An Unified Consumption and Production Model for a Closed Economy

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Abstract: The article presents an unified consumption and production model for a closed economy.

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1. Introduction

Let consider n goods: $G_1, ..., G_n$ whose elasticity of utility is constant, their prices being $p_1, ..., p_n$. For a consumer whose available income is V, the utility function corresponding to the consumption of x_p units of good G_p, $p=\overline{1,n}$: $U(x_1,..., x_n)=Ax_1^{\alpha_1}...x_n^{\alpha_n}$ where α_p is the elasticity of utility in relation to the good G_p, and A is a positive constant.

The issue of maximizing the utility relative to the restriction: $\sum_{k=1}^{n} p_k x_k \le V$ is:

$$\begin{cases} \max U(x_1,..., x_n) \\ \sum_{k=1}^{n} p_k x_k \le V \\ x_1,..., x_n \ge 0 \end{cases}$$

Considering the Lagrangeian:

$$\Phi(x_1,..., x_n,\lambda) = U(x_1,..., x_n) + \lambda\left(\sum_{k=1}^{n} p_k x_k - V\right)$$

the maximum condition with restrictions must satisfy:

$$\begin{cases} \frac{\partial \Phi}{\partial x_j} = \frac{\partial U}{\partial x_j} + \lambda p_j = 0, j=\overline{1,n} \\ \frac{\partial \Phi}{\partial \lambda} = \sum_{j=1}^{n} p_j x_j - V = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_j A x_1^{\alpha_1}...x_j^{\alpha_j -1}...x_n^{\alpha_n} + \lambda p_j = 0, j=\overline{1,n} \\ \sum_{j=1}^{n} p_j x_j - V = 0 \end{cases} \Leftrightarrow$$

$$-\sum_{j=1}^{n} \frac{\alpha_j A x_1^{\alpha_1}...x_j^{\alpha_j -1}...x_n^{\alpha_n}}{\lambda} x_j - V = 0 \Leftrightarrow \frac{A x_1^{\alpha_1}...x_j^{\alpha_j}...x_n^{\alpha_n}}{\lambda} \sum_{j=1}^{n} \alpha_j + V = 0 \Leftrightarrow$$

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$$\lambda = -\frac{Ax_1^{\alpha_1}...x_j^{\alpha_j}...x_n^{\alpha_n}\sum_{j=1}^{n}\alpha_j}{V}.$$

By re-replacing, we get the optimal solution to the problem:

$$x_j^* = \frac{\alpha_j V}{p_j\sum_{j=1}^{n}\alpha_j}, j = \overline{1,n}$$

Let us also consider a producer with a number of K capital units, having the price of p_K and L workers whose hourly wage is w for a working time t. If the elasticity of production in relation to capital and labor are constant, the function of production is: $Q(t,K,L) = CK^{\beta}L^{\gamma}$ where β and γ are the elasticities of production in relation to the capital, respectively the labor, C being a constant.

Total cost of production: $CT = p_K K + twL$ leads to a gross profit corresponding to a sales price p: $\pi(t,K,L) = pQ(t,K,L) - CT = CpK^{\beta}L^{\gamma} - p_K K - twL$.

For a given production Q_0, the profit maximization condition returns to minimizing the total cost, so to the problem: $\begin{cases} \min(p_K K + twL) \\ CK^{\beta}L^{\gamma} = Q_0 \\ K, L \geq 0 \end{cases}$.

Considering the Lagrangeian:

$$\Phi(x_1,...,x_n,\lambda) = p_K K + twL + \lambda(CK^{\beta}L^{\gamma} - Q_0)$$

the minimum condition with restrictions must satisfy:

$$\begin{cases} \frac{\partial\Phi}{\partial K} = p_K + \lambda\beta CK^{\beta-1}L^{\gamma} = 0 \\ \frac{\partial\Phi}{\partial L} = tw + \lambda\gamma CK^{\beta}L^{\gamma-1} = 0 \\ \frac{\partial\Phi}{\partial\lambda} = CK^{\beta}L^{\gamma} - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{p_K}{\beta CK^{\beta-1}L^{\gamma}} \\ \lambda = -\frac{tw}{\gamma CK^{\beta}L^{\gamma-1}} \\ CK^{\beta}L^{\gamma} - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{p_K}{\beta CK^{\beta-1}L^{\gamma}} = \frac{tw}{\gamma CK^{\beta}L^{\gamma-1}} \\ CK^{\beta}L^{\gamma} - Q_0 = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} \frac{p_K}{\beta L} = \frac{tw}{\gamma K} \\ CK^{\beta}L^{\gamma} - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} L^* = \frac{Q_0}{C}\left(\frac{p_K\gamma}{\beta tw}\right)^{\frac{\beta}{\beta+\gamma}} \\ K^* = \frac{Q_0}{C}\left(\frac{p_K\gamma}{\beta tw}\right)^{-\frac{\gamma}{\beta+\gamma}} \end{cases}$$

, and the minimum total cost:

$$CT^* = \frac{Q_0}{C}\left(p_K\left(\frac{p_K\gamma}{\beta tw}\right)^{-\frac{\gamma}{\beta+\gamma}} + tw\left(\frac{p_K\gamma}{\beta tw}\right)^{\frac{\beta}{\beta+\gamma}}\right).$$

The maximum profit is:

$$\pi^*\left(t,K^*,L^*\right)=pQ_0-\frac{Q_0}{C}\left(p_K\left(\frac{p_K\gamma}{\beta tw}\right)^{-\frac{\gamma}{\beta+\gamma}}+tw\left(\frac{p_K\gamma}{\beta tw}\right)^{\frac{\beta}{\beta+\gamma}}\right)$$

2. The Model

Suppose there are a number of n firms: $F_1, \ldots, F_n$ each having a number of L_i employees, $i=\overline{1,n}$ where we will include, for simplification, the entrepreneur of the firm. Let w_i - the average hourly wage for F_i, t_i - the working time during the analysis period in F_i. We will also assume that F_i produces a single good (of constant elasticity): G_i whose sales price is p_i.

From the total revenue received, each employee pays a tax quota to the state budget χ.

For health insurance, pensions and other services that will later be paid back to employees, they pay a share ξ of the salary received. Let us consider the providers of these services (a single service G_{n+j}, $j=\overline{1,m}$ for each firm) as being the firms $F_{n+1}, \ldots, F_{n+m}$ each having L_{n+j}, $j=\overline{1,m}$ employees (including the entrepreneur), with w_{n+j} - the average hourly wage corresponding to the company F_{n+j} and t_{n+j} - the working time worked during the analysis period in F_{n+j}. The service price offered by F_{n+j} will be p_{n+j}, $j=\overline{1,m}$.

Therefore, the tax paid by each employee will be: $T_b=\chi w_i t_i$, $i=\overline{1,n+m}$ and for public services: $T_s=\xi w_i t_i$, $i=\overline{1,n+m}$.

The revenues available to F_i staff are therefore (for each individual employee): $V_i=(1-\chi-\xi)w_i t_i$, $i=\overline{1,n+m}$.

On the other hand, the amount of salaries received by service providers comes from the share ξ therefore: $\xi\sum_{i=1}^{n+m}w_i t_i L_i=\sum_{j=1}^{m}w_{n+j}t_{n+j}L_{n+j}$ or: $\sum_{j=1}^{m}w_{n+j}t_{n+j}L_{n+j}=\frac{\xi}{1-\xi}\sum_{i=1}^{n}w_i t_i L_i$.

The entrepreneur of F_i, $i=\overline{1,n+m}$ will allocate the profits made for investments that will be considered as goods produced by firms. Let us consider the output of F_i as: $Q_i(t,K_i,L_i)=C_i K_i^{\beta_i} L_i^{\gamma_i}$ where β_i and γ_i represent the constantly assumed elasticities of production relative to K_i, respectively L_i, C_i - positive constant. At a price of capital p_{K_i}, the total cost of production in F_i becomes: $CT_i=p_{K_i}K_i+t_i w_i L_i$. Therefore, at a sale price p_i of the G_i asset, F_i's profit is:

$$\pi_i=p_i Q_i(t,K_i,L_i)-p_{K_i}K_i-t_i w_i L_i=p_i C_i K_i^{\beta_i}L_i^{\gamma_i}-p_{K_i}K_i-t_i w_i L_i$$

The F_i's firm's entrepreneur income will therefore be just that π_i, $i=\overline{1,n+m}$.

Let considering the set S of social assistants (pensioners, people without income etc.) with a number of M people whose incomes represents a share ρ of the taxes paid by employees (the remainder being allocated to government consumption, public works etc.). Their income will therefore be: $\rho(TP_b+TS_b)=\rho\chi\sum_{i=1}^{n+m}w_i t_i L_i$ and the average income per person: $\frac{\rho\chi}{M}\sum_{i=1}^{n+m}w_i t_i L_i$.

In the following, we will consider that the utility function of any employee of a production, service or social assistance company will be the same for all consumers within the category (it may be different from company to company – as an example, the utility of books is different for employees of an educational establishment and another for meat producers). In addition, we will assume that all the production of a company will be sold.

Consider the utility functions for an employee of F_i: $\tilde{U}_i(x_{i1},...,x_{i,n+m}) = A_i x_{i1}^{\alpha_{i1}} ... x_{i,n+m}^{\alpha_{i,n+m}}$, $i=\overline{1,n+m}$ where x_{ij} represents the quantity of good G_j consumed by an employee of F_i and for social assistants: $\tilde{U}_S(x_1,...,x_{n+m}) = A x_1^{\alpha_1} ... x_{n+m}^{\alpha_{n+m}}$ where x_j represents the amount of good G_j consumed by a social assistant.

The utility functions of the entrepreneurs in the investment activity will be: $\tilde{\tilde{U}}_i(y_{i1},...,y_{in}) = B_i y_{i1}^{\delta_{i1}} ... y_{in}^{\delta_{in}}$, $i=\overline{1,n+m}$ where y_{ij} represents the amount of good G_j consumed by the F_i's entrepreneur.

Every employee and entrepreneurs want to maximize their utilities in the context of disposable income, so problems arise:

- $$\begin{cases} \max \tilde{U}_i(x_{i1},...,x_{i,n+m}) = A_i x_{i1}^{\alpha_{i1}} ... x_{i,n+m}^{\alpha_{i,n+m}} \\ \sum_{k=1}^{n+m} p_k x_{ik} \le (1-\chi-\xi) w_i t_i \\ x_{i1},...,x_{i,n+m} \ge 0 \end{cases}$$ - for company employees F_i, $i=\overline{1,n+m}$;

- $$\begin{cases} \max \tilde{U}_S(x_1,...,x_{n+m}) = A x_1^{\alpha_1} ... x_{n+m}^{\alpha_{n+m}} \\ \sum_{k=1}^{n+m} p_k x_k \le \frac{\rho\chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i \\ x_{i1},...,x_{i,n+m} \ge 0 \end{cases}$$ - for social assistants;

- $$\begin{cases} \max \tilde{\tilde{U}}_i(y_{i1},...,y_{in}) = B_i y_{i1}^{\delta_{i1}} ... y_{in}^{\delta_{in}} \\ \sum_{k=1}^{n} p_k y_{ik} \le p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i \\ y_{i1},...,y_{in} \ge 0 \end{cases}$$, $i=\overline{1,n+m}$ - for entrepreneurs.

It follows from the above that the optimum quantities of products are:

- $x_{ij}^* = \dfrac{\alpha_{ij}(1-\chi-\xi) w_i t_i}{p_j \sum_{k=1}^{n+m} \alpha_{ik}}$, $j=\overline{1,n+m}$ - for company employees F_i, $i=\overline{1,n+m}$;

- $x_j^* = \dfrac{\alpha_j \frac{\rho\chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i}{p_j \sum_{k=1}^{n+m} \alpha_k}$, $j=\overline{1,n+m}$ - for social assistants;

- $y_{ij}^* = \dfrac{\delta_{ij} p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i}{p_j \sum_{k=1}^{n+m} \delta_{ik}}$, $j=\overline{1,n+m}$ - for entrepreneurs.

Therefore, the amount of required G_j needed is:

$$Q_j = \sum_{i=1}^{m+n} L_i x_{ij}^* + M\sum_{i=1}^{m+n} x_j^* + \sum_{i=1}^{m+n} y_{ij}^* =$$

$$\frac{1-\chi-\xi}{p_j}\sum_{i=1}^{m+n} L_i \frac{\alpha_{ij} w_i t_i}{\sum_{k=1}^{n+m}\alpha_{ik}} + \frac{\rho\chi\alpha_j(m+n)\sum_{i=1}^{n+m} w_i t_i L_i}{p_j\sum_{k=1}^{n+m}\alpha_k} + \frac{1}{p_j}\sum_{i=1}^{m+n}\frac{\delta_{ij} p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i}{\sum_{k=1}^{n+m}\delta_{ik}}, \; j=\overline{1,n+m}.$$

Returning to the problem of maximizing F_j's profit for the quantity Q_j we have:

$$\begin{cases} \min\left(p_{K_j} K_j + t_j w_j L_j\right) \\ C_j K_j^{\beta_j} L_j^{\gamma_j} = Q_j \qquad , \; j=\overline{1,n+m} \\ K_j, L_j \geq 0 \end{cases}$$

from where:

$$\begin{cases} L_j^* = \dfrac{Q_j}{C_j}\left(\dfrac{p_{K_j}\gamma_j}{\beta_j t_j w_j}\right)^{\frac{\beta_j}{\beta_j+\gamma_j}} \\ K_j^* = \dfrac{Q_j}{C_j}\left(\dfrac{p_{K_j}\gamma_j}{\beta_j t_j w_j}\right)^{\frac{\gamma_j}{\beta_j+\gamma_j}} \end{cases}$$

Noting for simplicity:

$$s_i = w_i t_i, \; r_{ij} = \frac{\alpha_{ij} s_i}{\sum_{k=1}^{n+m}\alpha_{ik}}, \; u_j = \frac{\rho\chi(m+n)\alpha_j}{\sum_{k=1}^{n+m}\alpha_k}, \; v_{ij} = \frac{\delta_{ij} p_i C_i}{\sum_{k=1}^{n+m}\delta_{ik}}, \; z_i = \frac{p_{K_i}}{\sum_{k=1}^{n+m}\delta_{ik}}, \; f_i = \frac{s_i}{\sum_{k=1}^{n+m}\delta_{ik}},$$

$$g_j = \frac{\left(\dfrac{p_{K_j}\gamma_j}{\beta_j t_j w_j}\right)^{\frac{\beta_j}{\beta_j+\gamma_j}}}{C_j p_j} \Rightarrow \frac{p_{K_j}\gamma_j}{\beta_j t_j w_j} = g_j^{\frac{\beta_j+\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\beta_j+\gamma_j}{\beta_j}} \text{ from where: } \frac{\left(\dfrac{p_{K_j}\gamma_j}{\beta_j t_j w_j}\right)^{\frac{\gamma_j}{\beta_j+\gamma_j}}}{C_j p_j} = g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1}$$

we find that:

$$\begin{cases} L_j^* = g_j\left(\left(1-\chi-\xi\right)\sum_{i=1}^{m+n} r_{ij} L_i^* + u_j\sum_{i=1}^{n+m} s_i L_i^* + \sum_{i=1}^{m+n}\left(v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - z_i K_i^* - f_i L_i^*\right)\right) \\ K_j^* = g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j e^{\mu_j t}\right)^{\frac{\gamma_j}{\beta_j}-1}\left(\left(1-\chi-\xi\right)\sum_{i=1}^{m+n} r_{ij} L_i^* + u_j\sum_{i=1}^{n+m} s_i L_i^* + \sum_{i=1}^{m+n}\left(v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - z_i K_i^* - f_i L_i^*\right)\right) \end{cases}$$

or, in other words:

$$\begin{cases} L_j^* = \sum_{i=1}^{m+n} g_j\left(r_{ij}\left(1-\chi-\xi\right) + u_j s_i - f_i\right)L_i^* + \sum_{i=1}^{m+n} g_j v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} g_j z_i K_i^* \\ K_j^* = \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1}\left(r_{ij}\left(1-\chi-\xi\right) + u_j s_i - f_i\right)L_i^* + \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1} v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1} z_i K_i^* \end{cases}$$

Noting again:

- $Y_{1,ij}= g_j\left(r_{ij}(1-\chi-\xi)+u_j s_i - f_i\right)$
- $Y_{2,ij}= g_j v_{ij}$
- $Y_{3,ij}= g_j z_i$
- $Y_{4,ij}= g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1}\left(r_{ij}(1-\chi-\xi)+u_j s_i - f_i\right)$
- $Y_{5,ij}= g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1} v_{ij}$
- $Y_{6,ij}= g_j^{\frac{\gamma_j}{\beta_j}}\left(C_j p_j\right)^{\frac{\gamma_j}{\beta_j}-1} z_i$

we obtain:

$$\begin{cases} L_j^* = \sum_{i=1}^{m+n} Y_{1,ij}L_i^* + \sum_{i=1}^{m+n} Y_{2,ij}K_i^{*\beta_i}L_i^{*\gamma_i} - \sum_{i=1}^{m+n} Y_{3,ij}K_i^* \\ K_j^* = \sum_{i=1}^{m+n} Y_{4,ij}L_i^* + \sum_{i=1}^{m+n} Y_{5,ij}K_i^{*\beta_i}L_i^{*\gamma_i} - \sum_{i=1}^{m+n} Y_{6,ij}K_i^* \end{cases}, j=\overline{1,n+m}$$

The system solution will provide the optimal number of employees of each firm as well as the required capital.

On the other hand, provided that: $\sum_{j=1}^{m} w_{n+j}t_{n+j}L_{n+j} = \frac{\xi}{1-\xi}\sum_{i=1}^{n} w_i t_i L_i$ follows:

$$\sum_{j=1}^{m} w_{n+j}t_{n+j}L_{n+j}^* = \frac{\xi}{1-\xi}\sum_{i=1}^{n} w_i t_i L_i^* .$$

By replacing the above optimal solutions, we obtain the link between the two quotas (the only ones that are imposed at government level): χ (tax) and ξ - for health insurance, pensions and other services.

3. References

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