An analysis of the substitution effect and of revenue effect in the case of the consumer's theory provided with an Allen utility function

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Abstract. In the consumer's theory, a crucial problem is to determine the substitution effect and the revenue effect in the case of one good price's modifing. There exists two theories due to John Richard Hicks and Eugen Slutsky which allocates differents shares of the total change of the consumption to these effects. The paper makes an analysis between the two effects, considering the general case of an Allen utility function.

Keywords: Allen, substitution, revenue, utility

1 Introduction

A central point of the producer's theory is the study of manufacturer production functions [1], [3], [6], [7].

In the consumer's theory, an important problem is to determine the substitution effect and the revenue effect in the case of one good price's modifing.

The theory due to John Richard Hicks consider, after a price modifing, first a new allocation of goods preserving the utility, but modifing the revenue and after taking into account that the revenue is the initial one the changing in allocation due to a different utility.

The theory of Eugen Slutsky consider a combined displacement of the relative consuming obtained a share of the substitution effect or of revenue effect depending only from the parameters of the utility.

The problem is to determine these shares for both methods and to inquire which effect is uppermost.

2 The analysis

Let two goods A and B with the initial prices p_A and p_B and an utility function of an Allen type:

$$U = d\sqrt{aX^2 + 2bXY + cY^2}\ ,\ a,b,c,d>0,\ b^2<ac$$

where X and Y are the consumed quantities of A and B respectively, in order to obtain an utility U. Let also, at a given time, V – the consumer's revenue.

In order to have the maximum utility for the revenue V it is known that we must have:

$$\begin{cases} \dfrac{U_{mA}}{U_{mB}} = \dfrac{p_A}{p_B} \\ V = p_A X + p_B Y \end{cases}$$

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where $U_{mA}=\frac{d(aX+bY)}{\sqrt{aX^2+2bXY+cY^2}}$ and $U_{mB}=\frac{d(bX+cY)}{\sqrt{aX^2+2bXY+cY^2}}$ are the marginal utilities corresponding to the two goods A and B respectively.

Let note now: $s=\frac{p_A}{p_B}$.

We have now:

$$\begin{cases} \frac{aX+bY}{bX+cY}=\frac{p_A}{p_B} \\ V=p_A X+p_B Y \end{cases}$$

from where:

$$\begin{cases} X_1=\frac{cp_A-bp_B}{cp_A^2-2bp_Ap_B+ap_B^2}V=\frac{s(cs-b)}{p_A(cs^2-2bs+a)}V \\ Y_1=\frac{ap_B-bp_A}{cp_A^2-2bp_Ap_B+ap_B^2}V=\frac{s(a-bs)}{p_A(cs^2-2bs+a)}V \end{cases}$$

and the utility:

$$U(X_1,Y_1)=Vd\sqrt{\frac{ac-b^2}{cp_A^2-2bp_Ap_B+ap_B^2}}=\frac{Vds}{p_A}\sqrt{\frac{ac-b^2}{cs^2-2bs+a}}$$

Let suppose now that it is a change in the price of one of the goods, let say B, from p_B to p'_B, but the revenue V remains constant. Let note now: $r=\frac{p'_B}{p_B}$ and obvious: $\frac{p_A}{p'_B}=\frac{s}{r}$.

We have, from the upper relations:

$$\begin{cases} X_3=\frac{cp_A-bp'_B}{cp_A^2-2bp_Ap'_B+ap'^2_B}V=\frac{s}{p_A}\frac{cs-br}{cs^2-2bsr+ar^2}V \\ Y_3=\frac{ap'_B-bp_A}{cp_A^2-2bp_Ap'_B+ap'^2_B}V=\frac{s}{p_A}\frac{ar-bs}{cs^2-2bsr+ar^2}V \end{cases}$$

and the utility:

$$U(X_3,Y_3)=Vd\sqrt{\frac{ac-b^2}{cp_A^2-2bp_Ap'_B+ap'^2_B}}=\frac{Vds}{p_A}\sqrt{\frac{ac-b^2}{cs^2-2bsr+ar^2}}$$

We shall apply now the Hicks method for our analysis.

At the modify of the price of B, we shall preserve the utility: $U(X_1,Y_1)=V\frac{s}{p_A}\sqrt{\frac{ac-b^2}{cs^2-2bsr+ar^2}}$ and we shall obtain the new revenue (in order to maximization):

$$V'=V\sqrt{\frac{cs^2-2bsr+ar^2}{cs^2-2bs+a}}$$

and the new allocation of goods:

$$\begin{cases} X_{2H} = \dfrac{s}{p_A} \dfrac{cs - br}{\sqrt{cs^2 - 2bsr + ar^2}\sqrt{cs^2 - 2bs + a}} V \\ Y_{2H} = \dfrac{s}{p_A} \dfrac{ar - bs}{\sqrt{cs^2 - 2bsr + ar^2}\sqrt{cs^2 - 2bs + a}} V \end{cases}$$

The substitution effect (which preserves the utility) gives us a difference:

$$\Delta_{1H}X = X_{2H} - X_1 = \frac{s}{p_A} \frac{cs - br}{\sqrt{cs^2 - 2bsr + ar^2}\sqrt{cs^2 - 2bs + a}} V - \frac{s(cs - b)}{p_A(cs^2 - 2bs + a)} V =$$

$$\frac{sV}{p_A\sqrt{cs^2 - 2bs + a}} \left(\frac{cs - br}{\sqrt{cs^2 - 2bsr + ar^2}} - \frac{cs - b}{\sqrt{cs^2 - 2bs + a}} \right)$$

$$\Delta_{1H}Y = Y_{2H} - Y_1 = \frac{s}{p_A} \frac{ar - bs}{\sqrt{cs^2 - 2bsr + ar^2}\sqrt{cs^2 - 2bs + a}} V - \frac{s(a - bs)}{p_A(cs^2 - 2bs + a)} V =$$

$$\frac{sV}{p_A\sqrt{cs^2 - 2bs + a}} \left(\frac{ar - bs}{\sqrt{cs^2 - 2bsr + ar^2}} - \frac{a - bs}{\sqrt{cs^2 - 2bs + a}} \right)$$

The difference caused by the revenue V instead V' is therefore:

$$\Delta_{2H}X = X_3 - X_{2H} = \frac{s}{p_A} \frac{cs - br}{cs^2 - 2bsr + ar^2} V - \frac{s}{p_A} \frac{cs - br}{\sqrt{cs^2 - 2bsr + ar^2}\sqrt{cs^2 - 2bs + a}} V =$$

$$\frac{s(cs - br)V}{p_A\sqrt{cs^2 - 2bsr + ar^2}} \left(\frac{1}{\sqrt{cs^2 - 2bsr + ar^2}} - \frac{1}{\sqrt{cs^2 - 2bs + a}} \right)$$

$$\Delta_{2H}Y = Y_3 - Y_{2H} = \frac{s}{p_A} \frac{ar - bs}{cs^2 - 2bsr + ar^2} V - \frac{s}{p_A} \frac{ar - bs}{\sqrt{cs^2 - 2bsr + ar^2}\sqrt{cs^2 - 2bs + a}} V =$$

$$\frac{s(ar - bs)V}{p_A\sqrt{cs^2 - 2bsr + ar^2}} \left(\frac{1}{\sqrt{cs^2 - 2bsr + ar^2}} - \frac{1}{\sqrt{cs^2 - 2bs + a}} \right)$$

named the revenue effect.

We shall apply now the Slutsky method for our analysis.

At the modify of the price of B, the revenue for the same optimal combination of goods is:

$$V' = p_A X_1 + p'_B Y_1 = \frac{cp_A^2 + ap_B p'_B - bp_A p_B - bp_A p'_B}{cp_A^2 - 2bp_A p_B + ap_B^2} V$$

therefore, in order to maximize the utility:

$$X_{2S} = \frac{(cp_A - bp'_B)(cp_A^2 + ap_B p'_B - bp_A p_B - bp_A p'_B)}{(cp_A^2 - 2bp_A p_B + ap_B^2)(cp_A^2 - 2bp_A p'_B + ap'^2_B)} V = \frac{s}{p_A} \frac{(cs - br)(cs^2 + ar - bs - bsr)}{(cs^2 - 2bs + a)(cs^2 - 2bsr + ar^2)} V$$

$$Y_{2S} = \frac{(ap'_B - bp_A)(cp_A^2 + ap_B p'_B - bp_A p_B - bp_A p'_B)}{(cp_A^2 - 2bp_A p_B + ap_B^2)(cp_A^2 - 2bp_A p'_B + ap'^2_B)} V = \frac{s}{p_A} \frac{(ar - bs)(cs^2 + ar - bs - bsr)}{(cs^2 - 2bs + a)(cs^2 - 2bsr + ar^2)} V$$

and the corresponding utility:

$$U_2 = d\sqrt{\frac{ac-b^2}{cp_A^2-2bp_Ap_B+ap_B^2}}V = \frac{Vds}{p_A}\sqrt{\frac{ac-b^2}{cs^2-2bs+a}}$$

The substitution effect after Slusky (which not preserves the utility) gives us a difference:

$$\Delta_{1S}X = X_{2S}-X_1 = \frac{s}{p_A}\frac{(cs-br)(cs^2+ar-bs-bsr)}{(cs^2-2bs+a)(cs^2-2bsr+ar^2)}V - \frac{s(cs-b)}{p_A(cs^2-2bs+a)}V = \frac{s^2r(1-r)(ac-b^2)V}{p_A(cs^2-2bs+a)(cs^2-2bsr+ar^2)}$$

$$\Delta_{1S}Y = Y_{2S}-Y_1 = \frac{s}{p_A}\frac{(cs-br)(cs^2+ar-bs-bsr)}{(cs^2-2bs+a)(cs^2-2bsr+ar^2)}V - \frac{s(cs-b)}{p_A(cs^2-2bs+a)}V = \frac{s^2r(1-r)(ac-b^2)V}{p_A(cs^2-2bs+a)(cs^2-2bsr+ar^2)}$$

and the revenue effect (after Slutsky):

$$\Delta_{2S}X = X_3-X_{2S} = \frac{s}{p_A}\frac{cs-br}{cs^2-2bsr+ar^2}V - \frac{s}{p_A}\frac{(cs-br)(cs^2+ar-bs-bsr)}{(cs^2-2bs+a)(cs^2-2bsr+ar^2)}V = \frac{s(cs-br)(1-r)(a-bs)V}{p_A(cs^2-2bsr+ar^2)(cs^2-2bs+a)}$$

$$\Delta_{2S}Y = Y_3-Y_{2S} = \frac{s}{p_A}\frac{ar-bs}{cs^2-2bsr+ar^2}V - \frac{s}{p_A}\frac{(ar-bs)(cs^2+ar-bs-bsr)}{(cs^2-2bs+a)(cs^2-2bsr+ar^2)}V = \frac{s(ar-bs)(1-r)(a-bs)V}{p_A(cs^2-2bsr+ar^2)(cs^2-2bs+a)}$$

We shall define, in what follows, the ratio:

$\alpha_Y = \frac{Y_2-Y_1}{Y_3-Y_1}$ - the share from the total consumption change for Y due to the substitution effect;

$\beta_Y = \frac{Y_3-Y_2}{Y_3-Y_1}$ - the share from the total consumption change for Y due to the revenue effect;

In the case of Hicks, we have:

- $$\alpha_{YH} = \frac{\frac{ar-bs}{\sqrt{cs^2-2bsr+ar^2}} - \frac{a-bs}{\sqrt{cs^2-2bs+a}}}{\sqrt{cs^2-2bs+a}\left(\frac{ar-bs}{cs^2-2bsr+ar^2} - \frac{a-bs}{cs^2-2bs+a}\right)}$$

- $\beta_{YH}=\frac{(ar-bs)\left(\frac{1}{\sqrt{cs^2-2bsr+ar^2}}-\frac{1}{\sqrt{cs^2-2bs+a}}\right)}{\sqrt{cs^2-2bsr+ar^2}\left(\frac{ar-bs}{cs^2-2bsr+ar^2}-\frac{a-bs}{cs^2-2bs+a}\right)}$

In the case of Slutsky, we have:

- $\alpha_{YS}=\frac{sr(ac-b^2)}{a(ar-cs^2)+bs(2bs-ar-a)}$
- $\beta_{YS}=\frac{sr(ac-b^2)}{a(ar-cs^2)+bs(2bs-ar-a)}$

3 Conclusions

From previous studies it is observed that the Slutsky effect causes, in the substitution effect, the same change of goods. For this reason, it can be seen that the Allen utility function may apply, if necessary, for complementary goods.

4 References

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