On the General Theory of Production Functions

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Abstract: In this paper we will study from an axiomatic point of view the production functions. Also we will define the main indicators of a production function, extending the classical definitions to n inputs and introducing other new. We will modify the notion of global average productivity and replace it with more realistic indicators. On the other hand, the notion of global rate of substitution will be introduced to the analysis of n goods.

Keywords: production function; productivity; marginal

JEL Classification: D01

1. Introduction

In any economic activity, obtaining a result of this means, implicitly, the existence of any number of resources required for a good deployment of the production process. We will assume that resources are indefinitely divisible, which implies the possibility of using specific tools of mathematical analysis to onset specific phenomena.

We then define on $\mathbf{R}^n$ the production space for n fixed resources as $SP=\{(x_1,...,x_n) \mid x_i \geq 0,\ i=\overline{1,n}\}$ where $x \in SP$, $x=(x_1,...,x_n)$ is an ordered set of resources.

Because inside a production process, depending on the nature of applied technology, but also its specificity, not any amount of resources is possible, we will restrict production space to a subset $D_p \subset SP$ – called the domain of production.

In the context of the existence domain of production, we will put the issue of determining its results (output) depending on the resources (inputs) of D_p.

We will call a production function an application:

$$Q:D_p \to \mathbf{R}_+,\ (x_1,...,x_n) \to Q(x_1,...,x_n) \in \mathbf{R}_+ \forall (x_1,...,x_n) \in D_p$$

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For an efficient and complex mathematical analysis of production functions, we will require a series of axioms (*but not all essential*) both its definition domain and its scope.

FP1. The production domain is convex.

D_p's convexity only means that if $x=(x_1,..., x_n)$, $y=(y_1,...,y_n)\in D_p$ then $\forall\lambda\in[0, 1]$ follows $\lambda x+(1-\lambda)y=(\lambda x_1+(1-\lambda)y_1,...,\lambda x_n+(1-\lambda)y_n)\in D_p$.

FP2. If all resources are zero then the output is zero.

The FP2 axiom states that $Q(0,..., 0)=0$.

FP3. The production function is continuous.

The continuity, purely mathematical, means that for any fixed point $(\overline{x}_1,...,\overline{x}_n)$ of the domain of production D_p and a range of inputs $(y_k)_{k\ge1}$, $y_k=(y_1^k,...,y_n^k)$ that converges to $(\overline{x}_1,...,\overline{x}_n)$, the production $Q(y_1^k,...,y_n^k)$ converges to $Q(\overline{x}_1,...,\overline{x}_n)$.

An axiom, not necessarily required, but very useful for obtaining significant results is:

FP4. The production function admits partial derivatives of order 2 and they are continuous (the function is of class C^2 on D_p).

Before commenting on this axiom, let note that all elementary functions are of class C^∞ on their domain of definition. Therefore, the class membership C^∞ is no way restrictive. It should also be noted that a function of class C^k, $k\ge0$ is continuous, therefore the axiom FP4 implies automatically FP3.

FP5. The production function is monotonically increasing in each variable.

The FP5 axiom says that, in caeteris paribus hypothesis for any $i=\overline{1,n}$, if $x_i\ge y_i$ then: $Q(\overline{x}_1,...\overline{x}_{i-1},x_i,\overline{x}_{i+1},\overline{x}_n)\ge Q(\overline{x}_1,...\overline{x}_{i-1},y_i,\overline{x}_{i+1},\overline{x}_n)\ \forall\ \overline{x}_k\ge0$, $k=\overline{1,n}$, $k\ne i$ such that $(\overline{x}_1,...\overline{x}_{i-1},x_i,\overline{x}_{i+1},\overline{x}_n)$, $(\overline{x}_1,...\overline{x}_{i-1},y_i,\overline{x}_{i+1},\overline{x}_n)\in D_p$.

If the function Q is at least class C^1, the increasing monotony is equivalent to: $\frac{\partial Q}{\partial x_i}\ge0$, $i=\overline{1,n}$.

From the axiom FP5 follows the global increasing related to the inequality relationship of $\mathbf{R}^n$:

FP5'. The production function is monotonically increasing with respect to the relationship of inequality of $\mathbf{R}^n$.

Indeed, if $x_1\ge y_1,...,x_n\ge y_n$ then:

$$Q(x_1, x_2,...,x_n) \geq Q(y_1, x_2,...,x_n) \geq Q(y_1, y_2,...,x_n) \geq ... \geq Q(y_1, y_2,...,y_n)$$

A condition often mentioned in the definition of production function is:

FP6. The production function is quasi-concave.

The quasi-concavity of a function means that:

$$Q(\lambda x+(1-\lambda)y) \geq \min(Q(x), Q(y))\ \forall\lambda\in[0, 1]\ \forall x, y\in D_p$$

Geometrically speaking, a quasi-concave function has the property to be above the lowest values recorded at the ends of some segment. This property is equivalent with the convexity of the set $Q^{-1}[a, \infty)\ \forall a\in\mathbf{R}$, where $Q^{-1}[a, \infty)= \{x\in R_p \mid Q(x)\geq a\}$.

On the other hand, let note that any monotone function defined on a convex set is quasi-concave, so the condition can be eliminated.

To simplify further considerations, however, we require an additional condition, namely:

FP6'. The production function is concave.

The concavity of a function means that:

$$Q(\lambda x+(1-\lambda)y) \geq \lambda Q(x)+(1-\lambda)Q(y)\ \forall\lambda\in[0, 1]\ \forall x, y\in D_p$$

or, in other words, its graph is above all straight line determined by any points of it.

Let note also, that a concave function defined on a convex domain is automatically quasi-concave (*but not each other*).

Following the concavity, we have that the production increases more slowly with amplification of production factors.

Considering a production function $Q:D_p\rightarrow\mathbf{R}_+$ and $\overline{Q}\in\mathbf{R}_+$ - fixed, the set of inputs which generate the production $\overline{Q}$ called isoquant. An isoquant is therefore characterized by: $\{(x_1,...,x_n)\in D_p \mid Q(x_1,...,x_n)=\overline{Q}\}$ or, in other words, it is the inverse image $Q^{-1}(\overline{Q})$.

We will say that a production function $Q:D_p\rightarrow\mathbf{R}_+$ is constant return to scale if $Q(\lambda x_1,...,\lambda x_n)=\lambda Q(x_1,...,x_n)$, with increasing return to scale if $Q(\lambda x_1,...,\lambda x_n)>\lambda Q(x_1,...,x_n)$ and decreasing return to scale if $Q(\lambda x_1,...,\lambda x_n)<\lambda Q(x_1,...,x_n)\ \forall\lambda\in(1, \infty)\ \forall(x_1,...,x_n)\in D_p$.

2. The Main Indicators of Production Functions

Let a production function:

$$Q:D_p\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+\forall(x_1,...,x_n)\in D_p$$

We will call the marginal physical production (*marginal productivity*) relative to a production factor x_i: $\eta_{x_i}=\frac{\partial Q}{\partial x_i}$ and represents the trend of variation of production at the variation of the factor x_i.

In particular, for a production function of the form: Q=Q(K, L) we have $\eta_K=\frac{\partial Q}{\partial K}$ - called the marginal efficiency of capital and $\eta_L=\frac{\partial Q}{\partial L}$ - called the marginal efficiency of labor.

We call the average physical production (*productivity*) relative to a production factor x_i: $w_{x_i}=\frac{Q}{x_i}$ and represents the value of production at the consumption of a unit of factor x_i.

In particular, for a production function of the form: Q=Q(K, L) we have: $w_K=\frac{Q}{K}$ - called the productivity (*efficiency*) of capital, and $w_L=\frac{Q}{L}$ - the productivity of labor.

In the general case of the variation of all inputs, for k_1 units of input 1,...,k_n units of input n, we will consider first the simple way $\gamma:[0, 1]\to\mathbf{R}^n$, $\gamma(t)=(tk_1,..., tk_n)$. This is nothing more than the large diagonal of the n-dimensional parallelepiped: $[0, k_1]\times...\times[0, k_n]$. Let also the differential form:

$$dQ=\frac{\partial Q}{\partial x_1}dx_1+...+\frac{\partial Q}{\partial x_n}dx_n$$

that is continuous everywhere after the C^2 character of Q. Along the path γ, the integral of dQ is defined by:

$$\int_\gamma dQ=\int_0^1\left(\frac{\partial Q}{\partial x_1}(\gamma_1(t),...,\gamma_n(t))\gamma'_1(t)+...+\frac{\partial Q}{\partial x_n}(\gamma_1(t),...,\gamma_n(t))\gamma'_n(t)\right)dt$$

Where $\gamma_1, ..., \gamma_n$ are the components of γ. The Leibniz-Newton's theorem for exact differential forms (forms with property $\exists Q$ such that $\omega=dQ$) states that: $\int_\gamma dQ = Q(\gamma(1))-Q(\gamma(0))$.

In the present case:

$$Q(k_1,..., k_n)-Q(0,..., 0)=\int_0^1\left(\frac{\partial Q}{\partial x_1}(k_1t,...,k_nt)k_1+...+\frac{\partial Q}{\partial x_n}(k_1t,...,k_nt)k_n\right)dt =$$

$$k_1\int_0^1\eta_{x_1}(k_1t,...,k_nt)dt+...+k_n\int_0^1\eta_{x_n}(k_1t,...,k_nt)dt$$

Because Q(0)=0, resulting the final formula:

$$Q(k_1,..., k_n)= k_1\int_0^1\eta_{x_1}(k_1t,...,k_nt)dt+...+k_n\int_0^1\eta_{x_n}(k_1t,...,k_nt)dt$$

The marginal coefficient of a factor x_i is $\gamma_{x_i}=\frac{\partial x_i}{\partial Q}$ and represents the trend of variation of x_i (caeteris paribus) relative to Q or, otherwise, the change production needs for an additional input at an infinitesimal variation of production.

In particular, for a production function of the form: Q=Q(K, L) we have: $\gamma_K=\frac{\partial K}{\partial Q}$ - the marginal capital coefficient and $\gamma_L=\frac{\partial L}{\partial Q}$ - the marginal coefficient of labor.

We will call also, the average coefficient of a production factor x_i: $\nu_{x_i}=\frac{x_i}{Q}$ and it is the necessary of factor (caeteris paribus) to achieve a given level of production.

In particular, for a production function of the form: Q=Q(K, L) we have: $\nu_K=\frac{K}{Q}$ - the average coefficient of capital, and $\nu_L=\frac{L}{Q}$ - the average coefficient of labor.

It is obvious that:

$$\nu_{x_i}=\frac{1}{w_{x_i}},\ w_{x_i}=\frac{1}{\nu_{x_i}}$$

and, if caeteris paribus hypothesis:

$$\eta_{x_i} = \frac{1}{\gamma_{x_i}},\ \gamma_{x_i} = \frac{1}{\eta_{x_i}}$$

In particular, for Q=Q(K, L) we have:

$$\nu_K = \frac{1}{w_K},\ w_K = \frac{1}{\nu_K},\ \nu_L = \frac{1}{w_L},\ w_L = \frac{1}{\nu_L},\ \eta_K = \frac{1}{\gamma_K},\ \gamma_K = \frac{1}{\eta_K},\ \eta_L = \frac{1}{\gamma_L},\ \gamma_L = \frac{1}{\eta_L}$$

It is called global average productivity the ratio of output produced and the sum of all factors of production used:

$$w_{av,\,g} = \frac{Q}{\sum_{i=1}^{n} x_i}$$

With the concept of the notion of average coefficient of a factor of production, we can write:

$$w_{av,\,g} = \frac{Q}{\sum_{i=1}^{n} x_i} = \frac{1}{\sum_{i=1}^{n} \frac{x_i}{Q}} = \frac{1}{\sum_{i=1}^{n} \nu_{x_i}}$$

By analogy with this notion, we will call global marginal productivity:

$$w_{marg,\,g} = \frac{1}{\sum_{i=1}^{n} \gamma_{x_i}}$$

In discrete terms, we have:

$$w_{marg,\,g} = \frac{1}{\sum_{i=1}^{n} \gamma_{x_i}} = \frac{1}{\sum_{i=1}^{n} \frac{\Delta x_i}{\Delta Q}} = \frac{\Delta Q}{\sum_{i=1}^{n} \Delta x_i}$$

therefore the marginal productivity represents the global changes obtained from the additional production from each factor.

In connection with these last two indicators ought to make some clarifications. On the one hand, the global average productivity and the overall marginal have the disadvantage of dividing the production to an amount of heterogeneous factors of production. On the other hand, geometrically speaking, the two types of productivity are not clear and have not an unambiguous representation as in the case of average productivity or marginal corresponding to a single factor.

For this reason, we will define another indicator of global average productivity, even if not appropriately respond to the objection above, will satisfactorily answer to the second requirement.

We will call global average productivity in the meaning of the Euclidean norm, the ratio of the production and the norm of the vector inputs:

$$w_{av,\,gn}=\frac{Q}{\sqrt{\sum_{i=1}^{n}x_i^2}}$$

We therefore have:

$$w_{av,\,gn}=\frac{Q}{\sqrt{\sum_{i=1}^{n}x_i^2}}=\frac{1}{\sqrt{\sum_{i=1}^{n}\left(\frac{x_i}{Q}\right)^2}}=\frac{1}{\sqrt{\sum_{i=1}^{n}\nu_{x_i}^2}}$$

Another useful formula can be obtained considering the angles that determine the input vector with the coordinate axes:

$$\cos\alpha_i=\frac{x_j}{\sqrt{\sum_{i=1}^{n}x_i^2}},\ i=\overline{1,n}$$

It follows:

$$w_{av,\,gn}=\frac{Q}{\sqrt{\sum_{i=1}^{n}x_i^2}}=\frac{Q}{x_j}\frac{x_j}{\sqrt{\sum_{i=1}^{n}x_i^2}}=w_{x_j}\frac{x_j}{\sqrt{\sum_{i=1}^{n}x_i^2}}=w_{x_j}\cos\alpha_j,\ j=\overline{1,n}$$

We will call now the overall marginal productivity in the meaning of the norm:

$$w_{marg,\,gn}=\frac{1}{\sqrt{\sum_{i=1}^{n}\gamma_{x_i}^2}}$$

In discrete terms, we have:

$$w_{marg,\,gn}=\frac{1}{\sqrt{\sum_{i=1}^{n}\gamma_{x_i}^2}}=\frac{1}{\sqrt{\sum_{i=1}^{n}\frac{(\Delta x_i)^2}{(\Delta Q)^2}}}=\frac{\Delta Q}{\sqrt{\sum_{i=1}^{n}(\Delta x_i)^2}}$$

Considering the factors i and j with $i \neq j$, we define the restriction of production area: $P_{ij}=\{(x_1,...,x_n) \mid x_k=a_k=\text{const},\ k=\overline{1,n},\ k\neq i,\ j,\ x_i,\ x_j \in D_p\}$ relative to the two factors when the others have fixed values. Also, let: $D_{ij}=\{(x_i,\ x_j) \mid (x_1,...,x_n)\in P_{ij}\}$ - the domain of production relative to factors i and j.

We define: $Q_{ij}:D_{ij}\to \mathbf{R}_+$ - the restriction of the production function to the factors i and j, i.e.: $Q_{ij}(x_i,\ x_j)=Q(a_1,...,\ a_{i-1},\ x_i,\ a_{i+1},...,\ a_{j-1},\ x_j,\ a_{j+1},...,a_n)$. The functions Q_{ij} define a surface in $\mathbf{R}^3$ for every pair of factors (i, j).

We will call partial marginal rate of technical substitution of the factors i and j, relative to D_{ij} (caeteris paribus), the opposite change in the amount of factor j to substitute a variation of the quantity of factor i in the situation of conservation production level.

We will note below:

$$\mathrm{RMS}(i,\ j,\ D_{ij})=-\frac{dx_j}{dx_i}$$

Since $Q_{ij}(x_i,\ x_j)=Q_0=\text{constant}$, we obtain by differentiation: $dQ_{ij}(x_i,\ x_j)=0$ that is: $\frac{\partial Q_{ij}}{\partial x_i}dx_i+\frac{\partial Q_{ij}}{\partial x_j}dx_j=0$ therefore:

$$-\frac{dx_j}{dx_i}=\frac{\frac{\partial Q_{ij}}{\partial x_i}}{\frac{\partial Q_{ij}}{\partial x_j}}=\frac{\left.\frac{\partial Q}{\partial x_i}\right|_{D_{ij}}}{\left.\frac{\partial Q}{\partial x_j}\right|_{D_{ij}}}=\frac{\left.\eta_{x_i}\right|_{D_{ij}}}{\left.\eta_{x_j}\right|_{D_{ij}}}$$

We can write: $\mathrm{RMS}(i,\ j,\ D_{ij})=\frac{\left.\eta_{x_i}\right|_{D_{ij}}}{\left.\eta_{x_j}\right|_{D_{ij}}}$ which is a function of x_i and x_j. In an arbitrary point $\overline{x}=\left(\overline{x}_1,...,\overline{x}_n\right)$:

$$\mathrm{RMS}(i,\ j,\ \overline{x})=\frac{\eta_{x_i}(\overline{x})}{\eta_{x_j}(\overline{x})}$$

Now consider the case in which all factors consumption varies. Let therefore be an arbitrary point $\overline{x}\in D_p$ such that $Q(\overline{x})=Q_0=\text{constant}$ and $\eta_{x_k}(\overline{x})\neq 0$, $k=\overline{1,n}$. Differentiating with respect to $\overline{x}$ we have: $0=dQ=\sum_{j=1}^{n}\frac{\partial Q}{\partial x_j}dx_j$ from where:

$\frac{\partial Q}{\partial x_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial Q}{\partial x_j}\frac{dx_j}{dx_i}$=0. In terms of marginal production, we can write: $\eta_{x_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}\eta_{x_j}\frac{dx_j}{dx_i}$=0. Noting $\frac{dx_j}{dx_i}$=y_j, j=$\overline{1,n}$, j≠i, follows: $\eta_{x_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}\eta_{x_j}y_j=0$.

With the partial substitution marginal rate introduced above, we get:

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\overline{x})}=-1$$

The above relationship is nothing but the equation of a hyperplane in $\mathbf{R}^{n-1}$ of coordinates ($y_1,\ldots,\hat{y}_i,\ldots,y_n$) (*the sign ^ meaning that that term is missing*) that intersects the coordinate axes in RMS(i, j, $\overline{x}$). This hyperplane is the locus of consumption factors variations relative to a change in the i-th factor consumption such that the production remains constant and is called the marginal hyperplane of technical substitution between factor i and the other factors (noted below $H_{mi,j}$).

In particular, for two factors, the marginal hyperplane of technical substitution between the factor i and the factor j from $\mathbf{R}_+$, is reduced to: $\frac{y_j}{RMS(i,j,\overline{x})}=-1$ where y_j=$\frac{dx_j}{dx_i}$. Therefore, $\frac{dx_j}{dx_i}$=-y_j=$-RMS(i,j,\overline{x})$ which is consistent with the definition of the partial marginal rate of technical substitution.

We will define now the global marginal rate of substitution between the i-th factor and the others as the distance from the origin to the marginal hyperplane of technical substitution, namely:

$$RMS(i,\overline{x})=\frac{1}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{RMS^2(i,j,\overline{x})}}}=\frac{1}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\eta_{x_j}^2(\overline{x})}{\eta_{x_i}^2(\overline{x})}}}=\frac{\eta_{x_i}(\overline{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\eta_{x_j}^2(\overline{x})}}$$

We note that for the particular case of two factors, is obtained, as above:

$$RMS(i,\overline{x})=\frac{\eta_{x_i}(\overline{x})}{\eta_{x_j}(\overline{x})}$$

Considering now v=(y_1,..., $\hat{y}_i$,..., y_n)$\in H_{mi,\ j}$ we have: $\|v\| = \sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n} y_j^2}$ and from the Cauchy-Schwarz inequality:

$$\frac{\|v\|}{\left|RMS(i,\bar{x})\right|} = \sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n} y_j^2}\ \sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{RMS^2(i,j,\bar{x})}} \geq \left|\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{y_j}{RMS(i,j,\bar{x})}\right| = 1$$

therefore $\|v\| \geq \left|RMS(i,\bar{x})\right|$.

Like a conclusion, the global marginal rate of technical substitution is the minimum (in the meaning of norm) of changes in consumption of factors so that the total production remain unchanged.

Considering now the marginal hyperplane of technical substitution: $\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{y_j}{RMS(i,j,\bar{x})} = -1$ the equation of the normal from origin to it, is:

$$\frac{y_1}{\frac{1}{RMS(i,1,\bar{x})}} = \ldots = \frac{y_{i-1}}{\frac{1}{RMS(i,i-1,\bar{x})}} = \frac{y_{i+1}}{\frac{1}{RMS(i,i+1,\bar{x})}} = \ldots = \frac{y_n}{\frac{1}{RMS(i,n,\bar{x})}}$$

from where:

$$\begin{cases} y_1 = \dfrac{\lambda}{RMS(i,1,\bar{x})} \\ \ldots \\ y_{i-1} = \dfrac{\lambda}{RMS(i,i-1,\bar{x})} \\ y_{i+1} = \dfrac{\lambda}{RMS(i,i+1,\bar{x})} \\ \ldots \\ y_n = \dfrac{\lambda}{RMS(i,n,\bar{x})} \end{cases}, \lambda \in \mathbf{R}$$

The intersection of the normal with the hyperplane, represents the coordinates of the point of minimal norm. We therefore have: $\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\lambda}{\mathrm{RMS}^2(i,j,\overline{x})}=-1$ from where:

$\lambda=-\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{\mathrm{RMS}^2(i,j,\overline{x})}}$ and the point of minimal norm has the coordinates:

$$-\mathrm{RMS}^2(i,\overline{x})\left(\frac{1}{\mathrm{RMS}(i,1,\overline{x})},\ldots,\frac{\hat{1}}{\mathrm{RMS}(i,i,\overline{x})},\ldots,\frac{1}{\mathrm{RMS}(i,n,\overline{x})}\right)=$$

$$-\frac{\eta_{x_i}(\overline{x})}{\sum_{\substack{j=1\\ j\neq i}}^{n}\eta_{x_j}^2(\overline{x})}\left(\eta_{x_1}(\overline{x}),\ldots,\hat{\eta}_{x_i}(\overline{x}),\ldots,\eta_{x_n}(\overline{x})\right)$$

which norm is nothing else that $\left|\mathrm{RMS}(i,\ \overline{x})\right|$.

The coordinates of the above point is no more than minimal vector (in the meaning of norm) of changes in consumption so that total output remains unchanged. We will say briefly that this is the minimal vector of technical substitution of the factor i.

In particular, for a production function of the form: Q=Q(K, L) we have:

$$\mathrm{RMS}(K,L)=\frac{\eta_K}{\eta_L},\ \mathrm{RMS}(L,K)=\frac{\eta_L}{\eta_K}$$

It is called elasticity of production in relation to a production factor x_i: $\varepsilon_{x_i}=\frac{\frac{\partial Q}{\partial x_i}}{\frac{Q}{x_i}}=$

$\frac{\eta_{x_i}}{w_{x_i}}$ - the relative variation of production at the relative variation of factor x_i.

In particular, for a production function of the form: Q=Q(K, L) we have $\varepsilon_K=\frac{\frac{\partial Q}{\partial K}}{\frac{Q}{K}}=\frac{\eta_K}{w_K}$ - called the elasticity of production in relation to the capital and $\varepsilon_L=\frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}}=\frac{\eta_L}{w_L}$ - the elasticity factor of production in relation to the labor.

3. Application

Considering now a production function $Q:D_p\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+\forall(x_1,..., x_n)\in D_p$ with constant return to scale, let note for an arbitrary factor (for example x_n):

$$\chi_i=\frac{x_i}{x_n},\ i=\overline{1,n-1}$$

We will have:

$$Q(x_1,..., x_n)=x_n Q\left(\frac{x_1}{x_n},...,\frac{x_{n-1}}{x_n},\frac{x_n}{x_n}\right)=x_n Q(\chi_1,...,\chi_{n-1},1)$$

Considering the restriction of the production function at $D_p\cap\mathbf{R}_+^{n-1}\times\{1\}$: $q(\chi_1,...,\chi_{n-1})=Q(\chi_1,...,\chi_{n-1},1)$ we can write:

$$Q(x_1,..., x_n)=x_n q(\chi_1,...,\chi_{n-1})$$

With the new function introduced, the above indicators are:

- $\eta_{x_i}=\frac{\partial q}{\partial \chi_i}$, $i=\overline{1,n-1}$
- $\eta_{x_n}=q-\sum_{i=1}^{n-1}\frac{\partial q}{\partial \chi_i}\chi_i$
- $w_{x_i}=\frac{q}{\chi_i}$, $i=\overline{1,n-1}$

- $w_{x_n} = q$

- $\text{RMS}(i, j) = \dfrac{\frac{\partial q}{\partial \chi_i}}{\frac{\partial q}{\partial \chi_j}}$, $i, j = \overline{1, n-1}$

- $\text{RMS}(i, n) = \dfrac{\frac{\partial q}{\partial \chi_i}}{q - \sum_{i=1}^{n-1} \frac{\partial q}{\partial \chi_i} \chi_i}$, $i = \overline{1, n-1}$

- $\text{RMS}(n, j) = \dfrac{q - \sum_{i=1}^{n-1} \frac{\partial q}{\partial \chi_i} \chi_i}{\frac{\partial q}{\partial \chi_j}}$, $j = \overline{1, n-1}$

- $\text{RMS}(i) = \dfrac{\frac{\partial q}{\partial \chi_i}}{\sqrt{\left(q - \sum_{j=1}^{n-1} \frac{\partial q}{\partial \chi_j} \chi_j\right)^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \left(\frac{\partial q}{\partial \chi_j}\right)^2}}$, $i = \overline{1, n-1}$

- $\text{RMS}(n) = \dfrac{q - \sum_{j=1}^{n-1} \frac{\partial q}{\partial \chi_j} \chi_j}{\sqrt{\sum_{j=1}^{n-1} \left(\frac{\partial q}{\partial \chi_j}\right)^2}}$

- $\varepsilon_{x_i} = \dfrac{\frac{\partial q}{\partial \chi_i}}{\frac{q}{\chi_i}}$, $i = \overline{1, n-1}$

- $\varepsilon_{x_n} = \frac{q - \sum_{i=1}^{n-1} \frac{\partial q}{\partial \chi_i} \chi_i}{q}$

4. Conclusion

After the above analysis, we have seen that the analysis of production functions, on the one hand from the axiomatic point of view and, on the other hand, on the general case of n variables, reveals very interesting aspects.

First, even a very restrictive axiomatization removes some of the common functions (Leontief case), it adds a more austerity to the notion, eliminating the use of, often negligent, of the production function.

On the other hand, the extension of the main indicators in the case of n inputs, allows the removal, on the one hand, of some absurd concepts from our point of view, such as the global average productivity and replacing them with more realistic indicators. On the other hand, the notion of global rate of substitution removes the usual drawback of partial substitutions that restrict the scope sometimes dramatically.

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