The behavior of a polynomial regression under a variables transformation

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Abstract: The paper treats a problem of a great importance in the author's view relatively to the behavior of polynomial regressions to linear transformations of the variables. If, from a theoretical point of view, the estimation of a process using regressions give the same results, regardless of the size of the exogenous variable, from a practical standpoint the results are not quite so.

Keywords: regression, polynomial.

1 Introduction

Let consider a real dataset (x_i,y_i), $i=\overline{1,n}$ and a polynomial regression of order $m\geq 1$ which will approximate through the least squares method the given set.

Let therefore $f(x)=a_mx^m+...+a_1x+a_0$, $x\in\mathbf{R}$ with unknown coefficients a_i, $i=\overline{0,m}$ to be determined from the condition that:

(1) $\sum_{i=1}^{n}(a_m x_i^m+...+a_1x_i+a_0-y_i)^2$ =minimum

Let $F(a_m,...,a_0)=\sum_{i=1}^{n}(a_m x_i^m+...+a_1x_i+a_0-y_i)^2$.

First, let us note that:

(2) $\frac{\partial F}{\partial a_k}=2\sum_{i=1}^{n}x_i^k(a_m x_i^m+...+a_1x_i+a_0-y_i)$, $k=\overline{0,m}$

(3) $\frac{\partial^2 F}{\partial a_k\partial a_p}=2\sum_{i=1}^{n}x_i^{k+p}$, $k,p=\overline{0,m}$

Considering the quadratic form: $H_F=2\sum_{k,p=0}^{m}\sum_{i=1}^{n}x_i^{k+p}da_kda_p=2\sum_{i=1}^{n}\sum_{k,p=0}^{m}x_i^{k+p}da_kda_p=2\sum_{i=1}^{n}\left(\sum_{k=0}^{m}x_i^k da_k\right)^2$ we can see that $H_F>0$ so F is strictly convex. As a result of strict convexity, it follows that if F has a local minimum point, it is unique.

The local minimum necessary condition for F is: $\frac{\partial F}{\partial a_k}=0$, $k=\overline{0,m}$. We have then:

(4) $2\sum_{i=1}^{n}x_i^k(a_m x_i^m+...+a_1x_i+a_0-y_i)=0$, $k=\overline{0,m}$

from where:

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(5) $a_m\sum_{i=1}^{n}x_i^{m+k}+...+a_1\sum_{i=1}^{n}x_i^{1+k}+a_0\sum_{i=1}^{n}x_i^{k}=\sum_{i=1}^{n}y_ix_i^{k}$, k= $\overline{0,m}$

The resulting system has solutions (polynomial regression coefficients of order m) ([2]):

(6)
$$a_k=\frac{\begin{array}{c}\text{column } m-k+1\\ \begin{vmatrix}\sum_{i=1}^{n}x_i^{m} & \sum_{i=1}^{n}x_i^{m-1} & \cdots & \sum_{i=1}^{n}y_i & \cdots & n\\ \sum_{i=1}^{n}x_i^{m+1} & \sum_{i=1}^{n}x_i^{m} & \cdots & \sum_{i=1}^{n}x_iy_i & \cdots & \sum_{i=1}^{n}x_i\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sum_{i=1}^{n}x_i^{2m} & \sum_{i=1}^{n}x_i^{2m-1} & \cdots & \sum_{i=1}^{n}x_i^{m}y_i & \cdots & \sum_{i=1}^{n}x_i^{m}\end{vmatrix}\end{array}}{\begin{vmatrix}\sum_{i=1}^{n}x_i^{m} & \sum_{i=1}^{n}x_i^{m-1} & \cdots & \sum_{i=1}^{n}x_i^{k} & \cdots & n\\ \sum_{i=1}^{n}x_i^{m+1} & \sum_{i=1}^{n}x_i^{m} & \cdots & \sum_{i=1}^{n}x_i^{k+1} & \cdots & \sum_{i=1}^{n}x_i\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sum_{i=1}^{n}x_i^{2m} & \sum_{i=1}^{n}x_i^{2m-1} & \cdots & \sum_{i=1}^{n}x_i^{k+m} & \cdots & \sum_{i=1}^{n}x_i^{m}\end{vmatrix}},\ k=\overline{0,m}$$

2 The determination of regression coefficients at linear transformations

Now consider linear transformations of the exogenous and endogenous variables. Let therefore:

- $x_i\to\alpha x_i+\beta$, i= $\overline{1,n}$, $\alpha,\beta\in\mathbf{R}$, $\alpha\neq0$;
- $y_i\to\mu y_i+\nu$, i= $\overline{1,n}$, $\mu,\nu\in\mathbf{R}$, $\mu\neq0$

and the function:

(7) $G(b_m,...,b_0)=\sum_{i=1}^{n}(b_m(\alpha x_i+\beta)^m+...+b_1(\alpha x_i+\beta)+b_0-\mu y_i-\nu)^2$, $\alpha,\beta,\mu,\nu\in\mathbf{R}$, $\alpha,\mu\neq0$

We have:

(8) $G(b_m,...,b_0)=$

$$\sum_{i=1}^{n}\left(c_mx_i^m+...+c_1x_i+c_0-\nu-\mu y_i\right)^2=\mu^2\sum_{i=1}^{n}\left(\frac{c_m}{\mu}x_i^m+...+\frac{c_1}{\mu}x_i+\frac{c_0-\nu}{\mu}-y_i\right)^2$$

where:

(9) $c_k = \sum_{j=k}^{m} C_j^k \alpha^k \beta^{j-k} b_j$, $k=\overline{0,m}$

From the uniqueness of local minimum point, follows:

(10) $c_k = \mu a_k + \nu \delta_{k0}, k = \overline{0,m}$

where $\delta_{ij}=1$ if i=j and $\delta_{ij}=0$ if $i \neq j$ is the Kronecker's symbol.

From the above formulas follows:

(11) $\sum_{j=k}^{m} C_j^k \alpha^k \beta^{j-k} b_j = \mu a_k + \nu \delta_{k0}, k = \overline{0,m}$

Considering the polynomial function: $P(x) = \sum_{j=0}^{m} b_j x^j$ we have:

(12) $P^{(k)}(x) = k! \sum_{j=k}^{m} C_j^k b_j x^{j-k} = \frac{k!}{x^k} \sum_{j=k}^{m} C_j^k b_j x^j$

Therefore:

$$\mu a_k + \nu \delta_{k0} = \sum_{j=k}^{m} C_j^k \alpha^k \beta^{j-k} b_j = \left(\frac{\alpha}{\beta}\right)^k \sum_{j=k}^{m} C_j^k b_j \beta^j = \left(\frac{\alpha}{\beta}\right)^k \frac{\beta^k P^{(k)}(\beta)}{k!} = \alpha^k \frac{P^{(k)}(\beta)}{k!}$$

from where:

(13) $a_k = \frac{\alpha^k}{\mu} \frac{P^{(k)}(\beta)}{k!} - \frac{\nu}{\mu} \delta_{k0}, k = \overline{0,m}$

Similarly, considering the changes of variables:

- $x_i \to \frac{1}{\alpha} x_i - \frac{\beta}{\alpha}$, $i=\overline{1,n}$
- $y_i \to \frac{1}{\mu} y_i - \frac{\nu}{\mu}$, $i=\overline{1,n}$

for $Q(x) = \sum_{j=0}^{m} a_j x^j$ we obtain the formulas:

(14) $b_k = \frac{\mu}{\alpha^k} \frac{Q^{(k)}\left(-\frac{\beta}{\alpha}\right)}{k!} - \nu \delta_{k0}, k = \overline{0,m}$

For $x' \in \mathbf{R}$ the estimation from the regression becomes: $Q(x')$ and after the transformation of the form:

- $x_i \to \alpha x_i + \beta$, $i=\overline{1,n}$;
- $y_i \to \mu y_i + \nu$, $i=\overline{1,n}$

we obtain:

$$P(\alpha x'+\beta)=\sum_{k=0}^{m} b_k(\alpha x'+\beta)^k=\sum_{k=0}^{m}\left(\frac{\mu}{\alpha^k}\frac{Q^{(k)}\left(-\frac{\beta}{\alpha}\right)}{k!}-\nu\delta_{k0}\right)(\alpha x'+\beta)^k=$$

$$\sum_{k=0}^{m}\alpha^k\left(\frac{\mu}{\alpha^k}\frac{Q^{(k)}\left(-\frac{\beta}{\alpha}\right)}{k!}-\nu\delta_{k0}\right)\left(x'+\frac{\beta}{\alpha}\right)^k=\sum_{k=0}^{m}\left(\frac{\mu Q^{(k)}\left(-\frac{\beta}{\alpha}\right)}{k!}-\nu\alpha^k\delta_{k0}\right)\left(x'+\frac{\beta}{\alpha}\right)^k=$$

$$\mu\sum_{k=1}^{m}\frac{Q^{(k)}\left(-\frac{\beta}{\alpha}\right)}{k!}\left(x'+\frac{\beta}{\alpha}\right)^k+\left(\mu Q\left(-\frac{\beta}{\alpha}\right)-\nu\right)=$$

$$\mu\sum_{k=0}^{m}\frac{Q^{(k)}\left(-\frac{\beta}{\alpha}\right)}{k!}\left(x'+\frac{\beta}{\alpha}\right)^k-\mu Q\left(-\frac{\beta}{\alpha}\right)+\left(\mu Q\left(-\frac{\beta}{\alpha}\right)-\nu\right)=$$

$$\mu Q(x')-\nu$$

We have therefore from the Taylor series expansion of Q in the point: $\left(-\frac{\beta}{\alpha}\right)$:

(15) $$P(\alpha x'+\beta)=\mu Q(x')-\nu$$

Therefore, at a linear transformation of the variable exogenous and at endogenous variables invariance (μ=1, ν=0), the expected value remains the same.
At a linear transformation of the endogenous variable, the estimated value will change multiplicative with the appropriate factor and additive with the opposite of the translation of endogenous data.

3 The determination of polynomial regression coefficients based on linear transformations of variables

Let us note first:

(16) $$\Delta(x_i)=\begin{vmatrix}\sum_{i=1}^{n}x_i^m & \sum_{i=1}^{n}x_i^{m-1} & \cdots & \sum_{i=1}^{n}x_i^k & \cdots & n\\ \sum_{i=1}^{n}x_i^{m+1} & \sum_{i=1}^{n}x_i^m & \cdots & \sum_{i=1}^{n}x_i^{k+1} & \cdots & \sum_{i=1}^{n}x_i\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sum_{i=1}^{n}x_i^{2m} & \sum_{i=1}^{n}x_i^{2m-1} & \cdots & \sum_{i=1}^{n}x_i^{k+m} & \cdots & \sum_{i=1}^{n}x_i^m\end{vmatrix}$$

$$\begin{array}{c} \text{column } m-k+1 \\ \end{array}$$

(17) $$\Delta_k(x_i,y_i)=\begin{vmatrix} \sum_{i=1}^{n}x_i^m & \sum_{i=1}^{n}x_i^{m-1} & \cdots & \sum_{i=1}^{n}y_i & \cdots & n \\ \sum_{i=1}^{n}x_i^{m+1} & \sum_{i=1}^{n}x_i^{m} & \cdots & \sum_{i=1}^{n}x_i y_i & \cdots & \sum_{i=1}^{n}x_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}x_i^{2m} & \sum_{i=1}^{n}x_i^{2m-1} & \cdots & \sum_{i=1}^{n}x_i^m y_i & \cdots & \sum_{i=1}^{n}x_i^m \end{vmatrix},\ k=\overline{0,m}$$

So we have:

(18) $$a_k=\frac{\Delta_k(x_i,y_i)}{\Delta(x_i)}$$

Let us note also:

(19) $$t_k=\sum_{i=1}^{n}x_i^k\ ,\ s_k=\sum_{i=1}^{n}x_i^k y_i$$

After the formula of determinants development, we get:

$$\Delta(\alpha x_i+\beta)=\begin{vmatrix} \sum_{i=1}^{n}(\alpha x_i+\beta)^m & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m-1} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^k & \cdots & n \\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{m+1} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^{k+1} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m-1} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^{k+m} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^m \end{vmatrix}=$$

$$\begin{vmatrix} \sum_{i=1}^{n}\sum_{j=0}^{m}C_m^j\alpha^{m-j}x_i^{m-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{m-1}C_{m-1}^j\alpha^{m-1-j}x_i^{m-1-j}\beta^j & \ldots & \sum_{i=1}^{n}\sum_{j=0}^{0}C_0^j\alpha^{0-j}x_i^{0-j}\beta^j \\ \sum_{i=1}^{n}\sum_{j=0}^{m+1}C_{m+1}^j\alpha^{m+1-j}x_i^{m+1-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{m}C_m^j\alpha^{m-j}x_i^{m-j}\beta^j & \ldots & \sum_{i=1}^{n}\sum_{j=0}^{1}C_1^j\alpha^{1-j}x_i^{1-j}\beta^j \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{i=1}^{n}\sum_{j=0}^{2m}C_{2m}^j\alpha^{2m-j}x_i^{2m-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m-1}C_{2m-1}^j\alpha^{2m-1-j}x_i^{2m-1-j}\beta^j & \ldots & \sum_{i=1}^{n}\sum_{j=0}^{m}C_m^j\alpha^{m-j}x_i^{m-j}\beta^j \end{vmatrix}=$$

$$\begin{vmatrix} \sum_{i=1}^{n}\sum_{j=0}^{2m}C_m^j\alpha^{m-j}x_i^{m-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m}C_{m-1}^j\alpha^{m-1-j}x_i^{m-1-j}\beta^j & \ldots & \sum_{i=1}^{n}\sum_{j=0}^{2m}C_0^j\alpha^{0-j}x_i^{0-j}\beta^j \\ \sum_{i=1}^{n}\sum_{j=0}^{2m}C_{m+1}^j\alpha^{m+1-j}x_i^{m+1-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m}C_m^j\alpha^{m-j}x_i^{m-j}\beta^j & \ldots & \sum_{i=1}^{n}\sum_{j=0}^{2m}C_1^j\alpha^{1-j}x_i^{1-j}\beta^j \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{i=1}^{n}\sum_{j=0}^{2m}C_{2m}^j\alpha^{2m-j}x_i^{2m-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m}C_{2m-1}^j\alpha^{2m-1-j}x_i^{2m-1-j}\beta^j & \ldots & \sum_{i=1}^{n}\sum_{j=0}^{2m}C_m^j\alpha^{m-j}x_i^{m-j}\beta^j \end{vmatrix}=$$

$$\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma)\left(\sum_{i_1=1}^{n}\sum_{j_1=0}^{2m}C_{m-1+\sigma(1)}^{j_1}\alpha^{m-1+\sigma(1)-j_1}x_{i_1}^{m-1+\sigma(1)-j_1}\beta^{j_1}\right)\left(\sum_{i_2=1}^{n}\sum_{j_2=0}^{2m}C_{m-2+\sigma(2)}^{j_2}\alpha^{m-2+\sigma(2)-j_2}x_{i_2}^{m-2+\sigma(2)-j_2}\beta^{j_2}\right)\cdots$$
$$\left(\sum_{i_{m+1}=1}^{n}\sum_{j_{m+1}=0}^{2m}C_{-1+\sigma(m+1)}^{j_{m+1}}\alpha^{-1+\sigma(m+1)-j_{m+1}}x_{i_{m+1}}^{-1+\sigma(m+1)-j_{m+1}}\beta^{j_{m+1}}\right)=$$

$$\alpha^{m(m+1)}\sum_{i_1,\dots,i_{m+1}=1}^{n}\sum_{j_1,\dots,j_{m+1}=0}^{2m}\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma)C_{m-1+\sigma(1)}^{j_1}C_{m-2+\sigma(2)}^{j_2}\cdots C_{-1+\sigma(m+1)}^{j_{m+1}}\left(\frac{\beta}{\alpha}\right)^{\sum_{k=1}^{m+1}j_k}x_{i_1}^{m-1+\sigma(1)-j_1}x_{i_2}^{m-2+\sigma(2)-j_2}\cdots x_{i_{m+1}}^{-1+\sigma(m+1)-j_{m+1}}$$

where we noted $C_a^b=\dfrac{a!}{b!(a-b)!}$ with the convention that $C_a^b=0$ if b>a and $C_0^0=1$.

Let also mention that $\varepsilon(\sigma)$ is the signature of the permutation $\sigma\in S_{m+1}$ – the group of permutations of order m+1 and we have: $\sum_{k=1}^{m+1}\sigma(k)=\dfrac{(m+1)(m+2)}{2}$.

With the above notations, we have:

$$\Delta(\alpha x_i+\beta)=\alpha^{m(m+1)}\sum_{j_1,\dots,j_{m+1}=0}^{2m}\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma)C_{m-1+\sigma(1)}^{j_1}C_{m-2+\sigma(2)}^{j_2}\cdots C_{-1+\sigma(m+1)}^{j_{m+1}}\left(\frac{\beta}{\alpha}\right)^{\sum_{k=1}^{m+1}j_k}t_{m-1+\sigma(1)-j_1}t_{m-2+\sigma(2)-j_2}\cdots t_{-1+\sigma(m+1)-j_{m+1}}$$

In particular ($\alpha=1$, $\beta=0$):

(20) $$\Delta(x_i)=\sum_{i_1,\dots,i_{m+1}=1}^{n}\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma)x_{i_1}^{m-1+\sigma(1)}x_{i_2}^{m-2+\sigma(2)}\cdots x_{i_{m+1}}^{-1+\sigma(m+1)}=$$
$$\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma)t_{m-1+\sigma(1)}t_{m-2+\sigma(2)}\cdots t_{-1+\sigma(m+1)}$$

Analogously:

$$\begin{array}{cc} & \qquad\qquad\qquad\qquad\qquad\qquad \text{column } m-k+1 \\ \Delta_k(\alpha x_i+\beta,\mu y_i+\nu)= & \begin{vmatrix}\sum_{i=1}^{n}(\alpha x_i+\beta)^m & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m-1} & \cdots & \sum_{i=1}^{n}(\mu y_i+\nu) & \cdots & n\\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{m+1} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)(\mu y_i+\nu) & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m-1} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^m(\mu y_i+\nu) & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^m\end{vmatrix}=\end{array}$$

$$\begin{vmatrix}\sum_{i=1}^{n}(\alpha x_i+\beta)^m & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m-1} & \cdots & \sum_{i=1}^{n}\mu y_i & \cdots & n\\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{m+1} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)\mu y_i & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m-1} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^m\mu y_i & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^m\end{vmatrix}+$$

$$\begin{vmatrix} \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m-1} & \cdots & \nu n & \cdots & n \\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{m+1} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \cdots & \nu\sum_{i=1}^{n}(\alpha x_i+\beta) & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m-1} & \cdots & \nu\sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} \end{vmatrix} =$$

$$\mu\begin{vmatrix} \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m-1} & \cdots & \sum_{i=1}^{n} y_i & \cdots & n \\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{m+1} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)y_i & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m} & \sum_{i=1}^{n}(\alpha x_i+\beta)^{2m-1} & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} y_i & \cdots & \sum_{i=1}^{n}(\alpha x_i+\beta)^{m} \end{vmatrix} =$$

$$\mu\begin{vmatrix} \sum_{i=1}^{n}\sum_{j=0}^{2m} C_m^j \alpha^{m-j} x_i^{m-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_{m-1}^j \alpha^{m-1-j} x_i^{m-1-j}\beta^j & \cdots & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_0^j \alpha^{0-j} x_i^{0-j} y_i\beta^j & \cdots & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_0^j \alpha^{0-j} x_i^{0-j}\beta^j \\ \sum_{i=1}^{n}\sum_{j=0}^{2m} C_{m+1}^j \alpha^{m+1-j} x_i^{m+1-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_m^j \alpha^{m-j} x_i^{m-j}\beta^j & \cdots & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_1^j \alpha^{1-j} x_i^{1-j} y_i\beta^j & \cdots & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_1^j \alpha^{1-j} x_i^{1-j}\beta^j \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}\sum_{j=0}^{2m} C_{2m}^j \alpha^{2m-j} x_i^{2m-j}\beta^j & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_{2m-1}^j \alpha^{2m-1-j} x_i^{2m-1-j}\beta^j & \cdots & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_m^j \alpha^{m-j} x_i^{m-j} y_i\beta^j & \cdots & \sum_{i=1}^{n}\sum_{j=0}^{2m} C_m^j \alpha^{m-j} x_i^{m-j}\beta^j \end{vmatrix} =$$

$$\mu\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma)\left(\sum_{i_1=1}^{n}\sum_{j_1=0}^{2m} C_{m-1+\sigma(1)}^{j_1}\alpha^{m-1+\sigma(1)-j_1} x_{i_1}^{m-1+\sigma(1)-j_1}\beta^{j_1}\right)\left(\sum_{i_2=1}^{n}\sum_{j_2=0}^{2m} C_{m-2+\sigma(2)}^{j_2}\alpha^{m-2+\sigma(2)-j_2} x_{i_2}^{m-2+\sigma(2)-j_2}\beta^{j_2}\right)\ldots$$
$$\left(\sum_{i_{m-k+1}=1}^{n}\sum_{j_{m-k+1}=0}^{2m} C_{-1+\sigma(m-k+1)}^{j_{m-k+1}}\alpha^{-1+\sigma(m-k+1)-j_{m-k+1}} x_{i_{m-k+1}}^{-1+\sigma(m-k+1)-j_{m-k+1}} y_{i_{m-k+1}}\beta^{j_{m-k+1}}\right)\ldots$$
$$\left(\sum_{i_{m+1}=1}^{n}\sum_{j_{m+1}=0}^{2m} C_{-1+\sigma(m+1)}^{j_{m+1}}\alpha^{-1+\sigma(m+1)-j_{m+1}} x_{i_{m+1}}^{-1+\sigma(m+1)-j_{m+1}}\beta^{j_{m+1}}\right) =$$

$$\mu\alpha^{m(m+1)-k}\sum_{j_1,\ldots,j_{m+1}=0}^{2m}\sum_{\sigma\in S_{m+1}}\varepsilon(\sigma) C_{-1+\sigma(m-k+1)}^{j_{m-k+1}}\prod_{\substack{t=1\\ t\neq m-k+1}}^{m+1} C_{m-t+\sigma(t)}^{j_t}\left(\frac{\beta}{\alpha}\right)^{\sum_{k=1}^{m+1} j_k} s_{-1+\sigma(m-k+1)-j_{m-k+1}}\prod_{\substack{t=1\\ t\neq m-k+1}}^{m+1} t_{m-t+\sigma(t)-j_t}$$

In particular $(\alpha=1, \beta=0, \mu=1, \nu=0)$:

(21) $$\Delta_k(x_i, y_i) = \sum_{\sigma\in S_{m+1}}\varepsilon(\sigma) t_{m-1+\sigma(1)} t_{m-2+\sigma(2)}\ldots s_{-1+\sigma(m-k+1)}\ldots t_{-1+\sigma(m+1)}$$

4 Error theory notions

Considering a real value x and an estimate of its $\overline{x}$ the absolute error is defined as: $e_x = x - \overline{x}$ and the relative error: $\varepsilon_x = \dfrac{e_x}{\overline{x}}$.

Let two values $x,y \in \mathbf{R}$. We have then ([1]):

- $e_{x+y} = e_x + e_y$
- $e_{x-y} = e_x - e_y$
- $e_{xy} = \overline{x}e_y + \overline{y}e_x$
- $e_{\frac{x}{y}} = \dfrac{1}{\overline{y}}e_x - \dfrac{\overline{x}}{\overline{y}^2}e_y$
- $\varepsilon_{x+y} = \dfrac{\overline{x}}{\overline{x}+\overline{y}}\varepsilon_x + \dfrac{\overline{y}}{\overline{x}+\overline{y}}\varepsilon_y$
- $\varepsilon_{x-y} = \dfrac{\overline{x}}{\overline{x}-\overline{y}}\varepsilon_x - \dfrac{\overline{y}}{\overline{x}-\overline{y}}\varepsilon_y$
- $\varepsilon_{xy} = \varepsilon_x + \varepsilon_y$
- $\varepsilon_{\frac{x}{y}} = \varepsilon_x - \varepsilon_y$

From the above formulas, it follows inductively:

- $e_{\sum_{i=1}^{n} x_i} = \sum_{i=1}^{n} e_{x_i} \quad \forall x_i \in \mathbf{R},\ i=\overline{1,n}$
- $e_{nx} = ne_x \quad \forall x \in \mathbf{R}$

Also for any $x_i \in \mathbf{R}$, $i=\overline{1,n}$, we have:

$$e_{\prod_{i=1}^{n} x_i} = \prod_{i=1}^{n} x_i - \prod_{i=1}^{n} \overline{x}_i = \prod_{i=1}^{n}\left(\overline{x}_i + e_{x_i}\right) - \prod_{i=1}^{n} \overline{x}_i = \sum_{\substack{i\in I \\ j\in J \neq \varnothing \\ I\cup J=\{1,\dots,n\} \\ I\cap J=\varnothing}} \prod_{i\in I} \overline{x}_i \prod_{j\in J} e_{x_j}$$

Neglecting the products of errors: $e_{\prod_{i=1}^{n} x_i} = \sum_{i=1}^{n} e_{x_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \overline{x}_j = \sum_{i=1}^{n} e_{x_i} \dfrac{\prod_{j=1}^{n} \overline{x}_j}{\overline{x}_i}$.

In particular, for $x_i = x \quad \forall i=\overline{1,n}$ we have: $e_{x^n} = \prod_{i=1}^{n}\left(\overline{x} + e_x\right) - \overline{x}^n = \sum_{k=1}^{n} C_n^k \overline{x}^{n-k} e_x^k$.

Neglecting the terms containing powers of the error, we get:

- $e_{x^n} = ne_x \overline{x}^{n-1}$

- $e_{t_p} = e_{\sum_{i=1}^{n} x_i^p} = \sum_{i=1}^{n} e_{x_i^p} = p\sum_{i=1}^{n} e_{x_i} \overline{x}_i^{p-1}$

For relative errors:

- $\varepsilon_{\sum_{i=1}^{n} x_i} = \frac{e_{\sum_{i=1}^{n} x_i}}{\sum_{i=1}^{n} \overline{x}_i} = \frac{\sum_{i=1}^{n} e_{x_i}}{\sum_{i=1}^{n} \overline{x}_i} = \sum_{i=1}^{n} \frac{\overline{x}_i}{\sum_{i=1}^{n} \overline{x}_i} \frac{e_{x_i}}{\overline{x}_i} = \sum_{i=1}^{n} \frac{\overline{x}_i}{\sum_{i=1}^{n} \overline{x}_i} \varepsilon_{x_i}$

from where:

- $\varepsilon_{nx} = \varepsilon_x$

- $\varepsilon_{\prod_{i=1}^{n} x_i} = \frac{e_{\prod_{i=1}^{n} x_i}}{\prod_{i=1}^{n} \overline{x}_i} = \frac{\sum_{i=1}^{n} e_{x_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} \overline{x}_j}{\prod_{i=1}^{n} \overline{x}_i} = \sum_{i=1}^{n} \frac{e_{x_i}}{\overline{x}_i} = \sum_{i=1}^{n} \varepsilon_{x_i}$

therefore:

- $\varepsilon_{x^n} = n\varepsilon_x$

Considering that the maximum error of the data from measurement or from computer data representation is e_{max} and that the sign of the error can not be ascertained, the above formulas become:

- $e_{x+y} \leq 2e_{max}$;

- $e_{x-y} \leq 2e_{max}$;

- $e_{xy} \leq \left(\left|\overline{x}\right| + \left|\overline{y}\right|\right) e_{max}$

- $e_{\frac{x}{y}} \leq \left(\left| \frac{1}{\overline{y}} \right| + \left| \frac{\overline{x}}{\overline{y}^2} \right| \right) e_{max}$

- $e_{\sum_{i=1}^{n} x_i} \leq ne_{max}$ $\forall x_i \in \mathbf{R}$, $i = \overline{1, n}$

- $e_{\prod_{i=1}^{n} x_i} \leq \sum_{i=1}^{n} \left| \frac{\prod_{j=1}^{n} \overline{x}_j}{\overline{x}_i} \right| e_{max}$

- $e_{x^n} \le n\left|\overline{x}\right|^{n-1} e_{max}$
- $e_{t_p} \le p\left|\overline{t}_{p-1}\right| e_{max}$

5 The determination of the maximum absolute error of polynomial regression determinants

We saw in section 3 that:

$$\Delta(x_i) = \sum_{\sigma \in S_{m+1}} \varepsilon(\sigma) t_{m-1+\sigma(1)} t_{m-2+\sigma(2)} \ldots t_{-1+\sigma(m+1)}$$

$$\Delta_k(x_i, y_i) = \sum_{\sigma \in S_{m+1}} \varepsilon(\sigma) t_{m-1+\sigma(1)} t_{m-2+\sigma(2)} \ldots s_{-1+\sigma(m-k+1)} \ldots t_{-1+\sigma(m+1)}$$

Suppose now that the absolute value of exogenous data belongs in the interval [a,b] and the endogenous data are in the interval [c,d]. We therefore have:

$$\left|\overline{t}_{p-1}\right| = \left|\sum_{i=1}^{n} \overline{x}_i^{p-1}\right| \le \sum_{i=1}^{n} \left|\overline{x}_i^{p-1}\right| = \sum_{i=1}^{n} \left|\overline{x}_i\right|^{p-1} \le nb^{p-1} \tag{22}$$

Therefore: $e_{t_p} \le npb^{p-1} e_{max}$ and as $\overline{s}_p = \sum_{i=1}^{n} x_i^p y_i \le nb^p d$ follows:

$$e_{s_p} =$$

$$e_{\sum_{i=1}^{n} x_i^p y_i} = \sum_{i=1}^{n} e_{x_i^p y_i} = \sum_{i=1}^{n} \left(\overline{x}_i^p e_{y_i} + \overline{y}_i e_{x_i^p} \right) \le \sum_{i=1}^{n} \left(\overline{x}_i^p e_y + \overline{y}_i p e_x \overline{x}_i^{p-1} \right) \le \sum_{i=1}^{n} \left(b^p + dpb^{p-1} \right) e_{max} =$$

$$nb^{p-1}(b + dp) e_{max}.$$

We now have:

$$e_{\Delta(x_i)} \le \sum_{\sigma \in S_{m+1}} e_{\prod_{q=1}^{m+1} t_{m-q+\sigma(q)}} = \sum_{\sigma \in S_{m+1}} \sum_{q=1}^{m+1} e_{t_{m-q+\sigma(q)}} \prod_{\substack{j=1 \\ j \ne q}}^{m+1} \overline{t}_{m-j+\sigma(j)} \le$$

$$\sum_{\sigma \in S_{m+1}} \sum_{q=1}^{m+1} n(m - q + \sigma(q)) b^{m-q+\sigma(q)-1} e_{max} \prod_{\substack{j=1 \\ j \ne q}}^{m+1} nb^{m-j+\sigma(j)} =$$

$$\sum_{\sigma \in S_{m+1}} \sum_{q=1}^{m+1} n(m - q + \sigma(q)) b^{m-q+\sigma(q)-1} e_{max} n^m b^{m^2 - \frac{(m+1)(m+2)}{2} + q + \frac{(m+1)(m+2)}{2} - \sigma(q)} =$$

$$n^{m+1} e_{max} \sum_{\sigma \in S_{m+1}} \sum_{q=1}^{m+1} (m - q + \sigma(q)) b^{m-q+\sigma(q)-1+m^2+q-\sigma(q)} =$$

$$n^{m+1} e_{max} \sum_{\sigma \in S_{m+1}} \sum_{q=1}^{m+1} (m - q + \sigma(q)) b^{m^2+m-1} =$$

$$n^{m+1} b^{m^2+m-1} m(m+1)(m+1)! e_{max}.$$

$$
\begin{aligned}
& e_{\Delta_k(x_i)} \le \sum_{\sigma\in S_{m+1}} e_{s_{-1+\sigma(m-k+1)} \prod\limits_{\substack{q=1\\ q\ne m-k+1}}^{m+1} t_{m-q+\sigma(q)}} = \\
& \sum_{\sigma\in S_{m+1}} \left(\bar{s}_{-1+\sigma(m-k+1)} e_{\prod\limits_{\substack{q=1\\ q\ne m-k+1}}^{m+1} t_{m-q+\sigma(q)}} + \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} \bar{t}_{m-q+\sigma(q)} e_{s_{-1+\sigma(m-k+1)}} \right) = \\
& \sum_{\sigma\in S_{m+1}} \left(\bar{s}_{-1+\sigma(m-k+1)} \sum_{\substack{q=1\\ q\ne m-k+1}}^{m+1} e_{t_{m-q+\sigma(q)}} \prod_{\substack{j=1\\ j\ne q}}^{m+1} \bar{t}_{m-j+\sigma(j)} + \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} \bar{t}_{m-q+\sigma(q)} e_{s_{-1+\sigma(m-k+1)}} \right) \le \\
& \sum_{\sigma\in S_{m+1}} \left(nb^{-1+\sigma(m-k+1)} d \sum_{\substack{q=1\\ q\ne m-k+1}}^{m+1} n(m-q+\sigma(q)) b^{m-q+\sigma(q)-1} \prod_{\substack{j=1\\ j\ne q}}^{m+1} nb^{m-j+\sigma(j)} e_{max} + \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} nb^{m-q+\sigma(q)} nb^{\sigma(m-k+1)-2} (b-d+d\sigma(m-k+1)) e_{max} \right) = \\
& \sum_{\sigma\in S_{m+1}} \left(n^{m+2} b^{-1+\sigma(m-k+1)} d \sum_{\substack{q=1\\ q\ne m-k+1}}^{m+1} (m-q+\sigma(q)) b^{m-q+\sigma(q)-1} b^{m^2+q-\sigma(q)} + n^{m+1} \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} b^{m-q+\sigma(q)+\sigma(m-k+1)-2} (b-d+d\sigma(m-k+1)) \right) e_{max} = \\
& n^{m+1} \sum_{\sigma\in S_{m+1}} \left(nb^{-1+\sigma(m-k+1)} d \sum_{\substack{q=1\\ q\ne m-k+1}}^{m+1} (m-q+\sigma(q)) b^{m^2+m-1} + \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} b^{m-q+\sigma(q)+\sigma(m-k+1)-2} (b-d+d\sigma(m-k+1)) \right) e_{max} = \\
& n^{m+1} \sum_{\sigma\in S_{m+1}} \left(nb^{\sigma(m-k+1)+m^2+m-2} d\left(m^2+(m-k+1)-\sigma(m-k+1)\right) + \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} b^{m-q+\sigma(q)+\sigma(m-k+1)-2} (b-d+d\sigma(m-k+1)) \right) e_{max} \le \\
& n^{m+1} \sum_{\sigma\in S_{m+1}} \left(nb^{\sigma(m-k+1)+m^2+m-2} d\left(m^2+(m-k+1)-\sigma(m-k+1)\right) + (b+dm)^m \prod_{\substack{q=1\\ q\ne m-k+1}}^{m+1} b^{m-q+\sigma(q)+\sigma(m-k+1)-2} \right) e_{max} = \\
& n^{m+1} \sum_{\sigma\in S_{m+1}} \left(nb^{\sigma(m-k+1)+m^2+m-2} d\left(m^2+(m-k+1)-\sigma(m-k+1)\right) + (b+dm)^m b^{m^2+m-k+1+(m-1)\sigma(m-k+1)-2m} \right) e_{max} = \\
& n^{m+1} \sum_{\sigma\in S_{m+1}} \left(nb^{\sigma(m-k+1)+m^2+m-2} d\left(m^2+(m-k+1)-\sigma(m-k+1)\right) + (b+dm)^m b^{m^2-m-k+1+(m-1)\sigma(m-k+1)} \right) e_{max} \le \\
& n^{m+1} \left(dn\left(m^2+m-k+1\right) b^{m^2+m-2} \sum_{\sigma\in S_{m+1}} b^{\sigma(m-k+1)} + (b+dm)^m b^{m^2-m-k+1} \sum_{\sigma\in S_{m+1}} b^{(m-1)\sigma(m-k+1)} \right) e_{max} = \\
& n^{m+1} \left(dn\left(m^2+m-k+1\right) b^{m^2+m-2+m+1} (m+1)! + (b+dm)^m b^{m^2-m-k+1} b^{(m-1)(m+1)} (m+1)! \right) e_{max} = \\
& n^{m+1} (m+1)! \left(dn\left(m^2+m-k+1\right) b^{m^2+2m-1} + (b+dm)^m b^{2m^2-m-k} \right) e_{max}
\end{aligned}
$$

It is observed from the above formulas that, with the increase of the maximum absolute values of the variables, the absolute error increases dramatically.

Thus, at a corresponding reduction of both exogenous variables and of the endogenous (decrease of b, or d) will lead to smaller maximum absolute errors. On the other hand, it should be kept in mind that if through a linear transformation of the form: $x_i \to \alpha x_i + \beta$, $i=\overline{1,n}$, $y_i \to \mu y_i + \nu$, $i=\overline{1,n}$, the values becomes in absolute value less than 1, then in determining of the polynomials t_p and s_p respectively, the truncation errors of computers will also lead to large errors.

6 A numerical example

Let consider the polynomial:
$P(x)=0{,}0000001x^8-0{,}00001x^7+0{,}00001x^6-0{,}00005x^5+0{,}00001x^4-0{,}000012x^3+0{,}00001x^2-0{,}000034x+0{,}00015$
and the data set: $(x_i, y_i)_{i=\overline{1,10}}$ where $x_i=99+i$, $y_i=P(x_i)$. The obtained data are as follows:

x	y
100	9.500.988
101	20.812.078
102	33.684.367
103	48.258.214
104	64.683.291
105	83.119.038
106	103.735.142
107	126.712.027
108	152.241.366
109	180.526.606

Because the law is of polynomial degree 8, the "forecast" for x=110 will be $y_{11}=P(x_{11})=P(110)=211.783.513$.
We now propose the determination of the 8th-order polynomial regression relative to the above data. Using the Add-Ins Data Analysis from Microsoft Excel, you can easily get:

Intercept	-1819,741535
X Variable 1 (X)	0
X Variable 2 (X^2)	0
X Variable 3 (X^3)	0
X Variable 4 (X^4)	0
X Variable 5 (X^5)	-4,13418E-05
X Variable 6 (X^6)	9,79622E-06
X Variable 7 (X^7)	-9,99834E-06
X Variable 8 (X^8)	9,99954E-08

The polynomial regression will be:
$Q(x)=9{,}99954\cdot10^{-8}x^8-9{,}99834\cdot10^{-6}x^7+9{,}79622\cdot10^{-6}x^6-4{,}13418\cdot10^{-5}x^5-1819{,}741535$

For x=110, the expected value is: Q(110)=211783512.8 with an absolute error of 0.2, the relative error being almost zero.
Transforming data by $x \rightarrow x-99$ we obtain the following set of values:

x	y
1	9.500.988
2	20.812.078
3	33.684.367
4	48.258.214
5	64.683.291
6	83.119.038
7	103.735.142
8	126.712.027
9	152.241.366
10	180.526.606

The regression analysis provides the following coefficients:

Intercept	-380382,6
X Variable 1 (X)	9208199,098
X Variable 2 (X^2)	653020,942
X Variable 3 (X^3)	19809,85892
X Variable 4 (X^4)	337,7886237
X Variable 5 (X^5)	2,844754844
X Variable 6 (X^6)	0,068831705
X Variable 7 (X^7)	-0,002310925
X Variable 8 (X^8)	4,96032E-05

The poolynomial regression will be:
$P(x)=4,96032\cdot10^{-5}x^8-0,002310925x^7+0,068831705x^6+2,844754844x^5+337,7886237x^4+19809,85892x^3+653020,942x^2+9208199,098x-380382,6$
For x=110-99=11, the forecast will be: P(11)=211783516.6 with an absolute error of 3.6 and again the relative error is almost zero.
But if the regression coefficients are determined using determinants, because enormous errors caused by large exponents, aberrant predicted values are obtained, in the case of original data the absolute error beaing greater than 10^{79} and in the case of the modified date than 10^{108}.
The determination of the regression polynomial based on the above formulas (13,14) shows some interesting aspects. Thus, based on the polynomial P, we have:

$$Q(x)=\sum_{j=0}^{8} a_j x^j \text{ cu } a_k=\frac{P^{(k)}(-99)}{k!}, k=\overline{0,8} \text{ from where:}$$

$Q(x)=0,00005x^8-0,041596644x^7+15,28280417x^6-3208,953433x^5+421069,3262x^4-35356807,74x^3+1855331468x^2-55626362954x+7,2957\cdot10^{11}$
the expected value being Q(110)=211783516.6 with an absolute error of 3.6, again the relative error being almost zero.

Considering now the polynomial Q, we have: $P(x)=\sum_{j=0}^{8} b_j x^j$ cu $b_k=\frac{Q^{(k)}(99)}{k!}, k=\overline{0,8}$ from where:

$P(x)=0,0000001x^8+6,9198\cdot10^{-5}x^7+0,020522478x^6+3,381332737x^5+334,2576304x^4+19823,76145x^3+652989,5629x^2+9208235,249x+376758,801$

the expected value being P(11)=212540669.86 with an absolute error of 757156.862, but the relative error being now significant: 0.36%.

7 References

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