Generalized Cobb-Douglas function for three inputs and linear elasticity

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Abstract. The article deals with a production function of three factors with constant scale return where each elasticity of two of the factors is a function of first degree. After the examination of parameters conditions according to the axioms of the production functions, there are computed the main indicators. Also, the combination of factors is determined in order to maximize the total output under a given cost.

Keywords: production function, Cobb-Douglas

1 General aspects of the production functions

In any economic activity, obtaining a result of it implies, by default, there is a certain number of resources, supposedly indivisible needed for the proper functioning of the production process.

We therefore define on $\mathbf{R}^3$ – the **production space** for three resources: K – capital, L - labor and T – land or natural resources as $SP=\{(K,L,T) \mid K,L,T\ge 0\}$ where $x\in SP$, $x=(K,L,T)$ is an **ordered set of resources**.

Because in a production process not any amount of resources are possible, we shall restrict the production area to a subset $D_p\subset SP$ called **domain of production**.

In a context of the existence of the domain of production, we put the question of determining its output depending on the level of inputs of D_p.

It is called **production function** an application $Q:D_p\to\mathbf{R}_+$, $(K,L,T)\to Q(K,L,T)\in\mathbf{R}_+$ $\forall(K,L,T)\in D_p$.

For an efficient and complex mathematical analysis of a production function, we impose a number of axioms both its definition and its scope.

FP1. The domain of production is convex;

FP2. $Q(0,0,0)=0$;

FP3. The production function is of class C^2 on D_p that is it admits partial derivatives of order 2 and they are continuous;

FP4. The production function is monotonically increasing in each variable;

FP5. The production function is quasiconcave that is: $Q(\lambda x+(1-\lambda)y)\ge\min(Q(x),Q(y))$ $\forall\lambda\in[0,1]$ $\forall x,y\in R_p$.

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From a geometric point of view, a quasiconcave function having the property of being above the lowest value recorded at the end of a certain segment. The property is equivalent to the convexity of the set $Q^{-1}[a,\infty)$ $\forall a\in\mathbf{R}$, where $Q^{-1}[a,\infty)=\{x\in R_p \mid Q(x)\geq a\}$.

2 The main indicators of production functions

Consider now a production function: $Q:D_p\to\mathbf{R}_+$, $(K,L,T)\to Q(K,L,T)\in\mathbf{R}_+$ $\forall(K,L,T)\in D_p$.

We call **marginal productivity** relative to an input x_i: $\eta_{x_i}=\frac{\partial Q}{\partial x_i}$ and represents the trend of variation of production to the variation of x_i.

We call **average productivity** relative to an input x_i: $w_{x_i}=\frac{Q}{x_i}$ the value of production at a consumption of a unit of factor x_i.

We call **partial marginal substitution rate** of factors i and j the opposite change in the amount of factor j as a substitute for the amount of change in the factor i in the case of a constant level of production and we have: $RMS(i,j)=\frac{\eta_{x_i}}{\eta_{x_j}}$.

We call **elasticity of output** with respect to an input x_i: $\varepsilon_{x_i}=\frac{\frac{\partial Q}{\partial x_i}}{\frac{Q}{x_i}}=\frac{\eta_{x_i}}{w_{x_i}}$ and represents the relative variation of production to the relative variation of the factor x_i.

3 The Generalized Cobb-Douglas function for three inputs

Consideringnow a production function $Q:D_p\to\mathbf{R}_+$, $(K,L,T)\to Q(K,L,T)\in\mathbf{R}_+$ $\forall(K,L,T)\in D_p$ with constant return to scale, let note $\chi_1=\frac{K}{T}$, $\chi_2=\frac{L}{T}$. Let suppose now that $\varepsilon_K=\rho_1(\chi_1)>0$, $\varepsilon_L=\rho_2(\chi_2)>0$.

Considering the function q such that: $Q(K,L,T)=T\,q(\chi_1,\chi_2)$ we have:

$$\varepsilon_K=\frac{\eta_K}{w_K}=\frac{\frac{\partial q}{\partial\chi_1}}{\frac{q}{\chi_1}}=\rho_1(\chi_1),\ \varepsilon_L=\frac{\eta_L}{w_L}=\frac{\frac{\partial q}{\partial\chi_2}}{\frac{q}{\chi_2}}=\rho_2(\chi_2)$$

From here we find that:

$$\frac{\partial q}{\partial\chi_1}=\frac{\rho_1(\chi_1)}{\chi_1}q,\ \frac{\partial q}{\partial\chi_2}=\frac{\rho_2(\chi_2)}{\chi_2}q$$

Let F_1 be a primitive function of $\frac{\rho_1(\chi_1)}{\chi_1}$ and F_2 of $\frac{\rho_2(\chi_2)}{\chi_2}$.

The equations become:

$$\frac{\partial q}{\partial \chi_1} = \frac{\partial F_1(\chi_1)}{\partial \chi_1} q,\ \frac{\partial q}{\partial \chi_2} = \frac{\partial F_2(\chi_2)}{\partial \chi_2} q$$

or, in other words:

$$\frac{\partial(\ln q - F_1(\chi_1))}{\partial \chi_1} = 0,\ \frac{\partial(\ln q - F_2(\chi_2))}{\partial \chi_2} = 0$$

Integrating with respect χ_1 and χ_2 we get:

$$\ln q - F_1(\chi_1) = f_1(\chi_2),\ \ln q - F_2(\chi_2) = f_2(\chi_1)$$

that is:

$$q = e^{F_1(\chi_1)+f_1(\chi_2)},\ q = e^{F_2(\chi_2)+f_2(\chi_1)}$$

But $\frac{\partial q}{\partial \chi_2} = e^{F_1(\chi_1)+f_1(\chi_2)} f_1'(\chi_2) = e^{F_2(\chi_2)+f_2(\chi_1)} F_2'(\chi_2)$ and from the condition that $F_2'(\chi_2) = \frac{\rho_2(\chi_2)}{\chi_2}$ we obtain:

$$e^{f_2(\chi_1)-F_1(\chi_1)} = e^{f_1(\chi_2)-F_2(\chi_2)} \frac{\chi_2 f_1'(\chi_2)}{\rho_2(\chi_2)}$$

Because each term is of different variable and these are independent we have:

$$f_2(\chi_1) - F_1(\chi_1) = C \text{ - constant}$$

But $f_2(\chi_1) = F_1(\chi_1) + C$ implies that $q = Ce^{F_1(\chi_1)+F_2(\chi_2)}$ (after an obvious renoting of C).

Finally:

$$Q(K,L,T)=CTe^{F_1\left(\frac{K}{T}\right)+F_2\left(\frac{L}{T}\right)}$$

If now: $\rho_1(\chi_1)=a\chi_1+b$, $\rho_2(\chi_2)=d\chi_2+g$ we get:

$$F_1(\chi_1) = \int \frac{\rho_1(\chi_1)}{\chi_1} d\chi_1 = \int \frac{a\chi_1 + b}{\chi_1} d\chi_1 = a\chi_1 + b\ln\chi_1,$$

$$F_2(\chi_2) = \int \frac{\rho_2(\chi_2)}{\chi_2} d\chi_2 = \int \frac{d\chi_2 + g}{\chi_2} d\chi_2 = d\chi_2 + g\ln\chi_2$$

therefore:

$$Q(K,L,T)=CK^b L^g T^{1-b-g} e^{\frac{aK+dL}{T}}$$

4 The Generalized Cobb-Douglas function for three inputs and linear elasticity

Consider now the production function: $Q(K,L,T)=CK^b L^g T^{1-b-g} e^{\frac{aK+dL}{T}}$, K,L,T>0, a,b,d,g,C>0.

Because the function is elementary follows that it is of class C^∞ on the definition domain.

We now have:

$$\frac{\partial Q}{\partial K}=\left(\frac{b}{K}+\frac{a}{T}\right)Q,\ \frac{\partial Q}{\partial L}=\left(\frac{g}{L}+\frac{d}{T}\right)Q,\ \frac{\partial Q}{\partial T}=-\frac{aK+dL-(1-b-g)T}{T^2}Q$$

Considering bordered Hessian matrix:

$$H^B(Q)=\begin{pmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} & \frac{\partial Q}{\partial T} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K\partial L} & \frac{\partial^2 Q}{\partial K\partial T} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K\partial L} & \frac{\partial^2 Q}{\partial L^2} & \frac{\partial^2 Q}{\partial L\partial T} \\ \frac{\partial Q}{\partial T} & \frac{\partial^2 Q}{\partial K\partial T} & \frac{\partial^2 Q}{\partial L\partial T} & \frac{\partial^2 Q}{\partial T^2} \end{pmatrix}$$

and the minors:

$$\Delta_1^B=\begin{vmatrix} 0 & \frac{\partial Q}{\partial K} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} \end{vmatrix},\ \Delta_2^B=\begin{vmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K\partial L} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K\partial L} & \frac{\partial^2 Q}{\partial L^2} \end{vmatrix},\ \Delta_3^B=\begin{vmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} & \frac{\partial Q}{\partial T} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K\partial L} & \frac{\partial^2 Q}{\partial K\partial T} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K\partial L} & \frac{\partial^2 Q}{\partial L^2} & \frac{\partial^2 Q}{\partial L\partial T} \\ \frac{\partial Q}{\partial T} & \frac{\partial^2 Q}{\partial K\partial T} & \frac{\partial^2 Q}{\partial L\partial T} & \frac{\partial^2 Q}{\partial T^2} \end{vmatrix}$$

it is known that if $\Delta_1^B<0$, $\Delta_2^B>0$, $\Delta_3^B<0$ the function is quasiconcave. Conversely, if the function is quasiconcave then: $\Delta_1^B\le 0$, $\Delta_2^B\ge 0$, $\Delta_3^B\le 0$.

In the present case:

$$\Delta_1^B=-\frac{(aK+bT)^2}{K^2T^2}Q^2,\ \Delta_2^B=\frac{a^2gK^2+2abgKT+b\left(d^2L^2+2dgLT+g(b+g)T^2\right)}{K^2L^2T^2}Q^3,\ \Delta_3^B=$$
$$\frac{a^2gK^2+2abgKT+b\left(d^2L^2+2dgLT-g(1-b-g)T^2\right)}{K^2L^2T^4}Q^4$$

It is obvious that $\Delta_1^B<0$, $\Delta_2^B>0$. For $\Delta_3^B<0$ we shall do a restriction of the domain D_p such that $a^2gK^2+2abgKT+b\left(d^2L^2+2dgLT-g(1-b-g)T^2\right)<0$.

Also, relative to the monotonically increasing in each variable, we have: $\frac{\partial Q}{\partial K}=\left(\frac{b}{K}+\frac{a}{T}\right)Q>0$, $\frac{\partial Q}{\partial L}=\left(\frac{g}{L}+\frac{d}{T}\right)Q>0$ and because $\frac{\partial Q}{\partial T}=-\frac{aK+dL-(1-b-g)T}{T^2}Q$ we must have also: $aK+dL-(1-b-g)T<0$.

Finally we have that the domain of production is:

$D_p=\{(K,L,T)\in \mathbf{R}_+^3 \mid a^2gK^2+2abgKT+b\left(d^2L^2+2dgLT-g(1-b-g)T^2\right)<0,\ aK+dL-(1-b-g)T\ \}$

5 Main indicators of the Generalized Cobb-Douglas function for three inputs and linear elasticity

We can compute, after section 2, the main indicators for the production function defined above. We have therefore:

- The marginal productivity:

$$\eta_K = \frac{\partial Q}{\partial K} = \left(\frac{b}{K} + \frac{a}{T}\right)Q,\ \eta_L = \frac{\partial Q}{\partial L} = \left(\frac{g}{L} + \frac{d}{T}\right)Q,\ \eta_T = \frac{\partial Q}{\partial T} = -\frac{aK + dL - (1-b-g)T}{T^2}Q$$

- The average productivity:

$$w_K = \frac{Q}{K},\ w_L = \frac{Q}{L},\ w_T = \frac{Q}{T}$$

- The partial marginal substitution rate:

$$\text{RMS(K,L)}=-\frac{L(aK+bT)}{K(dL+gT)},\ \text{RMS(K,T)}=-\frac{T(aK+bT)}{K(aK+dL-(1-b-g)T)},\ \text{RMS(L,T)}=-\frac{T(dL+gT)}{L(aK+dL-(1-b-g)T)}$$

- The elasticity of output:

$$\varepsilon_K = \frac{\frac{\partial Q}{\partial K}}{\frac{Q}{K}} = \frac{\eta_K}{w_K} = \frac{aK+bT}{T},\ \varepsilon_L = \frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}} = \frac{\eta_L}{w_L} = \frac{dL+gT}{T},\ \varepsilon_T = \frac{\frac{\partial Q}{\partial T}}{\frac{Q}{T}} = \frac{\eta_T}{w_T} = -\frac{aK+dL-(1-b-g)T}{T}$$

6 The problem of determining the maximum of production in terms of given total cost

Let now the following problem:

$$\begin{cases} \max Q(K,L,T) \\ p_K K + p_L L + p_T T = CT > 0 \\ K,L,T \geq 0 \end{cases}$$

where CT is the total cost of the production which is suppose to be a given constant.

From the Karush-Kuhn-Tucker conditions we have the necessary and sufficient conditions (taking into account that the restriction is affine):

$$\begin{cases} \frac{\frac{\partial Q}{\partial K}}{p_K} = \frac{\frac{\partial Q}{\partial L}}{p_L} = \frac{\frac{\partial Q}{\partial T}}{p_T} \\ p_K K + p_L L + p_T T = CT \end{cases}$$

From section 5 we get that the system becomes:

$$\begin{cases} \frac{aKL + bLT}{dKL + gKT} = \frac{p_K}{p_L} \\ -\frac{T(aK + bT)}{K(aK + dL - (1 - b - g)T)} = \frac{p_K}{p_T} \\ p_K K + p_L L + p_T T = CT \end{cases}$$

or, using $\chi_1 = \frac{K}{T}$, $\chi_2 = \frac{L}{T}$, we ind from the first two equations:

$$\begin{cases} \chi_2 = \frac{gp_K\chi_1}{(ap_L - dp_K)\chi_1 + bp_L} \\ ap_K\chi_1^2 + dp_K\chi_1\chi_2 - (1 - b - g)p_K\chi_1 + ap_T\chi_1 + bp_T = 0 \end{cases}$$

$$\begin{cases} \chi_2 = \frac{gp_K\chi_1}{(ap_L - dp_K)\chi_1 + bp_L} \\ ap_K\chi_1^2 + dp_K\chi_1\frac{gp_K\chi_1}{(ap_L - dp_K)\chi_1 + bp_L} - (1 - b - g)p_K\chi_1 + ap_T\chi_1 + bp_T = 0 \end{cases}$$

Solving the last equation for χ_1 and from the first obtaining χ_2 we shall find from:

$p_K\chi_1 + p_L\chi_2 + p_T = \frac{CT}{T}$ the value of T^* of T. Finally: $K^* = \chi_1 T^*$, $L^* = \chi_2 T^*$.

7 Conclusions

The Generalized Cobb-Douglas function for three inputs and linear elasticity is determined from the condition that linear elasticity of production with capital and labor are linear expressed. The problem of determining the factors of production that maximizes output is reduce to an equation of third degree.

8 References

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