A Study of Continued Fractions Using Software Tools – I

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Abstract. The paper treats some properties of continous fractions using appropriate software.

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1 Introduction

We call continued fractions a number which has the expression:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}}$$

where $a_0 \in \mathbf{Z}$, $a_i \in \mathbf{N}^*$ $\forall i \geq 1$.

We shall write: $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$.

With Euclid's algorithm every rational number has as a finite representation like a continued fraction. Because, for $\alpha \in \mathbf{Q}$, if $a_n \geq 2$: $\alpha = [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1]$ we shall consider the shorter decomposition for α therefore the first expression.

We shall remind, in what follows, from the [2], the principal properties of continous fractions.

Let note $\alpha_n = [a_0, a_1, \ldots, a_n]$ - the convergent of α of order n and $\alpha_n = \frac{p_n}{q_n}$, $(p_n, q_n) = 1$. p_n and q_n are called the continuants of α of order n.

It is easly to see that:

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{cases}, \ n \geq 1$$

where: $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$.

After these definitions and notations, we have:

$$\alpha_n = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_{n-1}}{q_{n-1}} + \frac{\frac{p_{n-2}}{q_{n-2}} - \frac{p_{n-1}}{q_{n-1}}}{a_n \frac{q_{n-1}}{q_{n-2}} + 1} = \alpha_{n-1} - \frac{\alpha_{n-1} - \alpha_{n-2}}{a_n \frac{q_{n-1}}{q_{n-2}} + 1}$$

therefore:

$$\alpha_n - \alpha_{n-1} = -\frac{\alpha_{n-1} - \alpha_{n-2}}{a_n \frac{q_{n-1}}{q_{n-2}} + 1} = -\frac{q_{n-2}}{a_n q_{n-1} + q_{n-2}} (\alpha_{n-1} - \alpha_{n-2}) = -\frac{q_{n-2}}{q_n} (\alpha_{n-1} - \alpha_{n-2})$$

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from where:

$$\alpha_n - \alpha_{n-1} = -\frac{q_{n-2}}{q_n}(\alpha_{n-1} - \alpha_{n-2}) = \frac{q_{n-2}}{q_n}\frac{q_{n-3}}{q_{n-1}}(\alpha_{n-2} - \alpha_{n-3}) = \ldots = (-1)^{n-1}\frac{q_{n-2}}{q_n}\frac{q_{n-3}}{q_{n-1}}\ldots\frac{q_0}{q_2}(\alpha_1 - \alpha_0) =$$

$$(-1)^{n-1}\frac{q_0 q_1}{q_{n-1}q_n}(\alpha_1 - \alpha_0) = (-1)^{n-1}\frac{a_1}{q_{n-1}q_n}\left(a_0 + \frac{1}{a_1} - a_0\right) = \frac{(-1)^{n-1}}{q_{n-1}q_n}$$

It is easy to see that:

- $q_n p_{n-1} - q_{n-1}p_n = (-1)^n$
- $(p_n, q_n) = 1$
- $\alpha_n - \alpha_{n-1} = \frac{(-1)^{n-1}}{q_{n-1}q_n}$
- $|\alpha_p - \alpha_n| < |\alpha_s - \alpha_n|$ $\forall s<p<n$
- $(\alpha_{2k})_{k\geq 1, 2k<n}$ is increasing, but $\alpha_{2k} < \alpha_n$
- $(\alpha_{2k+1})_{k\geq 0, 2k+1<n}$ is decreasing, but $\alpha_{2k+1} > \alpha_n$
- $\frac{1}{q_n(q_n + q_{n+1})} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$

After the last statement we can see that any convergent is closer to the value α of the continued fraction than any other rational whose denominator is less than that of the convergent.

Also, the theorem of Lagrange states that any quadratic irrational (that is a number which is root of an equation of second degree with discriminant not perfect square) has a periodic development as continued fraction, that is:

$$a + b\sqrt{c} = [a_0, a_1, \ldots, \overline{a_k, \ldots, a_n}]$$

which means that $a_{n+1}=a_k$, $a_{n+2}=a_{k+1}$ and so on.

The reciprocal is also true.

2 Solving algebraic equations with the aid of continuous fractions

Let now the polynomial $P\in\mathbf{Z}[X]$ and the equation P=0. First, we isolate the roots in intervals with whole ends. Let, for example: $x_1\in(a_0,a_0+1)$, $a_0\in\mathbf{Z}$. It is not necessary to have only one root in this interval.

The associate Taylor polynomial is:

$$P(a_0 + \alpha) = P(a_0) + \frac{P'(a_0)}{1!}\alpha + \frac{P''(a_0)}{2!}\alpha^2 + \ldots + \frac{P^{(n)}(a_0)}{n!}\alpha^n \quad \forall\alpha\in\mathbf{R}$$

Therefore:

$$P\left(a_0 + \frac{1}{\alpha}\right) = \frac{1}{\alpha^n}\left(P(a_0)\alpha^n + \frac{P'(a_0)}{1!}\alpha^{n-1} + \frac{P''(a_0)}{2!}\alpha^{n-2} + \ldots + \frac{P^{(n)}(a_0)}{n!}\right)$$

If we find α≥1 such that: $P\left(a_0 + \frac{1}{\alpha}\right)P\left(a_0 + \frac{1}{\alpha+1}\right) < 0$ that is:

$$\left(P(a_0)\alpha^n + \frac{P'(a_0)}{1!}\alpha^{n-1} + \frac{P''(a_0)}{2!}\alpha^{n-2} + \ldots + \frac{P^{(n)}(a_0)}{n!}\right)\cdot$$

$$\left(P(a_0)(\alpha+1)^n + \frac{P'(a_0)}{1!}(\alpha+1)^{n-1} + \frac{P''(a_0)}{2!}(\alpha+1)^{n-2} + \ldots + \frac{P^{(n)}(a_0)}{n!}\right) < 0$$

the root x_1 belongs in the interval: $\left(a_0+\frac{1}{\alpha+1}, a_0+\frac{1}{\alpha}\right)$.

Using the Horner's diagram, the computations are organized as follows, for $P=c_nX^n+...+c_0$

	c_n	c_{n-1}	...	c_2	c_1	c_0
a_0	c_n	b_{n-1}	...	b_2	b_1	$P(a_0)$
	...	...	...	...	$\frac{P'(a_0)}{1!}$	
	...	...	...	$\frac{P''(a_0)}{2!}$		
	...	...	...			
	$\frac{P^{(n)}(a_0)}{n!}$					

The new diagram has new coefficients:

	$P(a_0)$	$\frac{P'(a_0)}{1!}$	$\frac{P''(a_0)}{2!}$	...	$\frac{P^{(n)}(a_0)}{n!}$
α					

If at a step we find two or more α such that $P\left(a_0+\frac{1}{\alpha}\right)P\left(a_0+\frac{1}{\alpha+1}\right)<0$ it follows that in the interval (a_0,a_0+1) belongs more than one roots of the ploynomial.

3 Numbers expressed like ratio between prime numbers

All over in this paper, the software presented was written in Wolfram Mathematica 9.0.

Let now a number $r\in\mathbf{Q}^*_+$, $r=\frac{p_k}{p_s}$ where p_k – is the k-th prime number.

First, we will inquire into the existence of a development in continued fraction where all terms are primes or 1.

The next software find the development with maxim length for first 2000 prime numbers.

```
Clear["Global`*"];
numberprimes=2000;
nr=0;
For[k=2,k≤numberprimes,k++,For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
  a=ContinuedFraction[x];b=1;For[p=1,p≤Length[a],p++,x=Part[a,p];If[PrimeQ[x]||x==
1,b=b,b=0*b]];If[b==1,nr=nr+1;If[nr==1,c=a;d=k;e=s,If[Length[a]>Length[c],c=a;d=k;e=s]]]]]
Print[d,"---",e,"---",Prime[d],"/",Prime[e],"=",c]
```

The result of the execution is:

1430---1158---11933/9349={1,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2}

which means that $\frac{p_{1430}}{p_{1158}}=\frac{11933}{9349}$=[1,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2].

If we shall find the development in continued fraction where all terms are primes we find:

```
Clear["Global`*"];
```

```
numberprimes=2000;
nr=0;
For[k=2,k≤numberprimes,k++,For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
a=ContinuedFraction[x];b=1;For[p=1,p≤Length[a],p++,x=Part[a,p];If[PrimeQ[x],b=b,b=0*b]];
If[b==1,nr=nr+1;If[nr==1,c=a;d=k;e=s,If[Length[a]>Length[c],c=a;d=k;e=s]]]]]
Print[d,"---",e,"---",Prime[d],"/",Prime[e],"=",c]
```

and the result of the execution is:

1271---613---10357/4517={2,3,2,2,2,2,2,2,3,2}

which means that $\frac{p_{1271}}{p_{613}}=\frac{10357}{4517}=[2,3,2,2,2,2,2,2,3,2]$.

If we shall find the development in continued fraction where all terms are different primes we find, for first 5000 prime numbers:

```
Clear["Global`*"];
numberprimes=5000;
nr=0;
For[k=2,k≤numberprimes,k++,
 For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
  a=ContinuedFraction[x];b=1;
  For[p=1,p≤Length[a]-1,p++,
   x=Part[a,p];
   If[PrimeQ[x],
    For[i=p+1,i≤Length[a],i++,
     y=Part[a,i];
     If[PrimeQ[y],
      If[x== y,Goto[salt]],Goto[salt]]],Goto[salt]]];
  If[b==1,nr=nr+1;If[nr==1,c=a;d=k;e=s,If[Length[a]>Length[c],c=a;d=k;e=s]]];
  Print[k,"---",s,"---",Prime[k],"/",Prime[s],"=",a];
  Label[salt]]]
Print["Maximum length is for: ",d,"---",e,"---",Prime[d],"/",Prime[e],"=",c]
```

and the result of the execution is:

Maximum length is for: 3636---1335---33961/10993={3,11,5,7,13,2}

which means that $\frac{p_{3636}}{p_{1335}}=\frac{33961}{10993}=[3,11,5,7,13,2]$.

In what follows we shall try to find the numbers $r\in\mathbf{Q}^{*}_{+}$, $r=\frac{p_k}{p_s}$ where p_k – is the k-th prime number, $r=\left[a_0,a_1,\ldots,a_n\right]$ for which the reverse: $s=\left[a_n,a_{n-1},\ldots,a_0\right]$ is also of the form: $s=\frac{p_u}{p_v}$.

```
Clear["Global`*"];
numberprimes=2000;
nr=0;
For[k=2,k≤numberprimes,k++,
 For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
  a=ContinuedFraction[x];
  lung= Length[a];
  c=a;
  For[p=1,p≤lung,p++,
   c[[p]]=a[[lung+1-p]]];
```

```
 value=FromContinuedFraction[c];
If[PrimeQ[Numerator[value]]&&PrimeQ[Denominator[value]],nr=nr+1;If[nr==1,t=a;d=k;e=s,
If[Length[a]>Length[t],t=a;u=c;d=k;e=s]]]]]
Print[d,"---",e,"---",Prime[d],"/",Prime[e],"=",t," and reverse:",
Numerator[FromContinuedFraction[u]],"/",Denominator[FromContinuedFraction[u]],"=",u]
```

We found the maximum length for the first 2000 prime numbers:

1400---871---11657/6763={1,1,2,1,1,1,1,1,1,1,3,1,1,1,1,1,2}

and reverse: 11657/4447={2,1,1,1,1,1,3,1,1,1,1,1,1,1,2,1,1}

that is: $\frac{p_{1400}}{p_{871}} = \frac{11657}{6763}$=[1,1,2,1,1,1,1,1,1,1,3,1,1,1,1,1,2] and the reverse development is: $\frac{11657}{4447}$ =[2,1,1,1,1,1,3,1,1,1,1,1,1,1,2,1,1], both 11657 and 4447 being primes.

4 References

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