A Study of Integers Using Software Tools – II

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Abstract. The paper deals with a generalization of polite numbers that is of those numbers that are sums of consecutive integers.

Keywords: polite numbers, divisibility

1 Introduction

Let note for any $n\in\mathbf{N}^*$, $p\in\mathbf{N}^*$, $S_{n,p}=1^p+...+n^p$ and $S_{k,n,p}=k^p+...+n^p=S_{n,p}-S_{k-1,p}$, $k=\overline{2,n}$.

It is well known that in order to compute the expressions: $S_{n,p}$ we depart from the decomposition:

$$(k+1)^p=k^p+\sum_{i=1}^{p}C_p^i k^{p-i}\text{ , }k=\overline{1,n}$$

Summing for $k=\overline{1,n}$:

$$(n+1)^p-1=\sum_{k=1}^{n}(k+1)^p-\sum_{k=1}^{n}k^p=\sum_{i=1}^{p}C_p^i S_{n,p-i}$$

therefore:

$$S_{n,p}=\frac{(n+1)^{p+1}-1-\sum_{j=1}^{p}C_{p+1}^{j+1}S_{n,p-j}}{p+1}$$

and also:

$$S_{k,n,p}=\frac{(n+1)^{p+1}-1-\sum_{j=1}^{p}C_{p+1}^{j+1}S_{n,p-j}}{p+1}-\frac{k^{p+1}-1-\sum_{j=1}^{p}C_{p+1}^{j+1}S_{k-1,p-j}}{p+1}=$$

$$\frac{(n+1)^{p+1}-k^{p+1}-\sum_{j=1}^{p}C_{p+1}^{j+1}\left(S_{n,p-j}-S_{k-1,p-j}\right)}{p+1}=\frac{(n+1)^{p+1}-k^{p+1}-\sum_{j=1}^{p}C_{p+1}^{j+1}S_{k,n,p-j}}{p+1}$$

It is easy to see that the first 10 sums are:

$$S_{n,1}=1+...+n=\frac{n(n+1)}{2}$$

$$S_{n,2}=1^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_{n,3}=1^3+...+n^3=\frac{n^2(n+1)^2}{4}$$

$$S_{n,4}=1^4+...+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

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$S_{n,5}=1^5+...+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

$S_{n,6}=1^6+...+n^6=\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$

$S_{n,7}=1^7+...+n^7=\frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24}$

$S_{n,8}=1^8+...+n^8=\frac{n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)}{90}$

$S_{n,9}=1^9+...+n^9=\frac{n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)}{20}$

$S_{n,10}=1^{10}+...+n^{10}=$
$\frac{n(n+1)(2n+1)(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)}{66}$

We can compute these sums taking into account the symmetry after a middle term. We have therefore 2 cases:

Case 1 – the sum has an odd number of terms

Let therefore:$SC_{s,m,p}=(m-s)^p+...+(m-1)^p+m^p+(m+1)^p...+(m+s)^p=$ $S_{m+s,p}$-$S_{m-s-1,p}$.

We have now:

$SC_{s,m,1}=(m-s)+...+(m-1)+m+(m+1)...+(m+s)=m(2s+1)$

$SC_{s,m,2}=(m-s)^2+...+(m-1)^2+m^2+(m+1)^2...+(m+s)^2=\frac{(3m^2+s^2+s)(2s+1)}{3}$

$SC_{s,m,3}=(m-s)^3+...+(m-1)^3+m^3+(m+1)^3...+(m+s)^3=m(m^2+s^2+s)(2s+1)$

$SC_{s,m,4}=(m-s)^4+...+(m-1)^4+m^4+(m+1)^4...+(m+s)^4=$
$\frac{(15m^4+30m^2(s^2+s)+3s^4+6s^3+2s^2-s)(2s+1)}{15}$

$SC_{s,m,5}=(m-s)^5+...+(m-1)^5+m^5+(m+1)^5...+(m+s)^5=$
$\frac{m(3m^4+10m^2(s^2+s)+3s^4+6s^3+2s^2-s)(2s+1)}{3}$

and so on.

Case 2 – the sum has an even number of terms

Let therefore:$SC_{s,m,p}=(m-s+1)^p+...+(m-1)^p+m^p+(m+1)^p...+(m+s)^p=$ $S_{m+s,p}$-$S_{m-s-1,p}$.

We have now:

$SC_{s,m,1}=(m-s+1)+...+(m-1)+m+(m+1)...+(m+s)=s(2m+1)$

$SC_{s,m,2}=(m-s+1)^2+...+(m-1)^2+m^2+(m+1)^2...+(m+s)^2=$
$\frac{s(6m^2+6m+2s^2+1)}{3}$

$SC_{s,m,3}=(m-s+1)^3+...+(m-1)^3+m^3+(m+1)^3...+(m+s)^3=$
$s(m^2+m+s^2)(2m+1)$

$SC_{s,m,4}=(m-s+1)^4+...+(m-1)^4+m^4+(m+1)^4...+(m+s)^4=$
$\frac{s(30m^4+60m^3+30m^2(2s^2+1)+60ms^2+6s^4+10s^2-1)}{15}$

$SC_{s,m,5}=(m-s+1)^5+...+(m-1)^5+m^5+(m+1)^5...+(m+s)^5=$
$\frac{s(3m^4+6m^3+2m^2(5s^2+1)+m(10s^2-1)+3s^4)(2m+1)}{3}$
and so on.
All over in this paper, the software presented was written in Wolfram Mathematica 9.0.

2 Polite numbers

A natural number N greather than 2 is called polite number if it can be written as sum of two or more consecutive natural numbers.
If N is odd it is natural that for N=2k+1 we have N=k+(k+1) therefore each odd natural number is polite. Let therefore N=even, N=2M, M>2.
Let consider now the decomposition: $N=2^q a$ where $q\in\mathbf{N}^*$, a=odd.If the sum of integers has an odd number of terms, we have:

$$2^q a=m(2s+1) \text{ with } 1\le s\le m-1$$

Because 2s+1=odd we have that: $2^q | m$ therefore: $m=2^q b$, $b\in\mathbf{N}^*$. Now, from: $2^q a=2^q b(2s+1)$ we have: a=b(2s+1) therefore, for $N=2^q bc$, b,c=odd, we have: $m=2^q b$, 2s+1=c.

But 2s+1≥3 and 2s+1≤2m-1 impliy that: c≥3 and $c\le 2^{q+1}b-1$.

From $c\le 2^{q+1}b-1$ we have: $c^2\le 2N-c$ therefore: $3\le c\le\frac{\sqrt{1+8N}-1}{2}$, $b\ge\max\left\{1,\frac{c+1}{2^{q+1}}\right\}$

For example, for N=36 we have: $N=2^2 3^2$ therefore: $3\le c\le 8$, $b\ge\max\left\{1,\frac{c+1}{8}\right\}$ from where: q=2, b=3, c=3⇒m=12, s=1⇒N=36=11+12+13.If the sum of integers has an even number of terms, we have:

$$2^q a=s(2m+1) \text{ with } 1\le s\le m$$

With the same arguments like upper, we have that for $N=2^q bc$, b,c=odd, we have: $s=2^q b$, 2m+1=c. But s≥1 it is obvious and s≤m implies that: $2^{q+1}b\le c-1$ therefore $2N\le c^2-c$ that is: $c\ge\max\left\{3,\frac{1+\sqrt{1+8N}}{2}\right\}$ and $1\le b\le\frac{c-1}{2^{q+1}}$.

For example, for N=36 we have: $N=2^2 3^2$ therefore: $c\ge\max\{3,9\}=9$ and $1\le b\le\frac{c-1}{8}$ that is: q=2, c=9, b=1⇒m=4, s=4 therefore: N=36=1+2+3+4+5+6+7+8.If N is a power of 2, i.e. $N=2^q$ we then have a=1 and in each case we shall obtain s=0 or m=0 which will be a contradiction. After these considerations we have that no power of 2 can be expressed like a sum of consecutive natural numbers.

3 Almost polite numbers of order p

A natural number N greather than 2 willbe called almost polite number of order p if it can be written as sum of two or more consecutive of a same power p of natural numbers.The software for determining the almost polite numbers limited to 10000 and powers less than or equal with 30 is:

```
Clear["Global`*"];
limit=10000;
pmax=30;
```

```
S[0]=n;
(*The calculus of sums of powers from 1 to n*)
For[p=1,p≤pmax,p++,
 suma=0;
 For[j=1,j≤p,j++,suma=suma+Binomial[p+1,j+1]*S[p-j]];
 S[p]=Factor[((n+1)^(p+1)-1-suma)/(p+1)]
 ]
(*The calculus of sums of powers from k to n*)
For[p=1,p≤pmax,p++,sumpower[n_,p]=S[p]];
For[p=1,p≤pmax,p++,sumpowerkn[n_,k_,p]=Factor[Simplify[sumpower[n,p]-
sumpower[k-1,p]]]]
(*The analisys*)
For[number=2,number≤limit,number=number+1,
 For[p=2,p≤pmax,p++,
  For[n=2,n≤number^(1/p),n++,
   For[k=1,k≤n-1,k++,
    If[sumpowerkn[n,k,p]==number,
     Print[number,"=\[Sum](power=",p,") from ",k," to ",n]]]]]]
```

We find (first results):

5=[Sum](power=2) from 1 to 2
9=[Sum](power=3) from 1 to 2
13=[Sum](power=2) from 2 to 3
14=[Sum](power=2) from 1 to 3
17=[Sum](power=4) from 1 to 2
25=[Sum](power=2) from 3 to 4
29=[Sum](power=2) from 2 to 4
30=[Sum](power=2) from 1 to 4
33=[Sum](power=5) from 1 to 2
35=[Sum](power=3) from 2 to 3
36=[Sum](power=3) from 1 to 3
41=[Sum](power=2) from 4 to 5
50=[Sum](power=2) from 3 to 5
54=[Sum](power=2) from 2 to 5
55=[Sum](power=2) from 1 to 5
61=[Sum](power=2) from 5 to 6
65=[Sum](power=6) from 1 to 2
77=[Sum](power=2) from 4 to 6
85=[Sum](power=2) from 6 to 7
86=[Sum](power=2) from 3 to 6
90=[Sum](power=2) from 2 to 6
91=[Sum](power=2) from 1 to 6
91=[Sum](power=3) from 3 to 4
97=[Sum](power=4) from 2 to 3
98=[Sum](power=4) from 1 to 3
99=[Sum](power=3) from 2 to 4
100=[Sum](power=3) from 1 to 4

4 Almost polite numbers of order 2

Let consider now the problem of determining polite numbers of order 2. Let $N=2^q a$ where $q\in\mathbf{Z}$, $q\geq 0$, a=odd.If the sum has an odd number of terms, we have:

$$2^q a=\frac{(3m^2+s^2+s)(2s+1)}{3} \text{ with } 1\leq s\leq m-1$$

The equality becomes:

$$\left(3m^2+s^2+s\right)(2s+1)=3\cdot 2^q a$$

We have now two cases:

Case 1: q≥1

Because 2s+1=odd it follows that: $2^q \mid 3m^2+s^2+s$. But $s^2+s=s(s+1)$=even implies that: $2^q \mid m^2$ therefore: if q=even: m= $2^{\frac{q}{2}}b$ and if q=odd: m= $2^{\frac{q+1}{2}}b$, b∈$\mathbf{N}^*$. In both cases, we can write: m= $2^{\left[\frac{q+1}{2}\right]}b$, b∈$\mathbf{N}^*$, where $[\cdot]$ is the integer part. Also, $2^q \mid 3m^2+s^2+s$ implies now: $2^q \mid 3\cdot 2^{2\left[\frac{q+1}{2}\right]}b^2+s^2+s$ therefore: $2^q \mid s(s+1)$. Because $(s, s+1)=1$ we have that: s=2^qc or s=2^qc-1, c∈$\mathbf{N}^*$.

We have the following cases:

- m= $2^{\left[\frac{q+1}{2}\right]}b$, s=$2^q$c. Because s≤m-1 we have: $2^q c \le 2^{\left[\frac{q+1}{2}\right]}b-1$.
  - q=even: m= $2^{\frac{q}{2}}b$, s=2^qc⇒ $\left(3b^2+2^q c^2+c\right)\left(2^{q+1}c+1\right)=3a$ and $2^q c \le 2^{\frac{q}{2}}b-1$.
  - q=odd: m= $2^{\frac{q+1}{2}}b$, s=2^qc⇒ $\left(6b^2+2^q c^2+c\right)\left(2^{q+1}c+1\right)=3a$ and $2^q c \le 2^{\frac{q+1}{2}}b-1$.
- m= $2^{\left[\frac{q+1}{2}\right]}b$, s=$2^q$c-1. Because s≤m-1 we have: $2^q c \le 2^{\left[\frac{q+1}{2}\right]}b$.
  - q=even: m= $2^{\frac{q}{2}}b$, s=2^qc-1⇒ $\left(3b^2+2^q c^2-c\right)\left(2^{q+1}c-1\right)=3a$ and $2^q c \le 2^{\frac{q}{2}}b$.
  - q=odd: m= $2^{\frac{q+1}{2}}b$, s=2^qc-1⇒ $\left(6b^2+2^q c^2-c\right)\left(2^{q+1}c-1\right)=3\cdot 2^q a$ and $2^q c \le 2^{\frac{q+1}{2}}b$.

Like an example let consider N=140=2^2·35. We have q=2 therefore:

- m=2b, s=4c and $\left(3b^2+4c^2+c\right)(8c+1)=105=3\cdot 5\cdot 7$ ⇒c=13, b∉**Z**
- m= 2b, s=4c-1 and $\left(3b^2+4c^2-c\right)(8c-1)=3\cdot 5\cdot 7$ ⇒
  - c=1⇒b=2⇒m=4, s=3 – N=140=$1^2+2^2+3^2+4^2+5^2+6^2+7^2$.
  - c=2⇒b∉**N**

Case 2: q=0

We have now: $\left(3m^2+s^2+s\right)(2s+1)=3a$. Because 3a=odd and $s^2+s=s(s+1)$=even we must have m=odd.

Like an example let consider N=55. We have q=0 therefore: $\left(3m^2+s^2+s\right)(2s+1)=3\cdot 5\cdot 11$

- 2s+1=3⇒s=1⇒m∉**N**
- 2s+1=5⇒s=2⇒m=3⇒N=55=$1^2+2^2+3^2+4^2+5^2$
- 2s+1=11⇒s=5⇒ m∉**N**
- 2s+1=15⇒s=7⇒ m∉**N**
- 2s+1=33⇒s=16⇒ m∉**N**
- 2s+1=55⇒s=27⇒ m∉**N**
- 2s+1=165⇒s=82⇒ m∉**N**

If the sum has an even number of terms, we have:

$$2^q a = \frac{s(6m^2 + 6m + 2s^2 + 1)}{3} \text{ with } 1 \le s \le m$$

The equality becomes:

$$s(6m^2 + 6m + 2s^2 + 1) = 3 \cdot 2^q a$$

Because $2^q \mid s(6m^2 + 6m + 2s^2 + 1)$ and $(6m^2 + 6m + 2s^2 + 1)$=odd it follows that: $s=2^q b$, $b \ge 1$.

From $b(6m^2 + 6m + 2^{2q+1} b^2 + 1) = 3a$ we shall find m if it exists.

For example, let $N=126=2^1 \cdot 63$ we have: q=1, a=63. Therefore: s=2b and $b(6m^2 + 6m + 8b^2 + 1) = 3^3 \cdot 7$. We find after all cases that: b=1, m=5 and finally: $N=126=4^2+5^2+6^2+7^2$.

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