On Certain Conditions for Generating Production Functions - II

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Abstract: The article is the second in a series that will treat underlying conditions to generate a production function. The importance of production functions is fundamental to analyze and forecast the various indicators that highlights different aspects of the production process. How often forgets that these functions start from some premises, the article comes just meeting these challenges, analyzing different initial conditions. On the other hand, where possible, we have shown the concrete way of determining the parameters of the function.

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1. Introduction

We define on $\mathbf{R}^n$ the production space for n fixed resources as:

$$SP=\{(x_1,...,x_n) \mid x_i\geq 0, i=\overline{1,n}\}$$

where $x\in SP$, $x=(x_1,...,x_n)$ is an ordered set of inputs and a restriction of the production space to a subset $D_p\subset SP$ called production domain.

It is now called production function (output) an application:

$$Q:D_p\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D_p$$

which satisfies the following axioms:

1. The production domain D_p is convex i.e. $\forall x=(x_1,...,x_n)$, $y=(y_1,...,y_n)\in D_p$ $\forall\lambda\in[0,1]$ follows $(1-\lambda)x+\lambda y=((1-\lambda)x_1+\lambda y_1,...,(1-\lambda)x_n+\lambda y_n)\in D_p$

2. $Q(0,0,...,0)=0$

3. The production function is continuous

4. The production function is of class $C^2(D_p)$ i.e. admits 2nd order continous partial derivatives

5. The production function is monotonically increasing in each variable

6. The production function is quasi-concave that is $Q(\lambda x+(1-\lambda)y)\geq\min(Q(x),Q(y))$ $\forall\lambda\in[0,1]$ $\forall x,y\in R_p$

At the end of this introduction, let note that a function is called homogeneous if $\exists r\in\mathbf{R}$ such that: $Q(\lambda x_1,...,\lambda x_n)=\lambda^r Q(x_1,...,x_n)$ $\forall\lambda\in\mathbf{R}^*$. r is called the degree of homogeneity of the function. We will say that a production function $Q:D_p\to\mathbf{R}_+$ is with constant return to scale if $Q(\lambda x_1,...,\lambda x_n)=\lambda Q(x_1,...,x_n)$, with increasing return to scale if $Q(\lambda x_1,...,\lambda x_n)>\lambda Q(x_1,...,x_n)$ and with decreasing return to scale if $Q(\lambda x_1,...,\lambda x_n)<\lambda Q(x_1,...,x_n)$ $\forall\lambda\in(1,\infty)$ $\forall(x_1,...,x_n)\in D_p$.

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In what follows we will consider only functions of two variables: K – capital and L – labor: Q=Q(K,L) and we will note $\chi=\frac{K}{L}$.

We will call the marginal productivity relative to the capital K: $\eta_K=\frac{\partial Q}{\partial K}$ and with respect to the labor: $\eta_L=\frac{\partial Q}{\partial L}$.

Also, we define $w_K=\frac{Q}{K}$ - called the productivity of capital, and $w_L=\frac{Q}{L}$ - the productivity of labor being the average productivity relative to the production factor K, respectively L.

From [5], we have that in the general case of the variation of all inputs, for K_0 units of input K and L_0 units of input L, and Q(KL)=0:

$$Q(K_0,L_0)=K_0\int_0^1\eta_K(K_0t,L_0t)dt+L_0\int_0^1\eta_L(K_0t,L_0t)dt$$

We will call also the marginal rate of technical substitution of the factors K and J the opposite change in the amount of factor L to substitute a variation of the quantity of factor K in the situation of conservation production level and note: $RMS(K,L)=-\frac{dx_L}{dx_K}=\frac{\eta_K}{\eta_L}$ in an arbitrary point $\overline{x}=(\overline{K},\overline{L})$ and analogously: $RMS(L,K)=-\frac{dx_K}{dx_L}=\frac{\eta_L}{\eta_K}=\frac{1}{RMS(K,L)}$.

It is called elasticity of production in relation to K: $\varepsilon_K=\frac{\frac{\partial Q}{\partial K}}{\frac{Q}{K}}=\frac{\eta_K}{w_K}$ - the relative variation of production at the relative variation of K and also the elasticity of production in relation to L: $\varepsilon_L=\frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}}=\frac{\eta_L}{w_L}$ - the relative variation of production at the relative variation of L.

If the production function is homogenous of degree r, after Euler's relation: $K\frac{\partial Q}{\partial K}+L\frac{\partial Q}{\partial L}=rQ$ we obtain that $\varepsilon_K+\varepsilon_L=r$.

We finally define the marginal rate of substitution elasticity: $\sigma=\frac{\frac{\partial RMS(K,L)}{\partial\chi}}{\frac{RMS(K,L)}{\chi}}$.

Let suppose now that the function is homogenous of degree r.

Because $Q(K,L)=Q(\chi L,L)=L^rQ(\chi,1)$ we will note $q(\chi)=Q(\chi,1)$ and we have: $Q(K,L)=L^rq(\chi)$. We have therefore:

- $\eta_K = \frac{\partial Q}{\partial K} = L^r q'(\chi)\frac{\partial \chi}{\partial K} = L^{r-1}q'(\chi)$
- $\eta_L = \frac{\partial Q}{\partial L} = L^{r-1}q(\chi) + L^r q'(\chi)\frac{\partial \chi}{\partial L} = L^{r-1}(q(\chi) - \chi q'(\chi))$
- $w_K = \frac{Q}{K} = \frac{L^{r-1}q(\chi)}{\chi}$
- $w_L = \frac{Q}{L} = L^{r-1}q(\chi)$
- RMS(K,L)= $\frac{1}{RMS(L,K)} = \frac{\eta_K}{\eta_L} = \frac{q'(\chi)}{q(\chi) - \chi q'(\chi)}$
- $\sigma = \frac{\frac{\partial RMS(K,L)}{\partial \chi}}{\frac{RMS(K,L)}{\chi}} = \frac{\chi q(\chi) q''(\chi)}{q'(\chi)(q(\chi) - \chi q'(\chi))}$.

2. Conditions of Marginal Rate of Substitution Elasticity

Let suppose in what follows that: $\sigma=f(\chi)$, Q being homogenous of degree r.

We have the following differential equation: $\frac{\chi q(\chi) q''(\chi)}{q'(\chi)(q(\chi) - \chi q'(\chi))} = f(\chi)$ that is:

$$q''(\chi) = \frac{f(\chi)}{\chi} q'(\chi) - f(\chi)\frac{q'(\chi)^2}{q(\chi)} \text{ or } \frac{q''(\chi)}{q(\chi)} = \frac{f(\chi)}{\chi}\frac{q'(\chi)}{q(\chi)} - f(\chi)\left(\frac{q'(\chi)}{q(\chi)}\right)^2.$$

Let $y(\chi) = \frac{q'(\chi)}{q(\chi)}$. We have: $y'(\chi) = \frac{q''(\chi)q(\chi) - q'(\chi)^2}{q(\chi)^2} = \frac{q''(\chi)}{q(\chi)} - \left(\frac{q'(\chi)}{q(\chi)}\right)^2 = \frac{q''(\chi)}{q(\chi)} - y(\chi)^2$

The equation becomes:

$$y'(\chi) = \frac{f(\chi)}{\chi} y(\chi) - (1 + f(\chi)) y(\chi)^2$$

Again, if we note: $x(\chi) = \frac{1}{y(\chi)}$ we obtain by dividing at $y^2(\chi)$:

$\frac{y'(\chi)}{y(\chi)^2} = \frac{f(\chi)}{\chi}\frac{1}{y(\chi)} - (1 + f(\chi)) \Leftrightarrow x'(\chi) = -\frac{f(\chi)}{\chi} x(\chi) + (1 + f(\chi))$ - a linear equation of first degree.

The solution of homogenous equation $x'(\chi) = -\frac{f(\chi)}{\chi} x(\chi)$ is: $(\ln x(\chi))' = -\frac{f(\chi)}{\chi} \Leftrightarrow$ $\ln x(\chi) = -\int \frac{f(\chi)}{\chi} d\chi \Leftrightarrow x(\chi) = e^{-\int \frac{f(\chi)}{\chi} d\chi} + C$.

Returning at the nonhomogenous equation, let $x(\chi) = C(\chi)e^{-\int\frac{f(\chi)}{\chi}d\chi}$. We find that:

$$C'(\chi)e^{-\int\frac{f(\chi)}{\chi}d\chi} - C(\chi)\frac{f(\chi)}{\chi}e^{-\int\frac{f(\chi)}{\chi}d\chi} = -C(\chi)\frac{f(\chi)}{\chi}e^{-\int\frac{f(\chi)}{\chi}d\chi} + (1+f(\chi))$$

therefore $C'(\chi) = (1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}$ and finally:

$$C(\chi) = \int(1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}d\chi + C$$

We have now:

$$x(\chi) = e^{-\int\frac{f(\chi)}{\chi}d\chi}\int(1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}d\chi + Ce^{-\int\frac{f(\chi)}{\chi}d\chi}$$

and therefore: $y(\chi) = \dfrac{e^{\int\frac{f(\chi)}{\chi}d\chi}}{\int(1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}d\chi + C}$.

Returning to the definition of the function y:

$$(\ln q(\chi))' = \frac{e^{\int\frac{f(\chi)}{\chi}d\chi}}{\int(1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}d\chi + C} \Leftrightarrow \ln q(\chi) = \int\frac{e^{\int\frac{f(\chi)}{\chi}d\chi}}{\int(1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}d\chi + C}d\chi + D$$ therefore:

$$q(\chi) = De^{\int\frac{e^{\int\frac{f(\chi)}{\chi}d\chi}}{\int(1+f(\chi))e^{\int\frac{f(\chi)}{\chi}d\chi}d\chi+C}d\chi}$$

Let F be an indefinite integral of $\frac{f(\chi)}{\chi}$ that is: $F(\chi) = \int\frac{f(\chi)}{\chi}d\chi \Leftrightarrow F'(\chi) = \frac{f(\chi)}{\chi}$. We can write:

$$q(\chi) = De^{\int\frac{e^{F(\chi)}}{\int(1+f(\chi))e^{F(\chi)}d\chi+C}d\chi}.$$

But

$$\int(1+f(\chi))e^{F(\chi)}d\chi = \int(1+\chi F'(\chi))e^{F(\chi)}d\chi = \int e^{F(\chi)}d\chi + \int\chi\left(e^{F(\chi)}\right)'d\chi = \int e^{F(\chi)}d\chi + \chi e^{F(\chi)} - \int e^{F(\chi)}d\chi = \chi e^{F(\chi)} + C.$$

We have finally:

$$q(\chi) = De^{\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi}$$ where $F(\chi) = \int\frac{f(\chi)}{\chi}d\chi$ and, of course: $Q(K,L) = L^r q\left(\frac{K}{L}\right)$.

2.1. $f(\chi)=\alpha=$constant

$F(\chi)=\int\frac{\alpha}{\chi}d\chi=\alpha\ln\chi$ from where:

If $\alpha\neq-1$:

$$\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi=\int\frac{\chi^{\alpha}}{\chi^{\alpha+1}+C}d\chi=\frac{\ln\left(\chi^{\alpha+1}+C\right)}{\alpha+1}=\ln\left(\chi^{\alpha+1}+C\right)^{\frac{1}{\alpha+1}}$$

$$q(\chi)=De^{\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi}=D\left(\chi^{\alpha+1}+C\right)^{\frac{1}{\alpha+1}}$$

and $Q(K,L)=L^{r-1}D\left(K^{\alpha+1}+CL^{\alpha+1}\right)^{\frac{1}{\alpha+1}}$

For r=1 we obtain the CES production function.

If $\alpha=-1$:

$$\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi=\frac{1}{C+1}\int\frac{1}{\chi}d\chi=\frac{1}{C+1}\ln\chi$$

$$q(\chi)=De^{\frac{1}{C+1}\ln\chi}=D\chi^{\frac{1}{C+1}}$$

and $Q(K,L)=DK^{\frac{1}{C+1}}L^{r-\frac{1}{C+1}}$ - the Cobb-Douglas production function homogenous of degree r.

2.2. $f(\chi)=\alpha\chi+\beta$

$F(\chi)=\int\frac{f(\chi)}{\chi}d\chi=\int\frac{\alpha\chi+\beta}{\chi}d\chi=\alpha\chi+\beta\ln\chi$ from where:

$$\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi=\int\frac{\chi^{\beta}e^{\alpha\chi}}{\chi^{\beta+1}e^{\alpha\chi}+C}d\chi=$$

$$\frac{1}{\beta+1}\int\frac{\left(\chi^{\beta+1}e^{\alpha\chi}+C\right)'}{\chi^{\beta+1}e^{\alpha\chi}+C}d\chi-\frac{\alpha}{\beta+1}\int\frac{\chi^{\beta+1}e^{\alpha\chi}+C}{\chi^{\beta+1}e^{\alpha\chi}+C}d\chi+\frac{\alpha C}{\beta+1}\int\frac{1}{\chi^{\beta+1}e^{\alpha\chi}+C}d\chi=$$

$$\frac{1}{\beta+1}\ln\left|\chi^{\beta+1}e^{\alpha\chi}+C\right|-\frac{\alpha\chi}{\beta+1}+\frac{\alpha C}{\beta+1}\int\frac{1}{\chi^{\beta+1}e^{\alpha\chi}+C}d\chi=\ln\left|\chi^{\beta+1}e^{\alpha\chi}+C\right|^{\frac{1}{\beta+1}}-\frac{\alpha\chi}{\beta+1}+\frac{\alpha C}{\beta+1}G(\chi)\text{ where}$$

$$G(\chi)=\int\frac{1}{\chi^{\beta+1}e^{\alpha\chi}+C}d\chi$$

We have therefore:

$$q(\chi)=De^{\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi}=\frac{D\left|\chi^{\beta+1}e^{\alpha\chi}+C\right|^{\frac{1}{\beta+1}}}{e^{\frac{\alpha(\chi-CG(\chi))}{\beta+1}}}$$ and, of course: $$Q(K,L)=L^{r-1}\frac{D\left|K^{\beta+1}e^{\alpha\frac{K}{L}}+CL^{\beta+1}\right|^{\frac{1}{\beta+1}}}{e^{\frac{\alpha\left(\frac{K}{L}-CG\left(\frac{K}{L}\right)\right)}{\beta+1}}}.$$

In particular for $\beta=1$, we have, after developing in series:

$$G(\chi)=\int\frac{1}{\chi^2e^{\alpha\chi}+C}d\chi=\int\frac{1}{C}-\frac{1}{C^2}\chi^2-\frac{\alpha}{C^2}\chi^3+\ldots d\chi=\frac{1}{C}\chi-\frac{1}{3C^2}\chi^3-\frac{\alpha}{4C^2}\chi^4+\ldots$$

Now:

$$q(\chi)=De^{\int\frac{e^{F(\chi)}}{\chi e^{F(\chi)}+C}d\chi}=\frac{D\sqrt{\chi^2e^{\alpha\chi}+C}}{e^{\frac{\alpha}{6C}\chi^3+\frac{\alpha^2}{8C}\chi^4+\ldots}}$$ and, of course: $$Q(K,L)=L^{r-1}\frac{D\sqrt{K^2e^{\alpha\frac{K}{L}}+CL^2}}{e^{\frac{\alpha}{6C}\frac{K^3}{L^3}+\frac{\alpha^2}{8C}\frac{K^4}{L^4}+\ldots}}.$$

3. References

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