Mathematical and Quantative Methods

The Extreme of a Function Subject to Restraint Conditions

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Abstract: Many papers treat the classical problem of the determination function's extreme points subject to equality type restrictions. The well-known method of Lagrange's multipliers gives necessary conditions but not sufficient. In this paper, it is shown that in additional hypotesys (like the linearity) the nature of a stationary point remains the same for the restricted function and for the initial one.

Keywords: Lagrange; multipliers; extreme

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1 Introduction

In microeconomic theory, very often it is use the method of Lagrange multipliers in order to determine the extremes of a function whose variables satisfy additional restrictions. Unfortunately, many times, it is considered that the solutions of the system of equations derived from the method of Lagrange multipliers, are automatically extreme points. What is certain is that the method of Lagrange multipliers involve necessary conditions for the extreme, but not sufficient.

Given the importance of this method for a range of economic applications, such as the optimum choice when the consumer is given his income, the problem of the minimizing of the income providing a constant utility, minimizing the cost function under conditions of constant output or maximizing profit under restrictive conditions, we shall broach the problem of extreme points under additional

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relationships, without using Lagrange multipliers, obtaining, finally, sufficient conditions of extreme.

Let D an open subset in $\mathbf{R}^n$ and a function on D: $U:D\subset\mathbf{R}^n\to\mathbf{R}$, C^2-differentiable on D such that: $(x_1,...,x_n)\to U(x_1,...,x_n)\in\mathbf{R}$ for any $(x_1,...,x_n)\in D$.

Let also consider the restrictions: $g_k(x_1,...,x_n)=0$, $k=\overline{1,m}$, $m<n$ Let suppose that $\frac{D(g_1,...,g_m)}{D(x_{n-m+1},...,x_n)}\neq 0$.

From the implicit function theorem, there exist locally the family of functions φ_k, $k=\overline{1,m}$, such that: $x_{n-m+k}=\varphi_k(x_1,...,x_{n-m})$ and

$$\frac{\partial\varphi_k}{\partial x_i}=-\frac{\frac{D(g_1,...,g_m)}{D(x_{n-m+1},...,x_{n-m+k-1},x_i,x_{n-m+k+1},...,x_n)}}{\frac{D(g_1,...,g_m)}{D(x_{n-m+1},...,x_n)}},\ k=\overline{1,m}.$$

Substituting $x_{n-m+1}=\varphi_1(x_1,...,x_{n-m}),...,x_n=\varphi_m(x_1,...,x_{n-m})$ in U, we find that the function $u=U(x_1,...,x_{n-m},\varphi_1(x_1,...,x_{n-m}),...,\varphi_m(x_1,...,x_{n-m}))$ is the restriction of U at $D_1=\{(x_1,...,x_n)\in D \mid g_k(x_1,...,x_n)=0,\ k=\overline{1,m}\}$.

In what follows we shall inquire into the existence of a locally extreme of u, that is the locally extremum of U subject to the restrictions $g_k=0$, $k=\overline{1,m}$.

This problem it is classical and it is solved with Lagrange's multipliers method. The problem arises from the fact that this method does not gives sufficiently conditions for the extreme.

2 Main Theorem

In what follows, we shall compute the second differential of u.

(1) $\frac{\partial u}{\partial x_i}=\frac{\partial U}{\partial x_i}+\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\frac{\partial\varphi_k}{\partial x_i}$ and

(2) $$\frac{\partial^2 u}{\partial x_i\partial x_j}=\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{k=1}^{m}\frac{\partial^2 U}{\partial x_i\partial x_{n-m+k}}\frac{\partial\varphi_k}{\partial x_j}+\sum_{k=1}^{m}\frac{\partial^2 U}{\partial x_j\partial x_{n-m+k}}\frac{\partial\varphi_k}{\partial x_i}+\sum_{k,p=1}^{m}\frac{\partial^2 U}{\partial x_{n-m+p}\partial x_{n-m+k}}\frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_p}{\partial x_j}+\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}$$

The second differential is:

$$d^2u=\sum_{i,j=1}^{n-1}\frac{\partial^2 u}{\partial x_i\partial x_j}dx_i dx_j=\sum_{i,j=1}^{n-m}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j+2\sum_{i,j=1}^{n-m}\sum_{k=1}^{m}\frac{\partial^2 U}{\partial x_i\partial x_{n-m+k}}\frac{\partial\varphi_k}{\partial x_j}dx_i dx_j+$$

$$\sum_{i,j=1}^{n-m}\sum_{k,p=1}^{m}\frac{\partial^2 U}{\partial x_{n-m+p}\partial x_{n-m+k}}\frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_p}{\partial x_j}dx_i dx_j+\sum_{i,j=1}^{n-m}\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}dx_i dx_j=$$

$$\sum_{i,j=1}^{n-m}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j+2\sum_{k=1}^{m}\left(\sum_{j=1}^{n-m}\frac{\partial\varphi_k}{\partial x_j}dx_j\right)\sum_{i=1}^{n-m}\frac{\partial^2 U}{\partial x_i\partial x_{n-m+k}}dx_i+$$

$$\sum_{k,p=1}^{m}\frac{\partial^2 U}{\partial x_{n-m+p}\partial x_{n-m+k}}\left(\sum_{i=1}^{n-m}\frac{\partial\varphi_k}{\partial x_i}dx_i\right)\left(\sum_{j=1}^{n-m}\frac{\partial\varphi_p}{\partial x_j}dx_j\right)+\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\sum_{i,j=1}^{n-m}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}dx_i dx_j=$$

$$\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j-2\sum_{i=1}^{n-m}\sum_{k=1}^{m}\frac{\partial^2 U}{\partial x_i\partial x_{n-m+k}}dx_i dx_{n-m+k}-\sum_{k=1}^{m}\sum_{p=1}^{m}\frac{\partial^2 U}{\partial x_{n-m+k}\partial x_{n-m+p}}dx_{n-m+k}dx_{n-m+p}+$$

$$2\sum_{k=1}^{m}\left(\sum_{j=1}^{n-m}\frac{\partial\varphi_k}{\partial x_j}dx_j\right)\sum_{i=1}^{n-m}\frac{\partial^2 U}{\partial x_i\partial x_{n-m+k}}dx_i+\sum_{k,p=1}^{m}\frac{\partial^2 U}{\partial x_{n-m+p}\partial x_{n-m+k}}\left(\sum_{i=1}^{n-m}\frac{\partial\varphi_k}{\partial x_i}dx_i\right)\left(\sum_{j=1}^{n-m}\frac{\partial\varphi_p}{\partial x_j}dx_j\right)+$$

$$\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\sum_{i,j=1}^{n-m}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}dx_i dx_j=$$

$$\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j+2\sum_{k=1}^{m}\left(-dx_{n-m+k}+\sum_{j=1}^{n-m}\frac{\partial\varphi_k}{\partial x_j}dx_j\right)\sum_{i=1}^{n-m}\frac{\partial^2 U}{\partial x_i\partial x_{n-m+k}}dx_i+$$

$$\sum_{k,p=1}^{m}\frac{\partial^2 U}{\partial x_{n-m+p}\partial x_{n-m+k}}\left[\left(\sum_{i=1}^{n-m}\frac{\partial\varphi_k}{\partial x_i}dx_i\right)\left(\sum_{j=1}^{n-m}\frac{\partial\varphi_p}{\partial x_j}dx_j\right)-dx_{n-m+k}dx_{n-m+p}\right]+$$

$$\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\sum_{i,j=1}^{n-m}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}dx_i dx_j$$

Because $x_{n-m+k}=\varphi_k(x_1,...,x_{n-m})$ we find that: $dx_{n-m+k}=\sum_{i=1}^{n-m}\frac{\partial\varphi_k}{\partial x_i}dx_i$ therefore:

$$d^2u=\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j+\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}\sum_{i,j=1}^{n-m}\frac{\partial^2\varphi_k}{\partial x_i\partial x_j}dx_i dx_j$$

We can write therefore:

(3) $d^2u=d^2U+\sum_{k=1}^{m}\frac{\partial U}{\partial x_{n-m+k}}d^2\varphi_k$.

Theorem

Let D an open subset in $\mathbf{R}^n$ and a function on D: $U:D\subset\mathbf{R}^n\to\mathbf{R}$, C^2-differentiable on D such that: $(x_1,...,x_n)\to U(x_1,...,x_n)\in\mathbf{R}$ for any $(x_1,...,x_n)\in D$. Let also consider the restrictions: $g_k(x_1,...,x_n)=0$, $k=\overline{1,m}$, $m<n$ with $\frac{D(g_1,...,g_m)}{D(x_{n-m+1},...,x_n)}\neq 0$ and φ_k such that:

$x_{n-m+k}=\varphi_k(x_1,...,x_{n-m})$, $k=\overline{1,m}$.

a) If $\frac{\partial U}{\partial x_{n-m+k}}\geq 0$, $d^2\varphi_k$ – positive half-definite for any $k=\overline{1,m}$ and d^2U – positive definite then d^2u is positive definite;

b) If $\frac{\partial U}{\partial x_{n-m+k}}\geq 0$, $d^2\varphi_k$ – negative half-definite for any $k=\overline{1,m}$ and d^2U – negative definite then d^2u is negative definite;

c) If $\frac{\partial U}{\partial x_{n-m+k}}\leq 0$, $d^2\varphi_k$ – positive half-definite for any $k=\overline{1,m}$ and d^2U – negative definite then d^2u is negative definite;

d) If $\frac{\partial U}{\partial x_{n-m+k}}\leq 0$, $d^2\varphi_k$ – negative half-definite for any $k=\overline{1,m}$ and d^2U – positive definite then d^2u is positive definite.

Corrolary 1

If φ_k are affine functions for any $k=\overline{1,m}$ then $d^2u=d^2U$ that is the nature of stationary points are the same for the function and for those with restrictions.

Proof If φ_k are affine functions then $d^2\varphi_k=0$ for any $k=\overline{1,m}$.

3 Application

One spectacular application is in the economical theory of consumer. Let a function of utility U which is supposed to be of class C^2 and concave, that is d^2U is negative definite. If for the goods $G_1,...,G_n$ with the corresponding prices $p_1,...,p_n$ it is a limited budget V, the problem is to determine the optimal distribution in order to maximize the utility.

In this case, if we note with $x_1,...,x_n$ the desired quantities of $G_1,...,G_n$, we have the restriction $p_1x_1+...+p_nx_n=V$ therefore: $x_n=\frac{V}{p_n}-\frac{p_1}{p_n}x_1-...-\frac{p_{n-1}}{p_n}x_{n-1}$. The restriction of U related at this condition is therefore: u= $U\left(x_1,...,x_{n-1},\frac{V}{p_n}-\frac{p_1}{p_n}x_1-...-\frac{p_{n-1}}{p_n}x_{n-1}\right)$. Because U is concave, follows that if u has a stationary point this is a maximum point.

4 References

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