Mathematical and Quantative Methods

The Problem of Efficiency of the Consumption or Production

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Abstract: The consumption efficiency or production respectively, is of particular importance in the view of Pareto efficiency. The classical analysis for two goods or two factors of production, is based on the Edgeworth's box that allows determining the optimal quantity, and price equilibrium. The analysis which will follow, will deal this problem for the case of n goods or inputs, obtaining a general method for determining the optimal quantities, namely the equilibrium price using a series of relatively basic geometric tools.

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1. Introduction

The consumption efficiency or production respectively, is of particular importance in the view of Pareto efficiency.

The classical analysis for two goods or two factors of production, is based on the Edgeworth's box that allows determining the optimal quantity, and price equilibrium.

The analysis which will follow, will deal this problem for the case of n goods or inputs, obtaining a general method for determining the optimal quantities, namely the equilibrium price using a series of relatively basic geometric tools.

2. The Efficiency of the Consumption

Consider, in the following, two consumers A and B and a number of n goods $B_1,...,B_n$ available in the quantities $c_1,...,c_n$, for which we know the utility functions of A, respectively B as follows: $U=U_A(x_1,...,x_n)$ and $U=U_B(x_1,...,x_n)$ for consumption of x_k good units of B_k, $k=\overline{1,n}$.

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We assume below that the utility functions are of class C^2 on the inside space consumption $SC=\{(x_1,...,x_n) \mid x_i \geq 0, i=\overline{1,n}\}$. It is also known that the utility functions are concave, so the marginal utility is decreasing.

We will build in what follows **the Edgeworth's box** that is a n-dimensional parallelepiped: $[0,c_1]\times...\times[0,c_n]$, the quantities relative to A being relative to the origin O(0,...,0) and those appropriate to B to the point $F(c_1,...,c_n)$ on segments that define the n-dimensional parallelepiped.

Let an initial allocation of consumption for A and B:

$$x_A=(\alpha_1,...,\alpha_n),\ x_B=(\beta_1,...,\beta_n)$$

where $\alpha_k+\beta_k=c_k$, k= $\overline{1,n}$ (*A and B consume all the available products*).

The utilities consumption corresponding to the first two vectors are therefore: $U_{A,0}=U_A(\alpha_1,...,\alpha_n)$, respectively $U_{B,0}=U_B(\beta_1,...,\beta_n)$ relative to A, respectively F.

Since $\beta_k=c_k-\alpha_k$, k=$\overline{1,n}$ we have: $U_{B,0}=U_B(c_1-\alpha_1,...,c_n-\alpha_n)$. The B's utility function is then, relative to A:

$$U= U'_B(x_1,...,x_n)=U_B(c_1-x_1,...,c_n-x_n)$$

Let note that the notation U'_B is the expression of U_B relative to the origin of the coordinate axes and not the derivative of U_B (*which otherwise do not exist globally*). From now on, during the presentation of the problem, we consider the utility function of B of this form.

We now have:

$$U'_{Bm,i}=\frac{\partial U'_B}{\partial x_i}=\frac{\partial U_B(c_1-x_1,...,c_n-x_n)}{\partial x_i}=-\frac{\partial U_B}{\partial x_i}=-U_{Bm,i}\text{, i=}\overline{1,n}$$

$$\frac{\partial^2 U'_B}{\partial x_i \partial x_j}=\frac{\partial}{\partial x_j}\left(-\frac{\partial U_B(c_1-x_1,...,c_n-x_n)}{\partial x_i}\right)=\frac{\partial^2 U_B}{\partial x_i \partial x_j}\text{, i,j=}\overline{1,n}$$

so the function U'_B is still concave, but it has negative partial derivatives.

Considering the isoutilities hypersurfaces, follows that (relative to O) those of A is convex, while that of B is concave.

Let $ZU_{A,0}=\{(x_1,...,x_n)\in SC \mid U_A(x_1,...,x_n)\geq U_{A,0}\}$ – the consumer zone of A with higher utility than $U_{A,0}$ and $ZU_{B,0}=\{(x_1,...,x_n)\in SC \mid U'_B(x_1,...,x_n)\leq U'_{B,0}\}$ – the consumer zone of B with higher utility than $U_{B,0}$. Assume now that int $(ZU_{A,0}\cap ZU_{B,0})\neq\emptyset$ (*int - the inside of set, i.e. those points for which exists an n-dimensional cube centered in them and small enough side included in the given set*).

Let also $\Gamma(\gamma_1,...,\gamma_n) \in \text{int}\left(ZU_{A,0} \cap ZU_{B,0}\right)$ and consider the straight line which passing through the origin and Γ. Let note $\Psi(\psi_1,...,\psi_n)$ – the intersection with the utility hypersurface $U_A=U_{A,0}$ and with $\Omega(\omega_1,...,\omega_n)$ – the intersection with the utility hypersurface $U'_B = U'_{B,0}$. We therefore have: $U_A(\psi_1,...,\psi_n)=U_{A,0}$ and $U'_B(\omega_1,...,\omega_n)= U'_{B,0}$. Since $(\gamma_1,...,\gamma_n)>(\psi_1,...,\psi_n)$ from the P.7 axiom of the relationship of preference ([2]) follows that $\Gamma \succ \Psi$ (Γ is preferred to Ψ relative to the consumer A) so $U_A(\Gamma)>U_A(\Psi)=U_{A,0}$. Similarly, $(\gamma_1,...,\gamma_n)<(\omega_1,...,\omega_n)$ implies $\Gamma \succ \Omega$ relative to the consumer B) therefore $U'_B(\Gamma)< U'_B(\Omega)= U'_{B,0}$.

Following these considerations, we have that if $\text{int}\left(ZU_{A,0} \cap ZU_{B,0}\right) \neq \varnothing$ then each of the two consumers can enhance their utility, so the initial allocation is not optimal.

We will call Pareto efficiency where no consumer allocation can improve without affecting other interests.

From the above, Pareto efficiency is achieved that when the utility hypersurfaces become tangent.

The condition of tangency for $U=U_A(x_1,...,x_n)$ and $U= U'_B(x_1,...,x_n)= U_B(c_1-x_1,...,c_n-x_n)$ is reduced to determining those points $(x_1,...,x_n)$ for which:

$$\frac{\partial U_A}{\partial x_i} = \lambda \frac{\partial U'_B}{\partial x_i}, \ i=\overline{1,n}, \ \lambda \in \mathbf{R}$$

i.e. the points where the hypersurfaces will intersect and have the same tangent hyperplane (*the directors parameters are proportional*). In terms of utility of B, we have:

$$\frac{\partial U'_B}{\partial x_i}(x_1,...,x_n) = -\frac{\partial U_B}{\partial x_i}(c_1 - x_1,...,c_n - x_n)$$

from where the above condition becomes:

$$\frac{\partial U_A}{\partial x_i}(x_1,...,x_n) = \mu \frac{\partial U_B}{\partial x_i}(c_1 - x_1,...,c_n - x_n), \ i=\overline{1,n}, \ \mu \in \mathbf{R}$$

In marginal notation, we have:

$$U_{Am,i}(x_1,...,x_n) = \mu U_{Bm,i}(c_1 - x_1,...,c_n - x_n), \ i=\overline{1,n}, \ \mu \in \mathbf{R}$$

For two goods, the relations $U_{Am,1} = \mu U_{Bm,1}$ and $U_{Am,2} = \mu U_{Bm,2}$ are equivalent with: $\frac{U_{Am,1}}{U_{Am,2}} = \frac{U_{Bm,1}}{U_{Bm,2}}$. On the other hand, $\frac{U_{Am,1}}{U_{Am,2}} = \frac{dx_2}{dx_1}\Big|_A = RMS_A(1,2)$ – the marginal rate of substitution between the goods B_1 and B_2 for A, and $\frac{U_{Bm,1}}{U_{Bm,2}} = \frac{dx_2}{dx_1}\Big|_B$ =$RMS_B(1,2)$ – the marginal rate of substitution between the goods B_1 and B_2 for B. The above equality becomes:

$$RMS_A(1,2)= RMS_B(1,2)$$

All points where the allocation is Pareto efficient (*the solutions of the above problem*) generates the so-called curve of the contracts.

The contract curve represents all combinations of goods for which none of the parties can maximize utility without him diminish the other.

On the other hand, any point on the curve of the contracts represents a possible allocation. The problem is: if one of the two buyers wants a basket of products that maximizes his utility, another buyer agrees to buy what is left? The problem is very real and, fortunately, relatively easy to solve.

Consider then the n goods prices (*which we neglected to this moment*) $p_1,...,p_n$. For an income V, the budget hyperplane $\sum_{k=1}^{n} p_k x_k$ =V (*all combinations of goods that can be purchased by the amount V*) maximizes the utility (in the meaning of Walras) if it is tangent to its hypersurface, in which case, after the second law of Gossen, the marginal utilities are proportional to prices of goods.

As each consumer wants to maximize utility, results:

$$\frac{U_{Am,1}(x_1,...,x_n)}{p_1} = ... = \frac{U_{Am,n}(x_1,...,x_n)}{p_n}$$

$$\frac{U_{Bm,1}(c_1 - x_1,...,c_n - x_n)}{p_1} = ... = \frac{U_{Bm,n}(c_1 - x_1,...,c_n - x_n)}{p_n}$$

so the budget hyperplane will be tangent to the two utility hypersurfaces, that is will coincide with the common tangent hyperplane to them.

Considering the contract curve of the form:

$$x_1=f_1(\lambda),...,x_n=f_n(\lambda),\ \lambda\in\mathbf{R}$$

it follows that the price determination will be made from a single set of equality above (*the other derived from proportional marginal utilities on the curve of contracts*). So we get:

$$\frac{U_{Am,1}(f_1(\lambda),...,f_n(\lambda))}{p_1} = ... = \frac{U_{Am,n}(f_1(\lambda),...,f_n(\lambda))}{p_n}$$

from where:

$$p_k=\frac{U_{Am,k}(f_1(\lambda),...,f_n(\lambda))}{U_{Am,1}(f_1(\lambda),...,f_n(\lambda))}\nu,\ \nu>0,\ k=\overline{1,n}$$

We note that prices are determined to a multiplicative factor, which does not affect the outcome of the problem. We can consider $\nu=1$ due to the fact that a multiplication by a constant factor prices do not affect the parameters of the budget hyperplane, implicitly its orientation. If the initial allocation was $x_A=(\alpha_1,...,\alpha_n)$, $x_B=(\beta_1,...,\beta_n)$ follows that A's budget is: $V=\sum_{k=1}^{n} p_k\alpha_k$.

The new quantities (which satisfy also the budget equation $\sum_{k=1}^{n} p_k x_k = V$) implies that:

$$\sum_{k=1}^{n} p_k(\alpha_k - x_k) = 0$$

Substituting the expressions of p_k in this equation, it follows:

$$\sum_{k=1}^{n}\frac{U_{Am,k}(f_1(\lambda),...,f_n(\lambda))}{U_{Am,1}(f_1(\lambda),...,f_n(\lambda))}\left(\alpha_k - f_k(\lambda)\right)=0$$

hence $\lambda\in\mathbf{R}$. Substituting in the appropriate expressions, resulting p_k and x_k, $k=\overline{1,n}$.

Example

For B_1 and B_2 – two goods available in the quantities a and b, let consider two consumers A and B for which the utilities surfaces are of Cobb-Douglas type:

$U_A(x_1,x_2)=Cx_1^{\alpha}x_2^{1-\alpha}$, $\alpha\in(0,1)$, $C>0$, respectively $U_B(x_1,x_2)=Dx_1^{\beta}x_2^{1-\beta}$, $\beta\in(0,1)$, $D>0$

In order to obtain the curve of the contracts, we will determine first the transformed utility function of B. We have therefore:

$$U_B^{'}(x_1,x_2)=U_B(a-x_1,b-x_2)$$

from where:

$$U_B^{'}(x_1,x_2)=D(a-x_1)^{\beta}(b-x_2)^{1-\beta}$$

Also:

$$\frac{\partial U_A}{\partial x_1}(x_1,x_2)=C\alpha x_1^{\alpha-1}x_2^{1-\alpha},\ \frac{\partial U_A}{\partial x_2}(x_1,x_2)=C(1-\alpha)x_1^{\alpha}x_2^{-\alpha},$$

$$\frac{\partial U_B}{\partial x_1}(a-x_1,b-x_2)=D\beta(a-x_1)^{\beta-1}(b-x_2)^{1-\beta},$$
$$\frac{\partial U_B}{\partial x_2}(a-x_1,b-x_2)=D(1-\beta)(a-x_1)^{\beta}(b-x_2)^{-\beta}$$

The contracts curve points satisfying:

$$\begin{cases} C\alpha x_1^{\alpha-1}x_2^{1-\alpha}=\mu D\beta(a-x_1)^{\beta-1}(b-x_2)^{1-\beta} \\ C(1-\alpha)x_1^{\alpha}x_2^{-\alpha}=\mu D(1-\beta)(a-x_1)^{\beta}(b-x_2)^{-\beta} \end{cases}$$

from where:

$$x_2=f(x_1)=\frac{b\beta(1-\alpha)x_1}{a\alpha-a\alpha\beta+(\beta-\alpha)x_1}$$

In order to determine the prices of both goods that maximizes the utility of A and B in terms of an initial allocation for A: $x_1^*=\gamma$, $x_2^*=\delta$, we will write the contracts curve in the form:

$$\begin{cases} x_1=f_1(\lambda)=\lambda \\ x_2=f_2(\lambda)=\dfrac{b\beta(1-\alpha)\lambda}{a\alpha-a\alpha\beta+(\beta-\alpha)\lambda} \end{cases}$$

We have now:

$$p_1=\nu \text{ and } p_2=\frac{a\alpha-a\alpha\beta+(\beta-\alpha)\lambda}{\alpha\beta b}\nu$$

For $\nu=1$, we obtain: $p_1=\nu$ and $p_2=\dfrac{a\alpha-a\alpha\beta+(\beta-\alpha)\lambda}{\alpha\beta b}$.

Because the initial allocation was $x_1^*=\gamma$, $x_2^*=\delta$ it follows that the disposable income of A is: $V=p_1x_1^*+p_2x_2^*=p_1\gamma+p_2\delta$. On the other hand: $V=p_1x_1+p_2x_2$ therefore: $\lambda=\dfrac{\gamma\alpha\beta b+a\alpha\delta-a\alpha\beta\delta}{\beta b-\beta\delta+\alpha\delta}$. Therefore:

$$p_1=1,\ p_2=\frac{(a\alpha-a\alpha\beta)(\beta b-\beta\delta+\alpha\delta)+(\beta-\alpha)(\gamma\alpha\beta b+a\alpha\delta-a\alpha\beta\delta)}{\alpha\beta b(\beta b-\beta\delta+\alpha\delta)}$$

For these prices, the final allocation of goods for A is:

$$x_1=\frac{\gamma\alpha\beta b+a\alpha\delta-a\alpha\beta\delta}{\beta b-\beta\delta+\alpha\delta}$$

$$x_2=\frac{b\beta(1-\alpha)(\gamma\alpha\beta b+a\alpha\delta-a\alpha\beta\delta)}{(a\alpha-a\alpha\beta)(\beta b-\beta\delta+\alpha\delta)+(\beta-\alpha)(\gamma\alpha\beta b+a\alpha\delta-a\alpha\beta\delta)}$$

3. The Efficiency of the Production

Consider, in the following, two products A and B and a number of n inputs $F_1,...,F_n$, available in the quantities $s_1,...,s_n$ for which it is known the production functions of A and B as follows: $Q=Q_A(x_1,...,x_n)$, respectively $Q=Q_B(x_1,...,x_n)$ appropriate for the consumption of x_k units of factor F_k, $k=\overline{1,n}$.

Similarly to the case of consumer efficiency, we build the Edgeworth's box that is the n-dimensional parallelepiped: $[0,s_1]\times...\times[0,s_n]$, the quantities relative to A being referred to the origin O(0,...,0), while those at B to the point $F(s_1,...,s_n)$ on segments that define the n-dimensional parallelepiped.

Let an initial allocation of factors of production for A and B:

$$x_A=(\alpha_1,...,\alpha_n),\ x_B=(\beta_1,...,\beta_n)$$

where $\alpha_k+\beta_k=s_k$, $k=\overline{1,n}$ (*A and B consume all the available inputs*).

Doing similar as in the previous section it shows that the maximum efficiency in production is reached when the production hypersurfaces become tangent, which is to:

$$\eta_{A,i}(x_1,...,x_n)=\mu\eta_{B,i}(s_1-x_1,...,s_n-x_n),\ i=\overline{1,n},\ \mu\in\mathbf{R}$$

where $\eta_{A,i}=\frac{\partial Q_A}{\partial x_i}$ is the marginal productivity relative to the production factor x_i and analog for B.

For two inputs (K and L), the above relations are equivalent with: $\frac{\eta_{A,K}}{\eta_{A,L}}=\frac{\eta_{B,K}}{\eta_{B,L}}$. On the other hand, $\frac{\eta_{A,K}}{\eta_{A,L}}=\frac{dL}{dK}\Big|_A$ $=RMS_A(K,L)$ – the marginal rate of technical substitution of capital for A, and $\frac{\eta_{B,K}}{\eta_{B,L}}=\frac{dL}{dK}\Big|_B$ $=RMS_B(K,L)$ – the marginal rate of technical substitution of capital for B.

The above equality becomes:

$$RMS_A(K,L)=RMS_B(K,L)$$

All points where the allocation is Pareto efficient form the contracts production curve.

Considering now the prices of the n inputs being $r_1,...,r_n$ we get as above that a maximize of production requires that:

$$\frac{\eta_{A,1}(x_1,...,x_n)}{r_1}=...=\frac{\eta_{A,n}(x_1,...,x_n)}{r_n}$$

on the production contract curve.

Considering the production contract curve of the form:

$$x_1=g_1(\lambda),...,x_n=g_n(\lambda),\ \lambda\in\mathbf{R}$$

follows:

$$\frac{\eta_{A,1}(g_1(\lambda),...,g_n(\lambda))}{r_1}=...=\frac{\eta_{A,n}(g_1(\lambda),...,g_n(\lambda))}{r_n}$$

from where:

$$r_k=\frac{\eta_{A,k}(g_1(\lambda),...,g_n(\lambda))}{\eta_{A,1}(g_1(\lambda),...,g_n(\lambda))}\nu,\ \nu>0,\ k=\overline{1,n}$$

We note that prices are determined to a multiplicative factor, which does not affect the outcome of the problem and can therefore be considered $\nu=1$. If the initial allocation of factors of production has been $x_A=(\alpha_1,...,\alpha_n)$, $x_B=(\beta_1,...,\beta_n)$ follows that the total cost of production resulting value of A is: $CT=\sum_{k=1}^{n} r_k\alpha_k$.

The new quantities of factors (*which also satisfy the same total cost:* $\sum_{k=1}^{n} r_k x_k$ *=CT*) implies:

$$\sum_{k=1}^{n} r_k(\alpha_k - x_k)=0$$

Substituting the expressions of r_k in this equation, it follows:

$$\sum_{k=1}^{n}\frac{\eta_{A,k}(g_1(\lambda),...,g_n(\lambda))}{\eta_{A,1}(g_1(\lambda),...,g_n(\lambda))}\left(\alpha_k - g_k(\lambda)\right)=0$$

hence $\lambda\in\mathbf{R}$. Substituting the appropriate expressions, resulting r_k and x_k, $k=\overline{1,n}$.

4. Conclusion

From the analysis made above, we saw that in the treatment of Edgeworth's box, the reporting of all quantities to the same point and the n-dimensional approach allow for general conclusions and equations for determining the equilibrium prices and quantities.

Applications for n=2 have allowed the correlation of the known results with the conclusions of the theory presented above.

5. References

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