The Complete Theory of Generalized CES Production Function

Cătălin Angelo IOAN[1], Gina IOAN[2]

Abstract: The paper treats various aspects concerning the generalized CES production function. On the one hand were highlighted conditions for the existence of the generalized CES function. Also were calculated the main indicators of it and short and long-term costs. It has also been studied the dependence of long-term cost of the parameters of the production function. The determination of profit was made both for perfect competition market and maximizes its conditions. Also we have studied the effects of Hicks and Slutsky and the production efficiency problem.

Keywords: production function; generalized CES; Hicks; Slutsky

JEL Classification: C02; C65

1 Introduction

To conduct any economic activity is absolutely indispensable the existence of inputs, in other words of any number of resources required for a good deployment of the production process. We will assume that all resources are indefinitely divisible.

We define on $\mathbf{R}^n$ the production space for n fixed resources as $SP=\{(x_1,...,x_n) \mid x_i\geq 0,\ i=\overline{1,n}\}$ where $x\in SP$, $x=(x_1,...,x_n)$ is an ordered set of resources and, because inside a production process, depending on the nature of applied technology, not any amount of resources is possible, we will restrict production space to a convex subset $D_p\subset SP$ – called the domain of production.

We will call a production function an application:

$$Q:D_p\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D_p$$

which satisfies the following axioms:

A1. $Q(0,...,0)=0$;

A2. The production function is of class C^2 on D_p that is it admits partial derivatives of order 2 and they are continuous on D_p;

A3. The production function is monotonically increasing in each variable, that is: $\frac{\partial Q}{\partial x_i}\geq 0,\ i=\overline{1,n}$;

A4. The production function is quasi-concave, that is: $Q(\lambda x+(1-\lambda)y)\geq \min(Q(x),Q(y))\ \forall\lambda\in[0,1]\ \forall x,y\in D_p$

Considering a production function $Q:D_p\to\mathbf{R}_+$ and $Q_0\in\mathbf{R}_+$ - fixed, the set of inputs which generate the production Q_0 called isoquant. An isoquant is therefore characterized by: $\{(x_1,...,x_n)\in D_p \mid Q(x_1,...,x_n)=Q_0\}$ or, in other words, it is the inverse image $Q^{-1}(Q_0)$.

[1] Danubius University of Galati, Department of Finance and Business Administration, catalin_angelo_ioan@univ-danubius.ro.
[2] Danubius University of Galati, Department of Finance and Business Administration, ginaioan@univ-danubius.ro.

We will say that a production function $Q:D_p\to\mathbf{R}_+$ is constant return to scale if $Q(\lambda x_1,...,\lambda x_n)=\lambda Q(x_1,...,x_n)$, with increasing return to scale if $Q(\lambda x_1,...,\lambda x_n)>\lambda Q(x_1,...,x_n)$ and decreasing return to scale if $Q(\lambda x_1,...,\lambda x_n)<\lambda Q(x_1,...,x_n)$ $\forall\lambda\in(1,\infty)$ $\forall(x_1,...,x_n)\in D_p$.

2 The generalized CES production function

The generalized CES function has the following expression:

$$Q:D_p\subset\mathbf{R}_+^n-\{0\}\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)=\alpha\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D_p,\ \alpha,\beta_1...,\beta_n>0,$$

$$\gamma\in(-\infty,0)\cup(0,1),\ \gamma\rho>0$$

For $\gamma=\rho$ we have the classical CES production function.

Computing the partial derivatives of first and second order, we get:

$$Q'_{x_i}=\alpha\frac{\gamma}{\rho}\beta_i x_i^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^{\frac{1}{\rho}-1}=\frac{\gamma\beta_i x_i^{\gamma-1}Q}{\rho\sum_{k=1}^{n}\beta_k x_k^{\gamma}}\quad\forall i=\overline{1,n}$$

$$Q''_{x_ix_j}=\alpha\frac{\gamma^2}{\rho}\beta_i\beta_j x_i^{\gamma-1}x_j^{\gamma-1}\left(\frac{1}{\rho}-1\right)\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^{\frac{1}{\rho}-2}=-\frac{\beta_i\beta_j\gamma^2(\rho-1)x_i^{\gamma-1}x_j^{\gamma-1}Q}{\rho^2\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^2}\quad\forall i\neq j=\overline{1,n}$$

$$Q''_{x_ix_i}=\frac{\alpha\gamma\beta_i x_i^{\gamma-2}}{\rho^2}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^{\frac{1}{\rho}-2}\left(\rho(\gamma-1)\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)-\gamma(\rho-1)\beta_i x_i^{\gamma}\right)=\frac{\gamma\beta_i x_i^{\gamma-2}\left(\rho(\gamma-1)\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)-\gamma(\rho-1)\beta_i x_i^{\gamma}\right)Q}{\rho^2\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^2}$$

$\forall i=\overline{1,n}$

Let the bordered Hessian matrix:

$$H^B(Q)=\frac{\gamma Q}{\rho^2\left(\sum_{k=1}^{n}\beta_k x_k^{\rho}\right)^2}\begin{pmatrix}0 & \rho\beta_1 x_1^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right) & \rho\beta_2 x_2^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right) & \cdots & \rho\beta_n x_n^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)\\ \rho\beta_1 x_1^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right) & \beta_1 x_1^{\gamma-2}\left(\rho(\gamma-1)\left(\sum_{\substack{k=1\\k\neq1}}^{n}\beta_k x_k^{\gamma}\right)+(\gamma-\rho)\beta_1 x_1^{\gamma}\right) & -\beta_1\beta_2\gamma(\rho-1)x_1^{\gamma-1}x_2^{\gamma-1} & \cdots & -\beta_1\beta_n\gamma(\rho-1)x_1^{\gamma-1}x_n^{\gamma-1}\\ \rho\beta_2 x_2^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right) & -\beta_1\beta_2\gamma(\rho-1)x_1^{\gamma-1}x_2^{\gamma-1} & \beta_2 x_2^{\gamma-2}\left(\rho(\gamma-1)\left(\sum_{\substack{k=1\\k\neq2}}^{n}\beta_k x_k^{\gamma}\right)+(\gamma-\rho)\beta_2 x_2^{\gamma}\right) & \cdots & -\beta_2\beta_n\gamma(\rho-1)x_2^{\gamma-1}x_n^{\gamma-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \rho\beta_n x_n^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right) & -\beta_1\beta_n\gamma(\rho-1)x_1^{\gamma-1}x_n^{\gamma-1} & -\beta_2\beta_n\gamma(\rho-1)x_2^{\gamma-1}x_n^{\gamma-1} & \cdots & \beta_n x_n^{\gamma-2}\left(\rho(\gamma-1)\left(\sum_{\substack{k=1\\k\neq n}}^{n}\beta_k x_k^{\gamma}\right)+(\gamma-\rho)\beta_n x_n^{\gamma}\right)\end{pmatrix}$$

We find (not so easy): $\Delta_s^B=(-1)^s\alpha^{\rho s}\left(\frac{\gamma}{\rho}\right)^{s+1}(1-\gamma)^{s-1}\prod_{i=1}^{s}\beta_i\prod_{i=1}^{s}x_i^{\gamma-2}Q^{1-s(\rho-1)}$, $s=\overline{1,n}$.

Because $(-1)^s \Delta_s^B = \alpha^{\rho s}\left(\frac{\gamma}{\rho}\right)^{s+1}(1-\gamma)^{s-1}\prod_{i=1}^{s}\beta_i\prod_{i=1}^{s}x_i^{\gamma-2}Q^{1-s(\rho-1)}$ if $\left(\frac{\gamma}{\rho}\right)^{s+1}(1-\gamma)^{s-1}>0$, $k=\overline{1,n}$ it follows that the function is strictly quasi-concave. Also, if the function is quasi-concave we have that $\left(\frac{\gamma}{\rho}\right)^{s+1}(1-\gamma)^{s-1}\geq 0$. But from the definition this condition is equivalent with definition's conditions.

We have now: $q(\chi_1,...,\chi_{n-1})=Q(\chi_1,...,\chi_{n-1},1)=\alpha\left(\sum_{i=1}^{n-1}\beta_i\chi_i^{\gamma}+\beta_n\right)^{\frac{1}{\rho}}$ and the homogeneity degree: $r=\frac{\gamma}{\rho}$.

The main indicators are:

- $\eta_{x_i}=\alpha\frac{\gamma}{\rho}\beta_i x_i^{\gamma-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^{\frac{1}{\rho}-1}=\frac{\gamma\beta_i x_i^{\gamma-1}Q}{\rho\sum_{k=1}^{n}\beta_k x_k^{\gamma}}$, $i=\overline{1,n}$ - the marginal productivity relative to the production factor x_i;
- $w_{x_i}=\frac{\alpha}{x_i}\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}=\frac{Q}{x_i}$, $i=\overline{1,n}$ - the average productivity relative the production factor x_i;
- RMS(i,j)=$\frac{\beta_i}{\beta_j}\left(\frac{x_i}{x_j}\right)^{\gamma-1}$, $i,j=\overline{1,n}$ - the partial marginal rate of technical substitution of the factors i and j;
- RMS(i)=$\frac{\beta_i x_i^{\gamma-1}}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n-1}\beta_j^2 x_j^{2(\gamma-1)}}}$, $i=\overline{1,n}$ - the global marginal rate of substitution between the i-th factor and the others;
- $\varepsilon_{x_i}=\frac{\gamma\beta_i x_i^{\gamma}}{\rho\sum_{k=1}^{n}\beta_k x_k^{\gamma}}$, $i=\overline{1,n}$ - the elasticity of production in relation to the production factor x_i;
- $\sigma_{ij}=\gamma-1$, $i,j=\overline{1,n}$, $i\neq j$ - the relative variation of marginal rate of technical substitution relative to factors i and j at the relative variation of the factor endowment ratio with factor i relative to factor j.

Reciprocally, if for a homogenous production function of degree r: $\sigma_{ij}=\gamma-1$, $i,j=\overline{1,n}$, $i\neq j$, $\gamma\neq 1$ we have that: $\sigma_{ij}=x_i\frac{\partial \ln RMS(i,j)}{\partial x_i}=\gamma-1$.

But, in terms of $q(\chi_1,...,\chi_{n-1})$ we obtain:

$$(1)\ \sigma_{ij}=\chi_i\frac{\frac{\partial^2 q}{\partial\chi_i^2}\frac{\partial q}{\partial\chi_j}-\frac{\partial q}{\partial\chi_i}\frac{\partial^2 q}{\partial\chi_i\partial\chi_j}}{\frac{\partial q}{\partial\chi_i}\frac{\partial q}{\partial\chi_j}}=\gamma-1,\ i,j=\overline{1,n-1},\ i\neq j$$

$$(2)\ \sigma_{in}=\chi_i\frac{rq\frac{\partial^2 q}{\partial\chi_i^2}+(1-r)\left(\frac{\partial q}{\partial\chi_i}\right)^2+\frac{\partial q}{\partial\chi_i}\sum_{\substack{j=1\\ j\neq i}}^{n-1}\frac{\partial^2 q}{\partial\chi_j\partial\chi_i}\chi_j-\frac{\partial^2 q}{\partial\chi_i^2}\sum_{\substack{j=1\\ j\neq i}}^{n-1}\frac{\partial q}{\partial\chi_j}\chi_j}{\frac{\partial q}{\partial\chi_i}\left(rq-\sum_{j=1}^{n-1}\frac{\partial q}{\partial\chi_j}\chi_j\right)}=\gamma-1,\ i=\overline{1,n-1}$$

From the first relations, multiplying by χ_j and summing with $j\neq i$:

$$\chi_i\frac{\partial q}{\partial\chi_i}\sum_{\substack{j=1\\ j\neq i}}^{n-1}\frac{\partial^2 q}{\partial\chi_i\partial\chi_j}\chi_j=\chi_i\frac{\partial^2 q}{\partial\chi_i^2}\sum_{\substack{j=1\\ j\neq i}}^{n-1}\chi_j\frac{\partial q}{\partial\chi_j}-\frac{\partial q}{\partial\chi_i}(\gamma-1)\sum_{\substack{j=1\\ j\neq i}}^{n-1}\chi_j\frac{\partial q}{\partial\chi_j}$$

Replacing in (2) we find that:

$$(3)\ r\chi_i q\frac{\partial^2 q}{\partial\chi_i^2}+(1-r)\chi_i\left(\frac{\partial q}{\partial\chi_i}\right)^2=(\gamma-1)rq\frac{\partial q}{\partial\chi_i},\ i=\overline{1,n-1}$$

After multiplying (3) with $\frac{\partial q}{\partial\chi_j}$ it follows from (1):

$$(4)\ (1-r)\left(\frac{\partial q}{\partial\chi_i}\right)^2\frac{\partial q}{\partial\chi_j}+rq\frac{\partial q}{\partial\chi_i}\frac{\partial^2 q}{\partial\chi_i\partial\chi_j}=0,\ i,j=\overline{1,n-1},\ i\neq j$$

$$(5)\ r\chi_i q\frac{\partial^2 q}{\partial\chi_i^2}+(1-r)\chi_i\left(\frac{\partial q}{\partial\chi_i}\right)^2=(\gamma-1)rq\frac{\partial q}{\partial\chi_i},\ i=\overline{1,n-1}$$

Let note: $X_i=\frac{\frac{\partial q^\rho}{\partial\chi_i}}{q^\rho}=\frac{\rho\frac{\partial q}{\partial\chi_i}}{q}=\rho\frac{\partial\ln q}{\partial\chi_i}$, $i=\overline{1,n}$ where $\rho\in\mathbf{R}^*$. We have that:

$$(6)\ \frac{\partial X_i}{\partial\chi_j}=\rho\frac{\frac{\partial^2 q}{\partial\chi_i\partial\chi_j}q-\frac{\partial q}{\partial\chi_i}\frac{\partial q}{\partial\chi_j}}{q^2}=\frac{\partial X_j}{\partial\chi_i},\ i,j=\overline{1,n}$$

Because from (6) we have: $\frac{\partial^2 q}{\partial\chi_i\partial\chi_j}=\frac{q}{\rho}\frac{\partial X_i}{\partial\chi_j}+\frac{1}{\rho q}\frac{\partial q}{\partial\chi_i}\frac{\partial q}{\partial\chi_j}$ it follows from (4) and (5):

$$(7)\ \frac{\partial X_i}{\partial\chi_j}=-\frac{r+\rho-r\rho}{r\rho^2}X_iX_j$$

$$(8)\ \frac{\partial X_i}{\partial\chi_i}=\frac{\gamma-1}{\chi_i}X_i-\frac{r+\rho-r\rho}{r\rho^2}X_i^2$$

We have now the first differential of X_i:

$$dX_i=\sum_{\substack{j=1\\ j\neq i}}^{n-1}\frac{\partial X_i}{\partial\chi_j}d\chi_j+\frac{\partial X_i}{\partial\chi_i}d\chi_i=X_i\left(\frac{r\rho-\rho-r}{r\rho q}dq+\frac{\gamma-1}{\chi_i}d\chi_i\right)$$

therefore: $d\left(\ln X_i - \frac{r\rho-\rho-r}{r\rho}\ln q - (\gamma-1)\ln\chi_i\right)=0$ from where: $X_i = C_i q^{\frac{r\rho-\rho-r}{r\rho}}\chi_i^{\gamma-1}$. Taking into account the definition of X_i we finally find that:

(9) $q^{\frac{-r\rho+\rho+r}{r\rho}} = C_i\frac{\rho+r-r\rho}{r\gamma\rho^2}\chi_i^\gamma + A_i(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1})$

Noting g= $C_i\frac{\rho+r-r\rho}{r\gamma\rho^2}\chi_i^\gamma + A_i(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1})$ we obtained from (4), (5):

(10) $$\frac{r(\rho-1)}{\rho+r-r\rho}\frac{\partial g}{\partial\chi_i}\frac{\partial g}{\partial\chi_j}+\frac{\partial^2 g}{\partial\chi_i\partial\chi_j}=0$$

(11) $$\frac{r(\rho-1)}{\rho+r-r\rho}\chi_i\left(\frac{\partial g}{\partial\chi_i}\right)^2+g\chi_i\frac{\partial^2 g}{\partial\chi_i^2}=(\gamma-1)g\frac{\partial g}{\partial\chi_i}$$

But g= $C_i\frac{\rho+r-r\rho}{r\gamma\rho^2}\chi_i^\gamma + A_i(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1})$ implies that:

$$\frac{\partial g}{\partial\chi_i}=\frac{\rho+r-r\rho}{r\rho^2}C_i\chi_i^{\gamma-1},\ \frac{\partial g}{\partial\chi_j}=\frac{\partial A_i(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1})}{\partial\chi_j},\ \frac{\partial^2 g}{\partial\chi_i\partial\chi_j}=0,\ \frac{\partial^2 g}{\partial\chi_i^2}=\frac{\rho+r-r\rho}{r\rho^2}(\gamma-1)C_i\chi_i^{\gamma-2}$$

from where:

(12) $$\frac{(\rho-1)}{\rho^2}C_i\chi_i^{\gamma-1}\frac{\partial A_i(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1})}{\partial\chi_j}=0$$

therefore $\frac{\partial A_i(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1})}{\partial\chi_j}$=0 that is A=constant.

After these considerations: $q^{\frac{-r\rho+\rho+r}{r\rho}} = C_i\frac{\rho+r-r\rho}{r\gamma\rho^2}\chi_i^\gamma + A_i$ and summing for i=$\overline{1,n-1}$:

(13) $$q^{\frac{-r\rho+\rho+r}{r\rho}}=\frac{\rho+r-r\rho}{r\gamma\rho^2}\sum_{i=1}^{n-1}C_i\chi_i^\gamma+C_n$$

Denoting $\frac{r+\rho-r\rho}{r\rho}=\zeta$ we finally have:

(14) $$q=\left(\frac{\zeta}{\gamma\rho}\sum_{i=1}^{n-1}C_i\chi_i^\gamma+C_n\right)^{\frac{1}{\zeta}}$$

therefore q is a CES production function.

Considering now again the generalized CES production: $Q(x_1,...,x_n)=\alpha\left(\sum_{i=1}^{n}\beta_i x_i^\gamma\right)^{\frac{1}{\rho}}$ let search the dependence of the parameters $\beta_1,...,\beta_n,\gamma,\rho$.

We have:

$$\frac{\partial Q}{\partial \beta_j}=\frac{\alpha}{\rho}\left(\sum_{j=1}^{n}\beta_j x_j^{\gamma}\right)^{\frac{1}{\rho}-1} x_i^{\gamma}=\frac{Qx_i^{\gamma}}{\rho\sum_{j=1}^{n}\beta_j x_j^{\gamma}}$$

$$\frac{\partial Q}{\partial \gamma}=\frac{\alpha\gamma}{\rho}\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}-1}\sum_{i=1}^{n}\beta_i x_i^{\gamma-1}=\frac{\gamma Q\sum_{i=1}^{n}\beta_i x_i^{\gamma-1}}{\rho\sum_{i=1}^{n}\beta_i x_i^{\gamma}}$$

$$\frac{\partial Q}{\partial \rho}=-\frac{\alpha}{\rho^2}\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}\ln\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)=-\frac{Q\ln\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)}{\rho^2}$$

From these relations we have that at an increasing of a parameter β_j the production Q will increase also if $\rho>0$ and decreases if $\rho<0$. Because $\gamma\rho>0$ it follows that Q will increase if γ increases. On the restriction of production's domain at $D'_p=\{(x_1,...,x_n) \mid x_i\geq 0,\ i=\overline{1,n},\ \sum_{i=1}^{n}\beta_i x_i^{\gamma}\geq 1\}$ we have that Q will decrease at an increasing of ρ and on the restriction of production's domain at $D''_p=\{(x_1,...,x_n) \mid x_i\geq 0,\ i=\overline{1,n},\ \sum_{i=1}^{n}\beta_i x_i^{\gamma}\leq 1\}$ we have that Q will increase at an increasing of ρ.

In particular, for the generalized CES function related to capital K and labor L: $Q=\alpha\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)^{\frac{1}{\rho}}$ we have that the main indicators are:

- $\eta_K=\frac{\gamma\beta_K K^{\gamma-1}Q}{\rho\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)}$, $\eta_L=\frac{\gamma\beta_L L^{\gamma-1}Q}{\rho\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)}$
- $w_K=\frac{Q}{K}$, $w_L=\frac{Q}{L}$
- RMS(K,L)= RMS(K)= $\frac{\beta_K}{\beta_L}\left(\frac{K}{L}\right)^{\gamma-1}$, RMS(L,K)= RMS(L)= $\frac{\beta_L}{\beta_K}\left(\frac{L}{K}\right)^{\gamma-1}$
- $\varepsilon_K=\frac{\gamma\beta_K K^{\gamma}}{\rho\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)}$, $\varepsilon_L=\frac{\gamma\beta_L L^{\gamma}}{\rho\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)}$
- $\sigma=\gamma-1$

If $\gamma=\rho$ and $\beta_K=\beta$, $\beta_L=1-\beta$ we obtain the main indicators of the classical CES production function:

- $\eta_K=\frac{\beta K^{\gamma-1}Q}{\beta K^{\gamma}+(1-\beta)L^{\gamma}}$, $\eta_L=\frac{(1-\beta)L^{\gamma-1}Q}{\beta K^{\gamma}+(1-\beta)L^{\gamma}}$
- $w_K=\frac{Q}{K}$, $w_L=\frac{Q}{L}$
- RMS(K,L)= RMS(K)= $\frac{\beta}{1-\beta}\left(\frac{K}{L}\right)^{\gamma-1}$, RMS(L,K)= RMS(L)= $\frac{1-\beta}{\beta}\left(\frac{L}{K}\right)^{\gamma-1}$

- $\varepsilon_K = \dfrac{\beta K^{\gamma}}{\beta K^{\gamma} + (1-\beta)L^{\gamma}}$, $\varepsilon_L = \dfrac{(1-\beta)L^{\gamma}}{\beta K^{\gamma} + (1-\beta)L^{\gamma}}$
- $\sigma = \rho - 1$

3 The costs of the generalized CES production function

Considering now the problem of minimizing costs for a given production Q_0, where the prices of inputs are p_i, $i=\overline{1,n}$, we have:

$$\begin{cases} \min \sum_{k=1}^{n} p_k x_k \\ \alpha\left(\sum_{i=1}^{n} \beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}} \ge Q_0 \\ x_1,\dots,x_n \ge 0 \end{cases}$$

From the obvious relations: $\begin{cases} \dfrac{\beta_1 x_1^{\gamma-1}}{p_1 \sum_{k=1}^{n} \beta_k x_k^{\gamma}} = \dots = \dfrac{\beta_n x_n^{\gamma-1}}{p_n \sum_{k=1}^{n} \beta_k x_k^{\gamma}} \\ \alpha\left(\sum_{i=1}^{n} \beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}} = Q_0 \end{cases}$ we obtain: $\begin{cases} x_i = \left(\dfrac{\beta_n p_i}{\beta_i p_n}\right)^{\frac{1}{\gamma-1}} x_n, i=\overline{1,n-1} \\ \alpha\left(\sum_{i=1}^{n} \beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}} = Q_0 \end{cases}$

and from the second equation: $\alpha \dfrac{\beta_n^{\frac{\gamma}{\rho(\gamma-1)}}}{p_n^{\frac{\gamma}{\rho(\gamma-1)}}} x_n^{\frac{\gamma}{\rho}} \left(\sum_{i=1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\rho}} = Q_0$.

Noting r= $\sum_{k=1}^{n} \alpha_k$ we finally obtain: $\overline{x}_i = \dfrac{p_i^{\frac{1}{\gamma-1}}}{\alpha^{\frac{\rho}{\gamma}} \beta_i^{\frac{1}{\gamma-1}} \left(\sum_{i=1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}} Q_0^{\frac{\rho}{\gamma}}$, $i=\overline{1,n}$

The total cost is:

$$TC = \sum_{i=1}^{n} p_i \overline{x}_i = \frac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}} \left(\sum_{i=1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$$

At a price change of one factor, i.e. x_k, from the value p_k to $\overline{p}_k$ we have: $\overline{TC} = \dfrac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}} \left(\beta_k^{-\frac{1}{\gamma-1}} \overline{p}_k^{\frac{\gamma}{\gamma-1}} + \sum_{\substack{i=1 \\ i \ne k}}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$

where the relative variation of the total cost is:

$$\frac{\Delta TC}{TC}=\frac{\overline{TC}-TC}{TC}=\frac{\left(\beta_k^{-\frac{1}{\gamma-1}}\overline{p}_k^{\frac{\gamma}{\gamma-1}}+\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}-\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}}{\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}}$$

Let us now consider the behavior of the total cost of production function at a parameters variation. We have:

$$\frac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{\partial TC}{\partial\beta_k}=-\frac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\frac{1}{\gamma}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{-\frac{1}{\gamma}}p_k^{\frac{\gamma}{\gamma-1}}\beta_k^{-\frac{\gamma}{\gamma-1}}$$

Therefore, if $\gamma<0$ we have that $\frac{\partial TC}{\partial\beta_k}>0$ and if $\gamma>0$: $\frac{\partial TC}{\partial\beta_k}<0$.

If we consider now for a given output Q_0, the inputs $x_1,\ldots,x_n$ such that: $\alpha\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}$ let $x_k=\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}}$. We have $STC_k=\sum_{i=1}^{n}p_i x_i=\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i+p_k\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}}$ representing the short-term total cost when factors $x_1,\ldots,\hat{x}_k,\ldots,x_n$ remain constant (^ *means that the term is missing*).

We put now the question of determining the envelope of the family of hypersurfaces:

$$f\left(Q_0,x_1,\ldots,\hat{x}_k,\ldots,x_n\right)=\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i+p_k\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}}$$

Conditions to be met are:

$$\begin{cases}TC=f\left(Q_0,x_1,\ldots,\hat{x}_k,\ldots,x_n\right)\\ \frac{\partial f}{\partial x_i}=0,\ i=\overline{1,n},\ i\neq k\end{cases}$$

After the elimination of parameters $x_1,\ldots,\hat{x}_k,\ldots,x_n$ we have either the locus of singular points of hypersurfaces (which is not the case for the present issue) or envelope sought.
We have therefore:

$$\begin{cases} TC=\sum_{\substack{i=1\\i\neq k}}^{n}p_ix_i+p_k\dfrac{\left(\dfrac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\i\neq k}}^{n}\beta_ix_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}} \\ p_i-\dfrac{p_k\beta_i\left(\dfrac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\i\neq k}}^{n}\beta_ix_i^{\gamma}\right)^{\frac{1}{\gamma}-1}x_i^{\gamma-1}}{\beta_k^{\frac{1}{\gamma}}}=0,\ i=\overline{1,n},\ i\neq k \end{cases}$$

From the second equation, we have for $i=\overline{1,n}, i\neq k$: $\dfrac{x_i^{\gamma}}{\dfrac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\i\neq k}}^{n}\beta_ix_i^{\gamma}}=\left(\dfrac{p_k^{\gamma}\beta_i^{\gamma}}{p_i^{\gamma}\beta_k}\right)^{\frac{1}{1-\gamma}}$. Multiplying with β_i

and summing for all $i=\overline{1,n}, i\neq k$: $\dfrac{\sum_{\substack{i=1\\i\neq k}}^{n}\beta_ix_i^{\gamma}}{\dfrac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\i\neq k}}^{n}\beta_ix_i^{\gamma}}=\left(\dfrac{p_k^{\gamma}}{\beta_k}\right)^{\frac{1}{1-\gamma}}\sum_{\substack{i=1\\i\neq k}}^{n}\dfrac{\beta_i^{\frac{1}{1-\gamma}}}{p_i^{\frac{\gamma}{1-\gamma}}}$ therefore:

$$\sum_{\substack{i=1\\i\neq k}}^{n}\beta_ix_i^{\gamma}=\frac{\dfrac{Q_0^{\rho}}{\alpha^{\rho}}\left(\dfrac{p_k^{\gamma}}{\beta_k}\right)^{\frac{1}{1-\gamma}}\sum_{\substack{i=1\\i\neq k}}^{n}\dfrac{\beta_i^{\frac{1}{1-\gamma}}}{p_i^{\frac{\gamma}{1-\gamma}}}}{1+\left(\dfrac{p_k^{\gamma}}{\beta_k}\right)^{\frac{1}{1-\gamma}}\sum_{\substack{i=1\\i\neq k}}^{n}p_i^{\frac{\gamma}{\gamma-1}}\beta_i^{-\frac{1}{\gamma-1}}}.$$

Replacing we have now: $x_i=\left(\dfrac{p_k\beta_i}{p_i\beta_k^{\frac{1}{\gamma}}}\right)^{\frac{1}{1-\gamma}}\dfrac{\left(\dfrac{Q_0^{\rho}}{\alpha^{\rho}}\right)^{\frac{1}{\gamma}}}{\left(1+\left(\dfrac{p_k^{\gamma}}{\beta_k}\right)^{\frac{1}{1-\gamma}}\sum_{\substack{i=1\\i\neq k}}^{n}p_i^{\frac{\gamma}{\gamma-1}}\beta_i^{-\frac{1}{\gamma-1}}\right)^{\frac{1}{\gamma}}}$.

From the first equation: $TC=\dfrac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$.

We obtained so that the envelope of the family of hypersurfaces of the short-term total cost when all inputs are constant except one is just the long-term cost obtained from nonlinear optimization problem with respect to the minimizing of the cost for a given production.

Calculating the costs derived from the (long-term or short-term) total cost now, we have:

$ATC=\dfrac{TC}{Q_0}=\dfrac{Q_0^{\frac{\rho}{\gamma}-1}}{\alpha^{\frac{\rho}{\gamma}}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$ (*average long-term total cost*)

$$MTC=\frac{\partial TC}{\partial Q_0}=\frac{\rho Q_0^{\frac{\rho}{\gamma}-1}}{\gamma\alpha^{\frac{\rho}{\gamma}}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}=\frac{\rho}{\gamma}ATC$$ (*marginal long-term total cost*)

$$ASTC_k=\frac{STC_k}{Q_0}=\frac{\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i}{Q_0}+p_k\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}Q_0}$$ (*average short-term total cost*)

$$MC_k=\frac{\partial STC_k}{\partial Q_0}=\frac{\rho p_k\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}-1}Q_0^{\rho-1}}{\gamma\alpha^{\rho}\beta_k^{\frac{1}{\gamma}}}$$ (*marginal short-term total cost*)

$$VTC_k=p_k\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}}$$ (*variable short-term total cost*)

$$AVTC_k=\frac{VTC_k}{Q_0}=p_k\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_k^{\frac{1}{\gamma}}Q_0}$$ (*average variable short-term total cost*)

$$FTC_k=\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i$$ (*fixed short-term total cost*)

$$AFTC_k=\frac{FTC_k}{Q_0}=\frac{\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i}{Q_0}$$ (*average fixed short-term total cost*)

Finally we have:

$$\varepsilon_{p_k}=\frac{\frac{\partial TC}{\partial p_k}}{\frac{TC}{p_k}}=\frac{\beta_k^{-\frac{1}{\gamma-1}}p_k^{\frac{\gamma}{\gamma-1}}}{\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}}$$ - the coefficient of elasticity of long-term total cost with respect to the price factor k

$$\varepsilon_Q=\frac{\frac{\partial TC}{\partial Q_0}}{\frac{TC}{Q_0}}=\frac{\rho}{\gamma}$$ - the coefficient of elasticity of long-term total cost with respect to the production Q_0

$$\varepsilon_{av,p_k}=\frac{\frac{\partial ATC}{\partial p_k}}{\frac{ATC}{p_k}}=\frac{\beta_k^{-\frac{1}{\gamma-1}}p_k^{\frac{\gamma}{\gamma-1}}}{\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}}=\varepsilon_{p_k}$$ - the coefficient of elasticity of average long-term total cost with respect to the price factor k

$$\varepsilon_{marg,p_k}=\frac{\frac{\partial MTC}{\partial p_k}}{\frac{MTC}{p_k}}=\frac{\beta_k^{-\frac{1}{\gamma-1}}p_k^{\frac{\gamma}{\gamma-1}}}{\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}}=\varepsilon_{p_k}$$ - the coefficient of elasticity of marginal long-term total cost with respect to the price factor k

In particular, for the generalized CES function related to capital K and labor L: $Q=\alpha\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)^{\frac{1}{\rho}}$ we have:

$$\overline{K}=\frac{p_K^{\frac{1}{\gamma-1}}}{\alpha^{\frac{\rho}{\gamma}}\beta_K^{\frac{1}{\gamma-1}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}Q_0^{\frac{\rho}{\gamma}}$$

$$\overline{L}=\frac{p_L^{\frac{1}{\gamma-1}}}{\alpha^{\frac{\rho}{\gamma}}\beta_L^{\frac{1}{\gamma-1}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}Q_0^{\frac{\rho}{\gamma}}$$

$$TC=\frac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$$

On the short-term, we have for constancy of K: $STC_L=p_K K+p_L\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\beta_K K^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_L^{\frac{1}{\gamma}}}$ and

$$ATC=\frac{Q_0^{\frac{\rho}{\gamma}-1}}{\alpha^{\frac{\rho}{\gamma}}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$$

$$MTC=\frac{\rho Q_0^{\frac{\rho}{\gamma}-1}}{\gamma\alpha^{\frac{\rho}{\gamma}}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$$

$$ASTC_L=\frac{p_K K}{Q_0}+p_L\frac{\left(\frac{Q_0^{\rho}}{\alpha^{\rho}}-\beta_K K^{\gamma}\right)^{\frac{1}{\gamma}}}{\beta_L^{\frac{1}{\gamma}}Q_0}$$

$$MC_L=\frac{\rho p_L\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta_K K^\gamma\right)^{\frac{1}{\gamma}-1}Q_0^{\rho-1}}{\gamma\alpha^\rho\beta_L^{\frac{1}{\gamma}}}$$

$$VTC_L=p_L\frac{\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta_K K^\gamma\right)^{\frac{1}{\gamma}}}{\beta_L^{\frac{1}{\gamma}}}$$

$$AVTC_L=p_L\frac{\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta_K K^\gamma\right)^{\frac{1}{\gamma}}}{\beta_L^{\frac{1}{\gamma}}Q_0}$$

$$FTC_L=p_K K$$

$$AFTC_L=\frac{p_K K}{Q_0}$$

$$\varepsilon_{p_K}=\frac{\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}}{\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}},\quad \varepsilon_{p_L}=\frac{\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}}{\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}},\quad \varepsilon_Q=\frac{\rho}{\gamma},\quad \varepsilon_{av,p_K}=\varepsilon_{marg,p_K}=\varepsilon_{p_K},\quad \varepsilon_{av,p_L}=\varepsilon_{marg,p_L}=\varepsilon_{p_L}.$$

In particular, for $\gamma=\rho$ and $\beta_K=\beta$, $\beta_L=1-\beta$ we obtain for the classical CES production function:

$$\overline{K}=\frac{p_K^{\frac{1}{\rho-1}}}{\alpha\beta_K^{\frac{1}{\rho-1}}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}Q_0$$

$$\overline{L}=\frac{p_L^{\frac{1}{\rho-1}}}{\alpha\beta_L^{\frac{1}{\rho-1}}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}Q_0$$

$$TC=\frac{Q_0}{\alpha}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$STC_L=p_K K+p_L\frac{\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta K^\rho\right)^{\frac{1}{\rho}}}{(1-\beta)^{\frac{1}{\rho}}}$$

$$ATC=\frac{1}{\alpha}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$MTC=\frac{\rho}{\gamma\alpha}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$ASTC_L=\frac{p_K K}{Q_0}+p_L\frac{\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta K^\rho\right)^{\frac{1}{\rho}}}{(1-\beta)^{\frac{1}{\rho}}Q_0}$$

$$MC_L=\frac{\rho p_L\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta K^\rho\right)^{\frac{1}{\rho}-1}Q_0^{\rho-1}}{\gamma\alpha^\rho(1-\beta)^{\frac{1}{\rho}}}$$

$$VTC_L=p_L\frac{\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta K^\rho\right)^{\frac{1}{\rho}}}{(1-\beta)^{\frac{1}{\rho}}}$$

$$AVTC_L=p_L\frac{\left(\frac{Q_0^\rho}{\alpha^\rho}-\beta K^\rho\right)^{\frac{1}{\rho}}}{(1-\beta)^{\frac{1}{\rho}}Q_0}$$

$$FTC_L=p_K K$$

$$AFTC_L=\frac{p_K K}{Q_0}$$

$$\varepsilon_{p_K}=\frac{\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}}{\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}},\quad \varepsilon_{p_L}=\frac{(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}}{\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}},\quad \varepsilon_Q=1,\quad \varepsilon_{av,p_K}=\varepsilon_{marg,p_K}=\varepsilon_{p_K},$$

$$\varepsilon_{av,p_L}=\varepsilon_{marg,p_L}=\varepsilon_{p_L}.$$

4 The profit

Now consider a sale price of output Q_0: $p(Q_0)$. The profit is therefore:

$$\Pi(Q_0)=p(Q_0)\cdot Q_0-TC(Q_0)$$

It is known that in a market with perfect competition, the price is given and equals marginal cost. The profit on long-term becomes:

$$\Pi(Q_0)=p(Q_0)\cdot Q_0-TC(Q_0)=MTC(Q_0)\cdot Q_0-TC(Q_0)=ATC(Q_0)Q_0^2=$$

$$\left(\frac{\rho}{\gamma}-1\right)\frac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$$

In particular, for the generalized CES function related to capital K and labor L: Q= $\alpha\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)^{\frac{1}{\rho}}$ we have: $\Pi(Q_0)=\left(\frac{\rho}{\gamma}-1\right)\frac{Q_0^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$ and for the classical CES function ($\gamma=\rho$ and $\beta_K=\beta$, $\beta_L=1-\beta$): $\Pi(Q_0)=0$.

On short-term, when factors $x_1,...,\hat{x}_k,...,x_n$ remain constant, we have:

$$\Pi(Q_0)=p(Q_0)\cdot Q_0-STC_k(Q_0)=MTC(Q_0)\cdot Q_0-STC_k(Q_0)=AVTC_k'(Q_0)Q_0^2-FTC_k$$

therefore:

$$\Pi(Q_0)=p_k\left[\left(2\frac{\rho}{\gamma}-1\right)x_k+\frac{\rho x_k^{1-\gamma}}{\gamma\beta_k}\sum_{\substack{i=1\\i\neq k}}^{n}\beta_i x_i^{\gamma}\right]-\sum_{\substack{i=1\\i\neq k}}^{n}p_i x_i$$

For Q= $\alpha\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)^{\frac{1}{\rho}}$ we have that if K=constant:

$\Pi(Q_0)=p_L L\left[\left(2\frac{\rho}{\gamma}-1\right)+\frac{\rho\beta_K K^{\gamma}}{\gamma\beta_L L^{\gamma}}\right]-p_K K$ and if $\gamma=\rho$ and $\beta_K=\beta$, $\beta_L=1-\beta$ for classical CES: $\Pi(Q_0)=$ $p_L L\left(1+\frac{\beta K^{\rho}}{(1-\beta)L^{\rho}}\right)-p_K K$

The condition of profit maximization for an arbitrarily price p, depending on the factors of production, is: $\max\Pi(x_1,...,x_n)=\max\left(pQ(x_1,...,x_n)-\sum_{i=1}^{n}p_i x_i\right)$ from where $\frac{\partial Q}{\partial x_i}=\frac{p_i}{p}$, $i=\overline{1,n}$ or otherwise: $\frac{\gamma\beta_i x_i^{\gamma-1}Q}{\rho\sum_{k=1}^{n}\beta_k x_k^{\gamma}}=\frac{p_i}{p}$ from where: $x_i=\left(\frac{p_i\rho}{p\gamma\beta_i Q}\sum_{k=1}^{n}\beta_k x_k^{\gamma}\right)^{\frac{1}{\gamma-1}}$. But Q= $\alpha\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}$ implies:

$$\sum_{k=1}^{n}\beta_k x_k^{\gamma}=\frac{p\gamma Q^{\frac{\gamma+\rho\gamma-\rho}{\gamma}}}{\rho\alpha^{\frac{\rho(\gamma-1)}{\gamma}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}}\quad\text{therefore: }\overline{x}_i=\frac{p_i^{\frac{1}{\gamma-1}}Q^{\frac{\rho}{\gamma}}}{\beta_i^{\frac{1}{\gamma-1}}\alpha^{\frac{\rho}{\gamma}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}.$$

Because Q is quasi-concave the solution of the characteristic system is the unique point of maximum.

The maximum profit is: $\Pi(\overline{x}_1,...,\overline{x}_n)=pQ-\frac{Q^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}$.

For $Q=\alpha\left(\beta_K K^{\gamma}+\beta_L L^{\gamma}\right)^{\frac{1}{\rho}}$ we have that: $\overline{K}=\dfrac{p_K^{\frac{1}{\gamma-1}}Q^{\frac{\rho}{\gamma}}}{\beta_K^{\frac{1}{\gamma-1}}\alpha^{\frac{\rho}{\gamma}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}$,

$$\overline{L}=\frac{p_L^{\frac{1}{\gamma-1}}Q^{\frac{\rho}{\gamma}}}{\beta_L^{\frac{1}{\gamma-1}}\alpha^{\frac{\rho}{\gamma}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}},\ \Pi\left(\overline{K},\overline{L}\right)=pQ-\frac{Q^{\frac{\rho}{\gamma}}}{\alpha^{\frac{\rho}{\gamma}}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}}+\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}}.$$

If now $\gamma=\rho$ and $\beta_K=\beta$, $\beta_L=1-\beta$ for classical CES:

$$\overline{K}=\frac{p_K^{\frac{1}{\rho-1}}Q}{\alpha\beta^{\frac{1}{\rho-1}}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}},\quad \overline{L}=\frac{p_L^{\frac{1}{\rho-1}}Q}{\alpha\beta^{\frac{1}{\rho-1}}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}},\quad \Pi\left(\overline{K},\overline{L}\right)=$$

$$pQ-\frac{Q}{\alpha}\left(\beta^{-\frac{1}{\rho-1}}p_K^{\frac{\rho}{\rho-1}}+(1-\beta)^{-\frac{1}{\rho-1}}p_L^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}.$$

5 The Hicks and Slutsky effects for the generalized CES production function

Now consider the production function $Q(x_1,\ldots,x_n)=\alpha\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}$ and factor prices $\left(p_i\right)_{i=\overline{1,n}}$. The non-linear programming problem relative to maximize production at a given total cost (CT_0) is:

$$\begin{cases}\max\alpha\left(\sum_{i=1}^{n}\beta_i x_i^{\gamma}\right)^{\frac{1}{\rho}}\\ \sum_{k=1}^{n}p_k x_k=CT_0\\ x_1,\ldots,x_n\geq 0\end{cases}$$

Because the objective function is quasi-concave and also the restriction (being affine) and the partial derivatives are all positive we find that the Karush-Kuhn-Tucker conditions are also sufficient. Therefore, we have:

$$\begin{cases}\dfrac{\frac{\partial Q}{\partial x_1}\left(\overline{x}_1,\ldots,\overline{x}_n\right)}{p_1}=\ldots=\dfrac{\frac{\partial Q}{\partial x_n}\left(\overline{x}_1,\ldots,\overline{x}_n\right)}{p_n}\\ \sum_{k=1}^{n}p_k x_k=CT_0\end{cases}$$

From the first equations we obtain:

$$\begin{cases} \frac{\beta_1 x_1^{\gamma-1}}{p_1} = \ldots = \frac{\beta_n x_n^{\gamma-1}}{p_n} \\ \sum_{k=1}^{n} p_k x_k = CT_0 \end{cases}$$

therefore:

$$\begin{cases} x_k = \left(\frac{p_k \beta_n}{p_n \beta_k}\right)^{\frac{1}{\gamma-1}} x_n, \; k = \overline{1, n-1} \\ \sum_{k=1}^{n} p_k x_k = CT_0 \end{cases}$$

Substituting the first n-1 relations into the last we finally find that:

$$x_{0,k} = \frac{\beta_k^{-\frac{1}{\gamma-1}} p_k^{\frac{1}{\gamma-1}} CT_0}{\sum_{i=1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}}, \; k=\overline{1,n}$$ and the appropriate production: $Q_0(x_1,\ldots,x_n)= \alpha\left(\sum_{i=1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\rho}} CT_0^{\frac{\gamma}{\rho}}$.

Suppose now that some of the prices of factors of production (possibly after renumbering, we may assume that they are: $x_1,\ldots,x_s$) is modified to values $\overline{p}_1,,,.\overline{p}_s$, the rest remain constant.

From the above, it results:

$$\begin{cases} x_{f,k} = \frac{\beta_k^{-\frac{1}{\gamma-1}} \overline{p}_k^{\frac{1}{\gamma-1}} CT_0}{\sum_{i=1}^{s} \beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}}, \; k = \overline{1,s} \\ x_{f,k} = \frac{\beta_k^{-\frac{1}{\gamma-1}} p_k^{\frac{1}{\gamma-1}} CT_0}{\sum_{i=1}^{s} \beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}}, \; k = \overline{s+1,n} \\ Q_f = \alpha\left(\sum_{i=1}^{s} \beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\rho}} CT_0^{\frac{\gamma}{\rho}} \end{cases}$$

We will apply in the following, the method of Hicks. To an input price change, let consider that the production remains unchanged, leading thus to a change of the total cost. We therefore have:

$$\alpha\left(\sum_{i=1}^{s} \beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\rho}} \overline{CT}_0^{\frac{\gamma}{\rho}} = \alpha\left(\sum_{i=1}^{n} \beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\rho}} CT_0^{\frac{\gamma}{\rho}}$$

from where:

$$\overline{CT}_0 = \frac{\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}} CT_0$$

With the new total cost, the optimal amounts of inputs become:

$$\begin{cases} x_{int,k} = \dfrac{\beta_k^{-\frac{1}{\gamma-1}} \overline{p}_k^{\frac{1}{\gamma-1}} \left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}} CT_0, \; k = \overline{1,s} \\ x_{int,d} = \dfrac{\beta_d^{-\frac{1}{\gamma-1}} p_d^{\frac{1}{\gamma-1}} \left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}} CT_0, \; d = \overline{s+1,n} \end{cases}$$

The Hicks substitution effect which preserves the production is therefore:

$$\begin{cases} \Delta_{1H} x_k = x_{int,k} - x_{0,k} = \beta_k^{-\frac{1}{\gamma-1}} \overline{p}_k^{\frac{1}{\gamma-1}} \dfrac{\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}} - \left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}} \sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}} CT_0, \; k = \overline{1,s} \\ \Delta_{1H} x_d = x_{int,d} - x_{0,d} = \beta_d^{-\frac{1}{\gamma-1}} p_d^{\frac{1}{\gamma-1}} \dfrac{\left(\sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}} - \left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}} \overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}} \sum_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}} p_i^{\frac{\gamma}{\gamma-1}}} CT_0, \; d = \overline{s+1,n} \end{cases}$$

The difference caused by the old cost instead the new total cost one is therefore:

$$\begin{cases}\Delta_{2H}x_k = x_{f,k} - x_{int,k} = \beta_k^{-\frac{1}{\gamma-1}}\overline{p}_k^{\frac{1}{\gamma-1}}\dfrac{\left(\sum\limits_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum\limits_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}} - \left(\sum\limits_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\sum\limits_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum\limits_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}CT_0,\ k=\overline{1,s} \\ \Delta_{2H}x_d = x_{f,d} - x_{int,d} = \beta_d^{-\frac{1}{\gamma-1}}p_d^{\frac{1}{\gamma-1}}\dfrac{\left(\sum\limits_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum\limits_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}} - \left(\sum\limits_{i=1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\sum\limits_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum\limits_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}CT_0,\ d=\overline{s+1,n}\end{cases}$$

For $Q=\alpha\left(\beta_K K^{\gamma} + \beta_L L^{\gamma}\right)^{\frac{1}{\rho}}$ we have that, for s=1 (that is for a capital price change):

$$\begin{cases}\Delta_{1H}K = K_{int} - K_0 = \beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}}\dfrac{\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}} - \left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)}CT_0 \\ \Delta_{1H}L = L_{int} - L_0 = \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{1}{\gamma-1}}\dfrac{\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}} - \left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}\left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)}CT_0\end{cases}$$

$$\begin{cases}\Delta_{2H}K = K_f - K_{int} = \beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}}\dfrac{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}} - \left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}CT_0 \\ \Delta_{2H}L = L_f - L_{int} = \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{1}{\gamma-1}}\dfrac{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}} - \left(\beta_K^{-\frac{1}{\gamma-1}}p_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1-\gamma}{\gamma}}}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^{\frac{1}{\gamma}}}CT_0\end{cases}$$

We shall apply now the Slutsky method for our analysis.

At the modify of the price of the factors $x_1,\ldots,x_s$, the total cost for the same optimal combination of factors is:

$$CT_{int}=\frac{\sum_{j=1}^{s}p_j\beta_j^{-\frac{1}{\gamma-1}}\overline{p}_j^{\frac{1}{\gamma-1}}+\sum_{j=s+1}^{n}p_j\beta_j^{-\frac{1}{\gamma-1}}p_j^{\frac{1}{\gamma-1}}}{\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}}CT_0$$

therefore:

$$\begin{cases} x_{int,k}=\dfrac{\beta_k^{-\frac{1}{\gamma-1}}\overline{p}_k^{\frac{1}{\gamma-1}}\left(\sum_{j=1}^{s}p_j\beta_j^{-\frac{1}{\gamma-1}}\overline{p}_j^{\frac{1}{\gamma-1}}+\sum_{j=s+1}^{n}p_j\beta_j^{-\frac{1}{\gamma-1}}p_j^{\frac{1}{\gamma-1}}\right)CT_0}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^2},\ k=\overline{1,s} \\ x_{int,d}=\dfrac{\beta_d^{-\frac{1}{\gamma-1}}p_d^{\frac{1}{\gamma-1}}\left(\sum_{j=1}^{s}p_j\beta_j^{-\frac{1}{\gamma-1}}\overline{p}_j^{\frac{1}{\gamma-1}}+\sum_{j=s+1}^{n}p_j\beta_j^{-\frac{1}{\gamma-1}}p_j^{\frac{1}{\gamma-1}}\right)CT_0}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^2},\ d=\overline{s+1,n} \end{cases}$$

The appropriate production is:

$$Q_{int}\left(x_{int,1},\ldots,x_{int,n}\right)=\frac{\alpha\left(\sum_{j=1}^{s}\overline{p}_j\beta_j^{-\frac{1}{\gamma-1}}\overline{p}_j^{\frac{1}{\gamma-1}}+\sum_{j=s+1}^{n}p_j\beta_j^{-\frac{1}{\gamma-1}}p_j^{\frac{1}{\gamma-1}}\right)^{\frac{\gamma}{\rho}}CT_0^{\frac{\gamma}{\rho}}}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^{\frac{2\gamma-1}{\rho}}}$$

The Slutsky substitution effect which not preserves the production is therefore:

$$\begin{cases} \Delta_{1S}x_k=x_{int,k}-x_{0,k}= \\ \beta_k^{-\frac{1}{\gamma-1}}\dfrac{\overline{p}_k^{\frac{1}{\gamma-1}}\left(\sum_{j=1}^{s}p_j\beta_j^{-\frac{1}{\gamma-1}}\overline{p}_j^{\frac{1}{\gamma-1}}+\sum_{j=s+1}^{n}p_j\beta_j^{-\frac{1}{\gamma-1}}p_j^{\frac{1}{\gamma-1}}\right)-p_k^{\frac{1}{\gamma-1}}\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^2}CT_0,\ k=\overline{1,s} \\ \Delta_{1S}x_d=x_{int,d}-x_{0,d}=\beta_d^{-\frac{1}{\gamma-1}}p_d^{\frac{1}{\gamma-1}}\dfrac{\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{1}{\gamma-1}}\left(p_i-\overline{p}_i\right)}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}}+\sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^2}CT_0,\ d=\overline{s+1,n} \end{cases}$$

and the difference caused by the old production instead the new production one is therefore:

$$\begin{cases} \Delta_{2S}x_k = x_{f,k} - x_{int,k} = \beta_k^{-\frac{1}{\gamma-1}}\overline{p}_k^{\frac{1}{\gamma-1}} \dfrac{\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{1}{\gamma-1}}(\overline{p}_i - p_i)}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^2} CT_0, k = \overline{1,s} \\ \Delta_{2S}x_d = x_{f,d} - x_{int,d} = \beta_d^{-\frac{1}{\gamma-1}}p_d^{\frac{1}{\gamma-1}} \dfrac{\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{1}{\gamma-1}}(\overline{p}_i - p_i)}{\left(\sum_{i=1}^{s}\beta_i^{-\frac{1}{\gamma-1}}\overline{p}_i^{\frac{\gamma}{\gamma-1}} + \sum_{i=s+1}^{n}\beta_i^{-\frac{1}{\gamma-1}}p_i^{\frac{\gamma}{\gamma-1}}\right)^2} CT_0, d = \overline{s+1,n} \end{cases}$$

For $Q=\alpha\left(\beta_K K^\gamma + \beta_L L^\gamma\right)^{\frac{1}{\rho}}$ we have that, for s=1 (that is for a capital price change):

$$\begin{cases} \Delta_{1S}K = K_{int} - K_0 = \\ \beta_K^{-\frac{1}{\gamma-1}} \dfrac{\overline{p}_K^{\frac{1}{\gamma-1}}\left(p_K\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}} + p_L\beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{1}{\gamma-1}}\right) - p_k^{\frac{1}{\gamma-1}}\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^2} CT_0 \\ \Delta_{1S}L = L_{int} - L_0 = \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{1}{\gamma-1}} \dfrac{\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}}(p_K - \overline{p}_K)}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^2} CT_0 \end{cases}$$

$$\begin{cases} \Delta_{2S}K = K_f - K_{int} = \beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}} \dfrac{\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}}(\overline{p}_K - p_K)}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^2} CT_0 \\ \Delta_{2S}L = L_f - L_{int} = \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{1}{\gamma-1}} \dfrac{\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{1}{\gamma-1}}(\overline{p}_K - p_K)}{\left(\beta_K^{-\frac{1}{\gamma-1}}\overline{p}_K^{\frac{\gamma}{\gamma-1}} + \beta_L^{-\frac{1}{\gamma-1}}p_L^{\frac{\gamma}{\gamma-1}}\right)^2} CT_0 \end{cases}$$

6 Production efficiency of generalized CES production function

Let now two generalized CES production functions for two goods Φ, Ψ and a number of n inputs $F_1,\dots,F_n$ available in quantities $\overline{x}_1,\dots,\overline{x}_n$. Production functions of Φ or Ψ are:

$$Q_\Phi(x_1,\dots,x_n) = A\left(\sum_{i=1}^{n}\alpha_i x_i^\gamma\right)^{\frac{1}{\rho}},\ Q_\Psi(x_1,\dots,x_n) = B\left(\sum_{i=1}^{n}\beta_i x_i^\gamma\right)^{\frac{1}{\nu}}$$

appropriate to the consumption of x_k units of factor F_k, $k=\overline{1,n}$.

We have seen that: $\eta_{\phi,x_i} = \dfrac{\gamma\alpha_i x_i^{\gamma-1}Q_\Phi}{\rho\sum_{k=1}^{n}\alpha_k x_k^\gamma}$, $\eta_{\Psi,x_i} = \dfrac{\mu\beta_i x_i^{\gamma-1}Q_\Psi}{\nu\sum_{k=1}^{n}\beta_k x_k^\gamma}$, $i=\overline{1,n}$.

The production contract curve satisfies:

$$\frac{\eta_{\phi,x_i}}{\eta_{\Psi,x_i}} = \frac{A\gamma\alpha_i x_i^{\gamma-1}\nu\sum_{k=1}^{n}\beta_k\left(\overline{x}_k - x_k\right)^{\mu}\left(\sum_{i=1}^{n}\alpha_i x_i^\gamma\right)^{\frac{1}{\rho}}}{B\gamma\beta_i\left(\overline{x}_i - x_i\right)^{\gamma-1}\rho\sum_{k=1}^{n}\alpha_k x_k^\gamma\left(\sum_{i=1}^{n}\beta_i\left(\overline{x}_i - x_i\right)^\gamma\right)^{\frac{1}{\nu}}} = \mu,\ i=\overline{1,n}$$

Dividing for i≠j: $x_j = \dfrac{\overline{x}_j}{\left(\dfrac{\alpha_j\beta_i}{\alpha_i\beta_j}\right)^{\frac{1}{\gamma-1}}\dfrac{\overline{x}_i - x_i}{x_i}+1}$ and for i=1: $x_j = \dfrac{\overline{x}_j}{\left(\dfrac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\dfrac{\overline{x}_1 - x_1}{x_1}+1}$, $j=\overline{2,n}$. Finally, for $x_1=\lambda$ we have the equation of production contract curve:

$$\begin{cases} x_1 = \lambda \\ x_j = \dfrac{\overline{x}_j}{\left(\dfrac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\dfrac{\overline{x}_1 - \lambda}{\lambda}+1} \end{cases},\ \lambda \in \mathbf{R}$$

If we consider now the input prices: $p_1,\dots,p_n$ we have that for the production contract curve: $x_1=g_1(\lambda),\dots,x_n=g_n(\lambda)$, $\lambda\in\mathbf{R}$:

$$\begin{cases} x_1 = g_1(\lambda) = \lambda \\ x_j = g_j(\lambda) = \dfrac{\overline{x}_j}{\left(\dfrac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\dfrac{\overline{x}_1 - \lambda}{\lambda}+1} \end{cases},\ \lambda \in \mathbf{R}$$

and:

$$p_j = \frac{\eta_{\Phi,x_j}(g_1(\lambda),\dots,g_n(\lambda))}{\eta_{\Phi,x_1}(g_1(\lambda),\dots,g_n(\lambda))}\nu = \frac{\alpha_j\overline{x}_j^{\gamma-1}}{\alpha_1\lambda^{\gamma-1}\left(\left(\dfrac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\dfrac{\overline{x}_1-\lambda}{\lambda}+1\right)^{\gamma-1}}\nu,\ j=\overline{1,n}.$$

For ν=1 we then obtain: $p_1=1$, $p_j = \dfrac{\alpha_j\overline{x}_j^{\gamma-1}}{\alpha_1\lambda^{\gamma-1}\left(\left(\dfrac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\dfrac{\overline{x}_1-\lambda}{\lambda}+1\right)^{\gamma-1}}$, $j=\overline{2,n}$.

If the initial allocation of factors of production was $x_\Phi = (a_1,...,a_n)$ we have that $\sum_{j=1}^{n} p_j(a_j - x_j) = 0$

therefore: $$(a_1-\lambda)+\sum_{j=2}^{n}\frac{\alpha_j\overline{x}_j^{\gamma-1}\left(a_j\left(\frac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\frac{\overline{x}_1-\lambda}{\lambda}+a_j-\overline{x}_j\right)}{\alpha_1\lambda^{\gamma-1}\left(\left(\frac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\frac{\overline{x}_1-\lambda}{\lambda}+1\right)^{2(\gamma-1)}}=0$$ from where it follows λ^*.

For this value we find now the final allocation: $$\begin{cases} x_1 = \lambda^* \\ x_j = g_j(\lambda) = \dfrac{\overline{x}_j}{\left(\frac{\alpha_j\beta_1}{\alpha_1\beta_j}\right)^{\frac{1}{\gamma-1}}\frac{\overline{x}_1-\lambda^*}{\lambda}+1} \end{cases}$$

7 The concrete determination of the generalized CES production function

Considering an affine function: $f:\mathbf{R}^n \to \mathbf{R}$, $f(x_1,...,x_n)=\beta_1x_1+...+\beta_nx_n+\beta_{n+1}$ and a set of m>n+1 data: $(x_{1,k},...,x_{n,k},f_k)$, $k=\overline{1,m}$ the problem of determining β_i, $i=\overline{1,n+1}$ using the least square method is to minimize the expression: $\sum_{k=1}^{m}(\beta_1x_{1,k}+...+\beta_nx_{n,k}+\beta_{n+1}-f_k)^2$ that is to solve the system:

$$\begin{cases} \beta_1\sum_{k=1}^{m}x_{1,k}x_{i,k}+...+\beta_n\sum_{k=1}^{m}x_{n,k}x_{i,k}+\beta_{n+1}\sum_{k=1}^{m}x_{i,k}=\sum_{k=1}^{m}f_kx_{i,k}, i=\overline{1,n} \\ \beta_1\sum_{k=1}^{m}x_{1,k}+...+\beta_n\sum_{k=1}^{m}x_{n,k}+m\beta_{n+1}=\sum_{k=1}^{m}f_k \end{cases}$$

Considering the matrix:

$$\Theta=\begin{pmatrix} \sum_{k=1}^{m}(x_{1,k})^2 & \sum_{k=1}^{m}x_{1,k}x_{2,k} & ... & \sum_{k=1}^{m}x_{1,k} \\ \sum_{k=1}^{m}x_{1,k}x_{2,k} & \sum_{k=1}^{m}(x_{2,k})^2 & ... & \sum_{k=1}^{m}x_{2,k} \\ ... & ... & ... & ... \\ \sum_{k=1}^{m}x_{1,k} & \sum_{k=1}^{m}x_{2,k} & ... & m \end{pmatrix}$$

and Θ_{ij} the cofactor of the (i,j)-element in Θ we will obtain:

$$\beta_i=\frac{\Theta_{1i}\sum_{k=1}^{m}f_kx_{1,k}+...+\Theta_{ni}\sum_{k=1}^{m}f_kx_{n,k}+\Theta_{n+1,i}\sum_{k=1}^{m}f_k}{\det\Theta}, i=\overline{1,n+1}$$

Considering now a production function $Q(x_1,...,x_n)=A\left(\sum_{i=1}^{n}\alpha_ix_i^\gamma\right)^{\frac{1}{\rho}}$ we put the problem of concrete determination of the parameters A, γ, ρ, α_i, $i=\overline{1,n}$.

Let therefore a set of m>n+1 data: $(x_1^k,...,x_n^k,Q^k)$, $k=\overline{1,m}$.

Considering the equation of Q in the form: $Q^{\rho} = A^{\rho}\left(\sum_{i=1}^{n}\alpha_i x_i^{\gamma}\right)$ we will take for the beginning A=1 (because A will enter in the structure of α_i) and we have: $Q^{\rho} = \sum_{i=1}^{n}\alpha_i x_i^{\gamma}$.

Considering fixed ρ and γ, we will modify the data set to the new one: $\left(x_{1,k}^{\gamma},...,x_{n,k}^{\gamma},Q_k^{\rho}\right)$, $k=\overline{1,m}$.

From above, we will obtain the values of α_i, $i=\overline{1,n}$. For an accurate determination of Q we will vary the values of ρ and γ till we will find the maximum value of the correlation coefficient.

8 Conclusions

The above analysis reveals several aspects. On the one hand were highlighted conditions for the existence of the generalized CES function. Also were calculated the main indicators of it and short and long-term costs. It has also been studied the dependence of long-term cost of the parameters of the production function. The determination of profit was made both for perfect competition market and maximize its conditions. Also we have studied the effects of Hicks and Slutsky and the production efficiency problem.

9 References

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