV^n$ therefore: T>rV. In this case we have that the tax in relative value is advantageous for the consumer, while the opposite T<rV lead to preference for a tax in absolute terms.

Another problem is that of comparison, in terms of purchasing power of taxes. Suppose then that the state has a choice between imposing a level of value of all goods purchased r_V and a level of tax in relative value r. In the first case, the volume of consumer budget becomes: $V_t=\frac{V^n}{n!(1+r_v)^n p_1 \ldots p_n}$ and in the second: $V_i=\frac{(1-r)^n V^n}{n!p_1 \ldots p_n}$. We have that $V_t<V_i$ if and only if: $\frac{V^n}{n!(1+r_v)^n p_1 \ldots p_n}<\frac{(1-r)^n V^n}{n!p_1 \ldots p_n}$ or: $(1+r_v)^n(1-r)^n>1$ that is: $(1+r_v)(1-r)>1$. This condition is equivalent with: $r<\frac{r_v}{1+r_v}$ or $r_v>\frac{r}{1-r}$.

Like a conclusion, we have that if $r<\frac{r_v}{1+r_v}$ the value relative tax benefit consumers, and if $r>\frac{r_v}{1+r_v}$ the tax comes to stimulating the consumption value.

In the final let make the observation that the problem of subsidies is the same of the taxes but with opposite signs.

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