Spatial Distributions of Regional Economic Activity

The Equilibrium Analysis of a Closed Economy Model with Government and Money Market Sector

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Abstract: In this paper, we first study the static equilibrium of a a closed economy model in terms of dependence on national income and interest rate from the main factors namely the marginal propensity to consume, tax rate, investment rate and the rate of currency demand. In the second part, we study the dynamic equilibrium solutions in terms of stability. We thus obtain the variation functions of national income and interest rate variation and their limit values.

Keywords: equilibrium; demand; income

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1 Introduction

The purpose of this paper is to analyze a closed economy model so the situation when net exports are zero.

After formulation of classical assumptions of the model, we first study the static equilibrium in terms of dependence on national income and interest rate from the main factors namely the marginal propensity to consume, tax rate, investment rate and the rate of currency demand.

In the second part, we study the dynamic equilibrium solutions in terms of stability. We thus obtain the variation functions of national income and interest rate variation and their limit values.

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2 The Model Equations ([4])

The model equations are:

(1) $D=C+I+G$

(2) $C=c_Y V+C_0$, $C_0>0$, $c_Y\in(0,1)$

(3) $V=Y+TR-TI$, $TR>0$

(4) $TI=ri_Y Y+T_0$, $ri_Y\in(0,1)$, $T_0\in \mathbf{R}$

(5) $I=in_Y Y+i_r r+I_0$, $in_Y\in(0,1)$, $i_r<0$, $I_0>0$

(6) $G=\overline{G}$

(7) $D=Y$

(8) $MD=md_Y Y+m_r r+M_0\le M_0$, $md_Y>0$, $m_r<0$, $M_0>0$

(9) $MD=M$

(10) $\frac{dY}{dt}=\alpha(D-Y)$, $\alpha>0$

(11) $\frac{dr}{dt}=\beta(MD-M)$, $\beta>0$

where:

- D – the aggregate demand;
- C – the consumer demand (a concave function of V);
- I – the investment demand;
- G – the government spending;
- V – the disposable income;
- Y – the aggregate supply (national income);
- TR – the government transfers;
- TI – taxes;
- c_Y – the marginal propensity to consume, $c=\frac{dC}{dV}\in(0,1)$, $\frac{d^2c}{dV^2}\le 0$;
- ri_Y – the tax rate, $ri_Y\in(0,1)$;
- in_Y –the rate of investments, $in_Y\in(0,1)$;

- i_r – a factor of influence on the investment rate, $i_r<0$;
- r – the interest rate;
- MD – the money demand in the economy;
- md_Y – the rate of money demand in the economy;
- m_r – a factor of influencing the demand for currency from the interest rate, $m_r<0$;
- M – the money supply.

3 The Static Equilibrium

From (1), (2), (3), (4), (5), (6) we get:

(12) $D=c_Y(Y+TR-ri_YY-T_0)+C_0+in_YY+i_rr+I_0+\overline{G}$

The equilibrium condition D=Y in (7) implies: $Y=c_Y(Y+TR-ri_YY-T_0)+C_0+in_YY+i_rr+I_0+\overline{G}$ therefore:

(13) $Y=\frac{i_r}{1-c_Y(1-ri_Y)-in_Y}r+\frac{c_Y(TR-T_0)+C_0+I_0+\overline{G}}{1-c_Y(1-ri_Y)-in_Y}$

With the notation:

(14) $E=c_Y(TR-T_0)+C_0+I_0+\overline{G}>0$

we have:

(15) $Y=\frac{i_r}{1-c_Y(1-ri_Y)-in_Y}r+\frac{1}{1-c_Y(1-ri_Y)-in_Y}E$

From the fact that $ri_Y\in(0,1)$, $in_Y\in(0,1)$, $c_Y\in(0,1)$ we get that: $1-c_Y(1-ri_Y)-in_Y>0$ if and only if:

(16) $c_Y<\frac{1-in_Y}{1-ri_Y}$

Similarly, from equation (8): $MD=md_YY+m_rr+M_0=M$ therefore:

(17) $Y=-\frac{m_r}{md_Y}r+\frac{M-M_0}{md_Y}$

The condition of equilibrium on the two markets (goods and services and monetary):

$$(18)\quad \begin{cases} Y = \dfrac{i_r}{1 - c_Y(1 - ri_Y) - in_Y} r + \dfrac{1}{1 - c_Y(1 - ri_Y) - in_Y} E \\ Y = -\dfrac{m_r}{md_Y} r + \dfrac{M - M_0}{md_Y} \end{cases}$$

After solving the system results:

$$(19)\quad \begin{cases} r = \dfrac{(M - M_0)(1 - c_Y(1 - ri_Y) - in_Y) - md_Y E}{i_r md_Y + m_r(1 - c_Y(1 - ri_Y) - in_Y)} \\ Y = \dfrac{m_r E + i_r(M - M_0)}{i_r md_Y + m_r(1 - c_Y(1 - ri_Y) - in_Y)} \end{cases}$$

We will note below, for simplification:

$$(20)\quad \Lambda = i_r md_Y + m_r(1 - c_Y(1 - ri_Y) - in_Y) < 0$$

$$(21)\quad \Omega = i_r(M - M_0) + m_r E$$

$$(22)\quad \Gamma = \frac{\Omega}{\Lambda^2}$$

from where: $\mathrm{sgn}(\Gamma) = \mathrm{sgn}(\Omega)$.

From formulas (19) we have therefore:

$$(23)\quad \begin{cases} \dfrac{\partial r}{\partial c_Y} = -(1 - ri_Y) md_Y \dfrac{\Omega}{\Lambda^2} - \dfrac{md_Y(TR - T_0)}{\Lambda} \\ \dfrac{\partial r}{\partial ri_Y} = c_Y md_Y \dfrac{\Omega}{\Lambda^2} \\ \dfrac{\partial r}{\partial in_Y} = -md_Y \dfrac{\Omega}{\Lambda^2} \\ \dfrac{\partial r}{\partial md_Y} = -(1 - c_Y(1 - ri_Y) - in_Y) \dfrac{\Omega}{\Lambda^2} \end{cases}$$

$$\begin{cases} \dfrac{\partial Y}{\partial c_Y} = (1 - ri_Y)m_r \dfrac{\Omega}{\Lambda^2} + \dfrac{m_r(TR - T_0)}{\Lambda} \\ \dfrac{\partial Y}{\partial ri_Y} = -c_Y m_r \dfrac{\Omega}{\Lambda^2} \\ \dfrac{\partial Y}{\partial in_Y} = m_r \dfrac{\Omega}{\Lambda^2} \\ \dfrac{\partial Y}{\partial md_Y} = -i_r \dfrac{\Omega}{\Lambda^2} \end{cases} \tag{24}$$

Also:

$$\begin{cases} \dfrac{\partial \Gamma}{\partial c_Y} = 2m_r(1 - ri_Y)\dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial \Gamma}{\partial ri_Y} = -2m_r c_Y \dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial \Gamma}{\partial in_Y} = 2m_r \dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial \Gamma}{\partial md_Y} = -2i_r \dfrac{\Omega}{\Lambda^3} \end{cases} \tag{25}$$

We get now:

$$\begin{cases} \dfrac{\partial^2 r}{\partial c_Y^2} = -\dfrac{2m_r md_Y (1 - ri_Y)^2 \Omega}{\Lambda^3} - \dfrac{m_r md_Y (1 - ri_Y)(TR - T_0)}{\Lambda^2} \\ \dfrac{\partial^2 r}{\partial ri_Y^2} = -2m_r md_Y c_Y^2 \dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial^2 r}{\partial in_Y^2} = -2m_r md_Y \dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial^2 r}{\partial md_Y^2} = 2i_r(1 - c_Y(1 - ri_Y) - in_Y)\dfrac{\Omega}{\Lambda^3} \end{cases} \tag{26}$$

$$(27)\quad \begin{cases} \dfrac{\partial^2 Y}{\partial c_Y^2} = \dfrac{2m_r^2(1-ri_Y)^2\Omega}{\Lambda^3} + \dfrac{m_r^2(1-ri_Y)(TR-T_0)}{\Lambda^2} \\ \dfrac{\partial^2 Y}{\partial ri_Y^2} = 2m_r^2 c_Y^2 \dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial^2 Y}{\partial in_Y^2} = 2m_r^2 \dfrac{\Omega}{\Lambda^3} \\ \dfrac{\partial^2 Y}{\partial md_Y^2} = 2i_r^2 \dfrac{\Omega}{\Lambda^3} \end{cases}$$

In terms of the sign, we have:

$$(28)\quad \begin{cases} \operatorname{sgn}\left(\dfrac{\partial r}{\partial c_Y}\right) = \operatorname{sgn}\left(\dfrac{\partial Y}{\partial c_Y}\right) = -\operatorname{sgn}\left((1-ri_Y)\Omega + (TR-T_0)\Lambda\right) \\ \operatorname{sgn}\left(\dfrac{\partial r}{\partial ri_Y}\right) = \operatorname{sgn}\left(\dfrac{\partial Y}{\partial ri_Y}\right) = \operatorname{sgn}(\Omega) \\ \operatorname{sgn}\left(\dfrac{\partial r}{\partial in_Y}\right) = \operatorname{sgn}\left(\dfrac{\partial Y}{\partial in_Y}\right) = -\operatorname{sgn}(\Omega) \\ \operatorname{sgn}\left(\dfrac{\partial r}{\partial md_Y}\right) = \operatorname{sgn}\left(\dfrac{\partial Y}{\partial md_Y}\right) = \operatorname{sgn}(\Omega) \end{cases}$$

$$(29)\quad \begin{cases} \operatorname{sgn}\left(\dfrac{\partial^2 r}{\partial c_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\partial^2 Y}{\partial c_Y^2}\right) = \operatorname{sgn}\left(\dfrac{2(1-ri_Y)\Omega}{\Lambda} + (TR-T_0)\right) \\ \operatorname{sgn}\left(\dfrac{\partial^2 r}{\partial ri_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\partial^2 Y}{\partial ri_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\Omega}{\Lambda}\right) \\ \operatorname{sgn}\left(\dfrac{\partial^2 r}{\partial in_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\partial^2 Y}{\partial in_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\Omega}{\Lambda}\right) \\ \operatorname{sgn}\left(\dfrac{\partial^2 r}{\partial md_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\partial^2 Y}{\partial md_Y^2}\right) = \operatorname{sgn}\left(\dfrac{\Omega}{\Lambda}\right) \end{cases}$$

The inequality $\Omega = i_r(M-M_0) + m_r E > 0$ becomes $M < M_0 - \dfrac{m_r}{i_r} E$.

After these considerations:

Case 1 $M<M_0-\frac{m_r}{i_r}E$

- If $(1-ri_Y)\Omega+(TR-T_0)\Lambda>0$: $\frac{\partial r}{\partial c_Y}<0$, $\frac{\partial^2 r}{\partial c_Y^2}<0$ then r is decreasing and concave with respect to c_Y; $\frac{\partial Y}{\partial c_Y}<0$, $\frac{\partial^2 Y}{\partial c_Y^2}<0$ then Y is decreasing and concave with respect to c_Y;

- If $2(1-ri_Y)\Omega+(TR-T_0)\Lambda<0$: $\frac{\partial r}{\partial c_Y}>0$, $\frac{\partial^2 r}{\partial c_Y^2}>0$ then r is increasing and convex with respect to c_Y; $\frac{\partial Y}{\partial c_Y}>0$, $\frac{\partial^2 Y}{\partial c_Y^2}>0$ then Y is increasing and convex with respect to c_Y;

- If $(1-ri_Y)\Omega+(TR-T_0)\Lambda<0$ and $2(1-ri_Y)\Omega+(TR-T_0)\Lambda>0$: $\frac{\partial r}{\partial c_Y}>0$, $\frac{\partial^2 r}{\partial c_Y^2}<0$ then r is increasing and concave with respect c_Y; $\frac{\partial Y}{\partial c_Y}>0$, $\frac{\partial^2 Y}{\partial c_Y^2}<0$ then Y is increasing and concave with respect c_Y;

- $\frac{\partial r}{\partial ri_Y}>0$, $\frac{\partial^2 r}{\partial ri_Y^2}<0$ then r is increasing and concave with respect ri_Y; $\frac{\partial Y}{\partial ri_Y}>0$, $\frac{\partial^2 Y}{\partial ri_Y^2}<0$ then Y is increasing and concave with respect ri_Y;

- $\frac{\partial r}{\partial in_Y}<0$, $\frac{\partial^2 r}{\partial in_Y^2}<0$ then r is decreasing and concave with respect to in_Y; $\frac{\partial Y}{\partial in_Y}<0$, $\frac{\partial^2 Y}{\partial in_Y^2}<0$ then Y is decreasing and concave with respect to in_Y;

- $\frac{\partial r}{\partial md_Y}>0$, $\frac{\partial^2 r}{\partial md_Y^2}<0$ then r is increasing and concave with respect md_Y; $\frac{\partial Y}{\partial md_Y}>0$, $\frac{\partial^2 Y}{\partial md_Y^2}<0$ then Y is increasing and concave with respect md_Y.

Case 2 $M>M_0-\frac{m_r}{i_r}E$

- $\frac{\partial r}{\partial c_Y}>0$, $\frac{\partial^2 r}{\partial c_Y^2}>0$ then r is increasing and convex with respect to c_Y; $\frac{\partial Y}{\partial c_Y}>0$, $\frac{\partial^2 Y}{\partial c_Y^2}>0$ then Y is increasing and convex with respect to c_Y;
- $\frac{\partial r}{\partial ri_Y}<0$, $\frac{\partial^2 r}{\partial ri_Y^2}>0$ then r is decreasing and convex with respect to ri_Y; $\frac{\partial Y}{\partial ri_Y}<0$, $\frac{\partial^2 Y}{\partial ri_Y^2}>0$ then Y is decreasing and convex with respect to ri_Y;
- $\frac{\partial r}{\partial in_Y}>0$, $\frac{\partial^2 r}{\partial in_Y^2}>0$ then r is increasing and convex with respect to in_Y; $\frac{\partial Y}{\partial in_Y}>0$, $\frac{\partial^2 Y}{\partial in_Y^2}>0$ then Y is increasing and convex with respect to in_Y;
- $\frac{\partial r}{\partial md_Y}<0$, $\frac{\partial^2 r}{\partial md_Y^2}>0$ then r is decreasing and convex with respect to md_Y; $\frac{\partial Y}{\partial md_Y}<0$, $\frac{\partial^2 Y}{\partial md_Y^2}>0$ then Y is decreasing and convex with respect to md_Y.

4 A Result on the Stability of Solutions of a System of Differential Equations of First Order, Linear, with Constant Coefficients

Lemma

Let the system of differential equations:

$$\begin{pmatrix}\frac{dX}{dt}\\ \frac{dY}{dt}\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}+\begin{pmatrix}e\\ f\end{pmatrix}$$

, a,b,c,d,e,f$\in$**R**, $X(0)=X_0$, $Y(0)=Y_0$.

Then $\lim_{t\to\infty}X(t)=\tilde{X}$, $\lim_{t\to\infty}Y(t)=\tilde{Y}$, $\tilde{X},\tilde{Y}\in\mathbf{R}$ if and only if:

1. a=-d, d^2=-bc, b,c≠0: a= $\frac{ef}{eY_0 - fX_0}$, b= $-\frac{e^2}{eY_0 - fX_0}$, d= $-\frac{ef}{eY_0 - fX_0}$, c= $\frac{f^2}{eY_0 - fX_0}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

2. a=d=0, b≠0, c=0: f=0, b= $-\frac{e}{Y_0}$, e≠0 with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

3. a=d=0, b=0, c≠0: e=0, c= $-\frac{f}{X_0}$, f≠0 with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

4. a=b=c=d=0: e=f=0 with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

5. a=d<0, b=0: c,e,f∈ **R** with the solution:

$$\begin{cases} X = -\frac{e}{a} + \left(X_0 + \frac{e}{a} \right) e^{at} \\ Y = c\left(X_0 + \frac{e}{a} \right) te^{at} + \left(Y_0 - \frac{ce - af}{a^2} \right) e^{at} + \frac{ce - af}{a^2} \end{cases}$$

6. a=d>0, b=0: $a = -\frac{e}{X_0}$, $c = \frac{eY_0 - fX_0}{X_0^2}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

7. a=d<0, b≠0, c=0: e,f∈ **R** with the solution:

$$\begin{cases} X = \left(Y_0 + \frac{f}{a}\right) bte^{at} + \left(X_0 - \frac{bf - ae}{a^2}\right)e^{at} + \frac{bf - ae}{a^2} \\ Y = -\frac{f}{a} + \left(Y_0 + \frac{f}{a}\right)e^{at} \end{cases}$$

8. a=d>0, b≠0, c=0: $a = -\frac{f}{Y_0}$, $b = \frac{fX_0 - eY_0}{Y_0^2}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

9. a≠d, b,c≠0, $(a-d)^2+4bc=0$, a+d<0: e,f∈**R** with the solution:

$$\begin{cases} X = \left(\frac{a-d}{2}X_0 + bY_0 + \frac{2bf + e(a-d)}{a+d}\right)te^{\frac{a+d}{2}t} + \left(X_0 + 4\frac{de - bf}{(a+d)^2}\right)e^{\frac{a+d}{2}t} + 4\frac{bf - de}{(a+d)^2} \\ Y = \left[Y_0 + 4\frac{af - ce}{a+d}\right]e^{\frac{a+d}{2}t} - \left(\frac{a-d}{2}X_0 + bY_0 + \frac{2bf + e(a-d)}{a+d}\right)\frac{a-d}{2b}te^{\frac{a+d}{2}t} + 4\frac{ce - af}{b(a+d)^2} \end{cases}$$

10. a≠d, b,c≠0, $(a-d)^2+4bc=0$, a+d>0: $X_0 = 4\frac{bf - de}{(a+d)^2}$, $Y_0 = 4\frac{ce - af}{(a+d)^2}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

11. $(a-d)^2+4bc>0$, b≠0, a+d<0 and ad-bc>0 and $\lambda_1 \neq \lambda_2$ are roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: e,f∈**R** with the solution:

$$\begin{cases} X = k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t} - \frac{de - bf}{ad - bc} \\ Y = \frac{\lambda_1 - a}{b}k_1e^{\lambda_1 t} + \frac{\lambda_2 - a}{b}k_2e^{\lambda_2 t} + \frac{ce - af}{ad - bc} \end{cases}$$

where:

$$k_1 = \frac{(\lambda_2 - a)X_0 + (\lambda_2 - a)\frac{de - bf}{ad - bc} - bY_0 + b\frac{ce - af}{ad - bc}}{\lambda_2 - \lambda_1}$$

$$k_2 = \frac{bY_0 - b\frac{ce - af}{ad - bc} - (\lambda_1 - a)X_0 - (\lambda_1 - a)\frac{de - bf}{ad - bc}}{\lambda_2 - \lambda_1}$$

12. $(a-d)^2+4bc>0$, $b\neq 0$, $ad-bc<0$ and $\lambda_1<0$, $\lambda_2>0$ are roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: $bY_0 - b\frac{ce - af}{ad - bc} - (\lambda_1 - a)X_0 - (\lambda_1 - a)\frac{de - bf}{ad - bc}=0$ where $\lambda_1\in \mathbf{R}$ is the negative root, with the solution:

$$\begin{cases} X = \left(X_0 + \frac{de - bf}{ad - bc}\right)e^{\lambda_1 t} - \frac{de - bf}{ad - bc} \\ Y = \left(Y_0 - \frac{ce - af}{ad - bc}\right)e^{\lambda_1 t} + \frac{ce - af}{ad - bc} \end{cases}$$

13. $(a-d)^2+4bc>0$, $b\neq 0$, $a+d>0$, $ad-bc>0$ and $\lambda_1\neq\lambda_2$ are roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: $X_0 = \frac{bf - de}{ad - bc}$, $Y_0 = \frac{ce - af}{ad - bc}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

14. $(a-d)^2+4bc>0$, $b=0$, $c\neq 0$, $a,d<0$: $e,f\in \mathbf{R}$ with the solution:

$$\begin{cases} X = k_1\frac{a - d}{c}e^{at} + \frac{bf - de}{ad - bc} \\ Y = k_1e^{at} + k_2e^{dt} + \frac{ce - af}{ad - bc} \end{cases}$$

where:

$$k_1 = \frac{cX_0 - c\frac{bf - de}{ad - bc}}{a - d}$$

$$k_2 = \frac{(a - d)Y_0 - \frac{ce - af}{ad - bc}(a - d) - cX_0 + c\frac{bf - de}{ad - bc}}{a - d}$$

15. $(a-d)^2+4bc>0$, $b=0$, $c\neq 0$, $a<0$, $d>0$: $(a-d)Y_0 - cX_0 - \dfrac{ce-af+df}{d}=0$ with the solution:

$$\begin{cases} X = \left(X_0 + \dfrac{e}{a}\right)e^{at} - \dfrac{e}{a} \\ Y = \left(Y_0 - \dfrac{ce-af}{ad}\right)e^{at} + \dfrac{ce-af}{ad} \end{cases}$$

16. $(a-d)^2+4bc>0$, $b=0$, $c\neq 0$, $a>0$, $d<0$: $a=-\dfrac{e}{X_0}$ with the solution:

$$\begin{cases} X = -\dfrac{e}{a} \\ Y = \dfrac{(a-d)Y_0 - \dfrac{ce-af}{ad}(a-d) - cX_0 - \dfrac{ce}{a}}{a-d}e^{dt} + \dfrac{ce-af}{ad} \end{cases}$$

17. $(a-d)^2+4bc>0$, $b=0$, $c\neq 0$, $a,d>0$: $a=-\dfrac{e}{X_0}$ and $Y_0 = -\dfrac{cX_0+f}{d}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

18. $a\neq d$, $a,d<0$, $b=c=0$: $e,f\in \mathbf{R}$ with the solution::

$$\begin{cases} X = \left(X_0 + \dfrac{e}{a}\right)e^{at} - \dfrac{e}{a} \\ Y = \left(Y_0 + \dfrac{f}{d}\right)e^{dt} - \dfrac{f}{d} \end{cases}$$

19. $a\neq d$, $a<0$, $d>0$, $b=c=0$: $e\in \mathbf{R}$, $d=-\dfrac{f}{Y_0}$ with the solution:

$$\begin{cases} X = \left(X_0 + \dfrac{e}{a}\right)e^{at} - \dfrac{e}{a} \\ Y = Y_0 \end{cases}$$

20. $a\neq d$, $a>0$, $d<0$, $b=c=0$: $a=-\dfrac{e}{X_0}$ with the solution:

$$\begin{cases} X = -\dfrac{e}{a} \\ Y = \left(Y_0 + \dfrac{f}{d}\right)e^{dt} - \dfrac{f}{d} \end{cases}$$

21. a≠d, a,d>0, b=c=0: a= $-\dfrac{e}{X_0}$, d= $-\dfrac{f}{Y_0}$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

22. $(a-d)^2+4bc<0$, $b\neq 0$, $a+d<0$ and $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$, $\beta\neq 0$ are the roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: e,f∈ **R** with the solution:

$$\begin{cases} X = \left(X_0 + \dfrac{de-bf}{ad-bc}\right)e^{\alpha t}\cos\beta t - \\ \dfrac{(\alpha-a)(ad-bc)X_0 + (\alpha-a)(de-bf) - b(ad-bc)Y_0 + b(ce-af)}{\beta(ad-bc)}e^{\alpha t}\sin\beta t - \dfrac{de-bf}{ad-bc} \\ Y = \left(X_0 + \dfrac{de-bf}{ad-bc}\right)\dfrac{\alpha-a}{b}e^{\alpha t}\cos\beta t + \\ \dfrac{cX_0(ad-bc) + c(de-bf) + (ad-bc)(\alpha-a)Y_0 - (ce-af)(\alpha-a)}{\beta(ad-bc)}e^{\alpha t}\sin\beta t - \\ \dfrac{(\alpha-a)(ad-bc)X_0 + (\alpha-a)(de-bf) - b(ad-bc)Y_0}{b(ad-bc)} \end{cases}$$

23. $(a-d)^2+4bc<0$, $b\neq 0$, $a+d>0$ and $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$, $\beta\neq 0$ are the roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: $(ad-bc)X_0 = bf - de$, $(ad-bc)Y_0 = (ce-af)$ with the solution:

$$\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$$

24. $(a-d)^2+4bc<0$, $b\neq 0$, $a+d=0$ and $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$, $\beta\neq 0$ are the roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: e,f∈∅.

5 The Dynamic Equilibrium

Let the system of first order differential equations:

(30) $$\begin{cases} \frac{dY}{dt} = \alpha(D - Y) \\ \frac{dr}{dt} = \beta(MD - M) \end{cases}, \alpha,\beta>0$$

From (12) we have: $D=c_Y Y-c_Y ri_Y Y+in_Y Y+i_r r+E$

from where:

(31) $$\begin{cases} \frac{dY}{dt} = \alpha(c_Y Y - c_Y ri_Y Y + in_Y Y + i_r r + E - Y) \\ \frac{dr}{dt} = \beta(md_Y Y + m_r r + M_0 - M) \end{cases}$$

or otherwise:

(32) $$\begin{cases} \frac{dY}{dt} = -\alpha\chi_Y Y + \alpha i_r r + \alpha E \\ \frac{dr}{dt} = \beta md_Y Y + \beta m_r r + \beta(M_0 - M) \end{cases}$$

where we note $\chi_Y=1-c_Y(1-ri_Y)-in_Y>0$

In matrix notation, the system becomes:

(33) $$\begin{pmatrix} \frac{dY}{dt} \\ \frac{dr}{dt} \end{pmatrix} = \begin{pmatrix} -\alpha\chi_Y & \alpha i_r \\ \beta md_Y & \beta m_r \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} + \begin{pmatrix} \alpha E \\ \beta(M_0 - M) \end{pmatrix}$$

Using the above lemma, it follows that: $\lim_{t\to\infty} Y(t) = \tilde{Y}$, $\lim_{t\to\infty} r(t) = \tilde{r}$, $\tilde{Y}, \tilde{r} \in \mathbf{R}_+$ if and only if:

1. $(\alpha\chi_Y+\beta m_r)^2+4\alpha\beta i_r md_Y=0$ then:

$$\begin{cases} Y = \left(-\dfrac{\alpha\chi_Y + \beta m_r}{2} Y_0 + \alpha i_r r_0 - \alpha \dfrac{2 i_r \beta (M_0 - M) - E(\alpha\chi_Y + \beta m_r)}{\alpha\chi_Y - \beta m_r}\right) t e^{\frac{-\alpha\chi_Y + \beta m_r}{2} t} + \\ \left(Y_0 + 4\alpha\beta \dfrac{m_r E - i_r (M_0 - M)}{(\alpha\chi_Y - \beta m_r)^2}\right) e^{\frac{-\alpha\chi_Y + \beta m_r}{2} t} + 4\alpha\beta \dfrac{i_r (M_0 - M) - m_r E}{(\alpha\chi_Y - \beta m_r)^2} \\ r = \left[r_0 + 4\alpha\beta \dfrac{\chi_Y (M_0 - M) + m d_Y E}{\alpha\chi_Y - \beta m_r} \right] e^{\frac{-\alpha\chi_Y + \beta m_r}{2} t} - \\ \left(\dfrac{\alpha\chi_Y + \beta m_r}{2} Y_0 - \alpha i_r r_0 + \alpha \dfrac{2 i_r \beta (M_0 - M) - E(\alpha\chi_Y + \beta m_r)}{\alpha\chi_Y - \beta m_r}\right) \dfrac{\alpha\chi_Y + \beta m_r}{2\alpha i_r} t e^{\frac{-\alpha\chi_Y + \beta m_r}{2} t} - \\ \quad 4\beta \dfrac{m d_Y E + \chi_Y (M_0 - M)}{i_r (\alpha\chi_Y - \beta m_r)^2} \end{cases}$$

and:
$$\begin{cases} \tilde{Y} = 4\alpha\beta \dfrac{i_r (M_0 - M) - m_r E}{(\alpha\chi_Y - \beta m_r)^2} \\ \tilde{r} = -4\beta \dfrac{m d_Y E + \chi_Y (M_0 - M)}{i_r (\alpha\chi_Y - \beta m_r)^2} \end{cases}$$

2. $(\alpha\chi_Y+\beta m_r)^2+4\alpha\beta i_r m d_Y>0$ and $\lambda_1\neq\lambda_2$ are roots of the equation: $\lambda^2+(\alpha\chi_Y -\beta m_r)\lambda-\alpha\beta(\chi_Y m_r+i_r m d_Y)=0$ then:

$$\begin{cases} Y = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + \dfrac{m_r E - i_r (M_0 - M)}{\chi_Y m_r + i_r m d_Y} \\ r = \dfrac{\lambda_1 + \alpha\chi_Y}{\alpha i_r} k_1 e^{\lambda_1 t} + \dfrac{\lambda_2 + \alpha\chi_Y}{\alpha i_r} k_2 e^{\lambda_2 t} - \dfrac{m d_Y E + \chi_Y (M_0 - M)}{\chi_Y m_r + i_r m d_Y} \end{cases}$$

where:

$$k_1 = \frac{(\lambda_2 + \alpha\chi_Y) Y_0 - (\lambda_2 + \alpha\chi_Y) \dfrac{m_r E - i_r (M_0 - M)}{\chi_Y m_r + i_r m d_Y} - \alpha i_r r_0 - \alpha i_r \dfrac{m d_Y E + \chi_Y (M_0 - M)}{\chi_Y m_r + i_r m d_Y}}{\lambda_2 - \lambda_1}$$

$$k_2 = \frac{\alpha i_r r_0 + \alpha i_r \dfrac{m d_Y E + \chi_Y (M_0 - M)}{\chi_Y m_r + i_r m d_Y} - (\lambda_1 + \alpha\chi_Y) Y_0 + (\lambda_1 + \alpha\chi_Y) \dfrac{m_r E - i_r (M_0 - M)}{\chi_Y m_r + i_r m d_Y}}{\lambda_2 - \lambda_1}$$

$$\begin{cases}\tilde{Y}=\dfrac{m_rE-i_r\left(M_0-M\right)}{\chi_Y m_r+i_r md_Y}\\ \tilde{r}=-\dfrac{md_YE+\chi_Y\left(M_0-M\right)}{\chi_Y m_r+i_r md_Y}\end{cases}$$

and:

3. $(\alpha\chi_Y+\beta m_r)^2+4\alpha\beta i_r md_Y<0$ and $\lambda_1=\mu+i\nu$, $\lambda_2=\mu-i\nu$, $\nu\neq 0$ are roots of the equation: $\lambda^2+(\alpha\chi_Y-\beta m_r)\lambda-\alpha\beta(\chi_Y m_r+i_r md_Y)=0$ then:

$$\begin{cases}Y=\left(Y_0-\dfrac{m_rE-i_r\left(M_0-M\right)}{\chi_Y m_r+i_r md_Y}\right)e^{\mu t}\cos\nu t-\\ \dfrac{\left(\chi_Y m_r+i_r md_Y\right)\left[\left(\mu+\alpha\chi_Y\right)Y_0-\alpha i_r r_0\right]-\left(\mu+\alpha\chi_Y\right)m_rE+\mu i_r\left(M_0-M\right)-\alpha i_r md_YE}{\nu\left(\chi_Y m_r+i_r md_Y\right)}e^{\mu t}\sin\nu t+\\ \dfrac{m_rE-i_r\left(M_0-M\right)}{\chi_Y m_r+i_r md_Y}\\ r=\left(Y_0-\dfrac{m_rE-i_r\left(M_0-M\right)}{\chi_Y m_r+i_r md_Y}\right)\dfrac{\mu+\alpha\chi_Y}{\alpha i_r}e^{\mu t}\cos\nu t+\\ \dfrac{\left(\chi_Y m_r+i_r md_Y\right)\left[\beta md_YY_0+\left(\mu+\alpha\chi_Y\right)r_0\right]-\beta md_Y\left(m_rE-i_r\left(M_0-M\right)\right)+\left(md_YE+\chi_Y\left(M_0-M\right)\right)\left(\mu+\alpha\chi_Y\right)}{\nu\left(\chi_Y m_r+i_r md_Y\right)}e^{\mu t}\sin\nu t+\\ \dfrac{\left(\chi_Y m_r+i_r md_Y\right)\left[\alpha i_r r_0-\left(\mu+\alpha\chi_Y\right)Y_0\right]+\left(\mu+\alpha\chi_Y\right)\left(m_rE-i_r\left(M_0-M\right)\right)}{\alpha i_r\left(\chi_Y m_r+i_r md_Y\right)}\end{cases}$$

$$\begin{cases}\tilde{Y}=\dfrac{m_rE-i_r\left(M_0-M\right)}{\chi_Y m_r+i_r md_Y}\\ \tilde{r}=\dfrac{\left(\chi_Y m_r+i_r md_Y\right)\left[\alpha i_r r_0-\left(\mu+\alpha\chi_Y\right)Y_0\right]+\left(\mu+\alpha\chi_Y\right)\left(m_rE-i_r\left(M_0-M\right)\right)}{\alpha i_r\left(\chi_Y m_r+i_r md_Y\right)}\end{cases}$$

and:

6 Conclusions

The above analysis highlights the following issues:

- At an offer of money M upper limited by $M_0-\frac{m_r}{i_r}E$ where $E=c_Y(TR-T_0)+C_0+I_0+\overline{G}$ is obtained that if $(1-ri_Y)\Omega+\left(TR-T_0\right)\Lambda>0$ then the interest rate and the national income are decreasing and concave in relation to the marginal propensity to consume; if $2(1-ri_Y)\Omega+\left(TR-T_0\right)\Lambda<0$ then the interest rate and the national income are increasing and convex in relation to the marginal propensity to consume; if $(1-ri_Y)\Omega+\left(TR-T_0\right)\Lambda<0$ and $2(1-ri_Y)\Omega+\left(TR-T_0\right)\Lambda>0$ then the interest rate and the national income are increasing and concave in relation to the marginal propensity to consume. Also, the interest rate and the national income are increasing and concave with respect to tax rates, are decreasing and concave in relation to the investment rate, and increasing and concave in relation to the rate of currency demand.

- At an offer of money M lower limited by $M_0-\frac{m_r}{i_r}E$ where $E=c_Y(TR-T_0)+C_0+I_0+\overline{G}$ we get that the interest rate and the national income are increasing and convex in relation to the marginal propensity to consume, decreasing and convex with respect to tax rates, increasing and convex in relation to the investment rate, decreasing and convex in relation to the rate of currency demand.
- In relation to time, the national income and the interest rate have a tendency to stabilize, their evolution and limit values being specified above.

7 References

Dornbusch, R.; Fischer, S.; Startz R. (1997). *Macroeconomics*. Richard D. Irwin Publishers.

Mankiw, N.G. (2010). *Macroeconomics*. Worth Publishers.

Romer, D. (1996). *Advanced Macroeconomics*. McGraw-Hill.

Stancu S. & Mihail N. (2009). *Macroeconomie. Modele statice și dinamice de comportament. Teorie și aplicații/Macroeconomics. Static and dynamic models of behavior. Theory and applications.* Bucharest: Economică.