Microeconomics

Discussions on n Substitutable Goods Production and Consumption

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Abstract: The analysis takes into account the issue of production of n consumer goods whose destination is either the mass of workers who have contributed to them or third parties such as social categories, the directly unproductive or abroad. In the analysis, we considered, for simplicity, utility and production functions of Cobb-Douglas type that allowed finally getting interesting conclusions on the relationship between the appropriate elasticities.

Keywords: production; consumption; Cobb-Douglas

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1 Introduction

The problem about equilibrium between production and consumption is particularly delicate.

Currently there are several theoretical models at both microeconomic and macroeconomic level trying to provide solutions to balance the production so as not appear an over or underproduction.

This analysis takes into account the issue of production of n consumer goods whose destination is either the mass of workers who have contributed to them or third parties such as social categories, the directly unproductive or abroad.

In the analysis, we considered, for simplicity, utility and production functions of Cobb-Douglas type that allowed finally getting interesting conclusions on the relationship between the appropriate elasticities. All items considered were assumed to be perfect substitutes, competition being also perfect.

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2 Theoretical Analysis

Let consider a number of n goods, perfect substitutes, G_i, $i=\overline{1,n}$ produced by n companies F_i, $i=\overline{1,n}$ that have a number of workers – L_i and capital – K_i.

We will consider that the production function for the good G_i is Cobb-Douglas type:

(1) $Q_i= A_i K_i^{\alpha_i} L_i^{\beta_i}$, $\alpha_i,\beta_i\in(0,1)$, $A_i>0$, $i=\overline{1,n}$.

For each company F_i, let the price of the labor L_i – p_i and the price of capital K_i – q_i.

The total cost of production of the good G_i is:

(2) $CT_i(K_i,L_i)= p_iL_i+q_iK_i$.

Now consider that each firm sets a good production $\overline{Q}_i$ for the good G_i, $i=\overline{1,n}$.

The minimizing of the total cost for production of G_i leads to:

$$(3)\ \begin{cases} \min(p_iL_i + q_iK_i) \\ Q_i(K_i,L_i) \ge \overline{Q}_i \\ K_i, L_i \ge 0 \end{cases}.$$

The above nonlinear programming problem is subject to Karush-Kuhn-Tucker conditions which states that the problem:

$$(4)\ \begin{cases} \min f(x_1,...,x_n) \\ g_i(x_1,...,x_n) \le 0, i=\overline{1,p} \\ h_j(x_1,...,x_n) = 0, j=\overline{1,q} \\ x_1,...,x_n \ge 0 \end{cases}$$

where $f,g,h\in C^2(D)$, D – domain, has the solution $(\overline{x}_1,...,\overline{x}_n)$ if $\exists\lambda_i\in\mathbf{R}_+$, $i=\overline{1,p}$ $\exists\nu_j\in\mathbf{R}$, $j=\overline{1,q}$ such that:

$$(5)\ \begin{cases} \nabla f(\overline{x}_1,...,\overline{x}_n) + \sum_{i=1}^{p}\lambda_i\nabla g_i(\overline{x}_1,...,\overline{x}_n) + \sum_{j=1}^{q}\nu_j\nabla h_j(\overline{x}_1,...,\overline{x}_n) = 0 \\ g_i(\overline{x}_1,...,\overline{x}_n) \le 0, i=\overline{1,p} \\ h_j(\overline{x}_1,...,\overline{x}_n) = 0, j=\overline{1,q} \\ \lambda_i g_i(\overline{x}_1,...,\overline{x}_n) = 0, i=\overline{1,p} \end{cases}$$

where ∇F is the gradient of F defined by: $\nabla F=\left(\frac{\partial F}{\partial x_1},...,\frac{\partial F}{\partial x_n}\right)$.

On detail, the Karush-Kuhn-Tucker conditions becomes:

$$(6)\begin{cases}\frac{\partial f}{\partial x_k}(\overline{x}_1,...,\overline{x}_n)+\sum_{i=1}^{p}\lambda_i\frac{\partial g_i}{\partial x_k}(\overline{x}_1,...,\overline{x}_n)+\sum_{j=1}^{q}\nu_j\frac{\partial h_j}{\partial x_k}(\overline{x}_1,...,\overline{x}_n)=0,\ k=\overline{1,n}\\ g_i(\overline{x}_1,...,\overline{x}_n)\le 0,\ i=\overline{1,p}\\ h_j(\overline{x}_1,...,\overline{x}_n)=0,\ j=\overline{1,q}\\ \lambda_i g_i(\overline{x}_1,...,\overline{x}_n)=0,\ i=\overline{1,p}\end{cases}$$

The Karush-Kuhn-Tucker conditions are sufficient if f,g_i, $i=\overline{1,p}$ are convex of class C^2, and h_j, $j=\overline{1,q}$ are affine functions.

In the particular case of our problem, we have:

$$(7)\begin{cases}q_i-\lambda\frac{\partial Q_i}{\partial K_i}=0\\ p_i-\lambda\frac{\partial Q_i}{\partial L_i}=0\\ Q_i(K_i,L_i)\ge\overline{Q}_i\\ \lambda\left(\overline{Q}_i-Q_i(K_i,L_i)\right)=0\end{cases}$$

where, as p_i, $q_i\neq 0$ follows:

$$(8)\begin{cases}q_i-\lambda\frac{\partial Q_i}{\partial K_i}=0\\ p_i-\lambda\frac{\partial Q_i}{\partial L_i}=0\\ Q_i(K_i,L_i)=\overline{Q}_i\end{cases}$$

After the removal of λ:

$$(9)\begin{cases}q_i\frac{\partial Q_i}{\partial L_i}=p_i\frac{\partial Q_i}{\partial K_i}\\ Q_i(K_i,L_i)=\overline{Q}_i\end{cases}$$

From (1) and (9) follows:

$$(10)\quad \begin{cases} K_i^* = \dfrac{p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} \alpha_i^{\frac{\beta_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}}{A_i^{\frac{1}{\alpha_i+\beta_i}} q_i^{\frac{\beta_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}} \\ L_i^* = \dfrac{q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}}{A_i^{\frac{1}{\alpha_i+\beta_i}} p_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}}} \end{cases}$$

Also, from (2) and (10), the total cost function becomes:

$$(11)\quad CT_i = p_i L_i^* + q_i K_i^* = \frac{(\alpha_i+\beta_i) p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}}$$

Now consider the selling price r_i of the good G_i. The received income is: $V_i = r_i \overline{Q}_i$, and the profit:

$$(12)\quad \Pi_i(\overline{Q}_i) = r_i \overline{Q}_i - \frac{(\alpha_i+\beta_i) p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}}$$

The extreme condition of the profit: $\Pi_i'(\overline{Q}_i) = 0$ implies:

$$(13)\quad r_i = \frac{p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}-1}}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}}$$

We will assume below, that not all of the amount produced is consumed by the workers, some of which being for those who are not directly productive (social assisted, educational, health, public administration etc.) or those not being part of production of the considered goods. We will denote by $\mu_i \in (0,1)$ the share of the good G_i consumption reported to the total production $\overline{Q}_i$.

We will also consider that the total income of a worker is allocated to consumption of goods in some $\sigma \in (0,1)$, the difference being allocated to pay taxes or consumption of foreign goods in other manufacturing companies.

Let consider now the utility function, the same for all consumers, of Cobb-Douglas type:

(14) $U(x_1,...,x_n)= Bx_1^{\gamma_1}...x_n^{\gamma_n}$, $\gamma_i \in (0,1)$, $i=\overline{1,n}$, $\gamma_1+...+\gamma_n=1$, $B>0$

corresponding to the n goods.

The total disposable income (for purchase of goods G_i, $i=\overline{1,n}$) of the $L=\sum_{i=1}^{n} L_i^*$ workers is:

(15) $$\sum_{i=1}^{n}\sigma p_i L_i^* = \sum_{i=1}^{n}\sigma \frac{p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}}}$$

Proceeding as above, from the Karush-Kuhn-Tucker conditions, it follows that the utility maximization under budget constraint: $\sum_{i=1}^{n} r_i \mu_i \overline{Q}_i \le \sum_{i=1}^{n}\sigma p_i L_i^*$ satisfies:

(16) $$\begin{cases} \dfrac{\frac{\partial U}{\partial \overline{Q}_1}}{r_1} = ... = \dfrac{\frac{\partial U}{\partial \overline{Q}_n}}{r_n} \\ \sum_{i=1}^{n} r_i \mu_i \overline{Q}_i = \sum_{i=1}^{n}\sigma p_i L_i^* \end{cases}$$

Using (10), (13), (14) we get:

(17) $$\begin{cases} \dfrac{\gamma_1 A_1^{\frac{1}{\alpha_1+\beta_1}} \alpha_1^{\frac{\alpha_1}{\alpha_1+\beta_1}} \beta_1^{\frac{\beta_1}{\alpha_1+\beta_1}}}{\mu_1 p_1^{\frac{\beta_1}{\alpha_1+\beta_1}} q_1^{\frac{\alpha_1}{\alpha_1+\beta_1}} \overline{Q}_1^{\frac{1}{\alpha_1+\beta_1}}} = ... = \dfrac{\gamma_i A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}}{\mu_i p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}} = ... = \dfrac{\gamma_n A_n^{\frac{1}{\alpha_n+\beta_n}} \alpha_n^{\frac{\alpha_n}{\alpha_n+\beta_n}} \beta_n^{\frac{\beta_n}{\alpha_n+\beta_n}}}{\mu_n p_n^{\frac{\beta_n}{\alpha_n+\beta_n}} q_n^{\frac{\alpha_n}{\alpha_n+\beta_n}} \overline{Q}_n^{\frac{1}{\alpha_n+\beta_n}}} \\ \sum_{i=1}^{n} \dfrac{p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}} (\mu_i - \sigma\beta_i)}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}} = 0 \end{cases}$$

Noting with λ the common value of the ratios, we obtain:

(18)
$$\begin{cases} \dfrac{\gamma_i A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}}{\mu_i p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}}} = \lambda, i = \overline{1,n} \\ \sum_{i=1}^{n} \dfrac{p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}} (\mu_i - \sigma\beta_i)}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}} = 0 \end{cases}$$

from where:

(19)
$$\begin{cases} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}} = \dfrac{\gamma_i A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}}{\lambda \mu_i p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}}}, i = \overline{1,n} \\ \sum_{i=1}^{n} \dfrac{p_i^{\frac{\beta_i}{\alpha_i+\beta_i}} q_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \overline{Q}_i^{\frac{1}{\alpha_i+\beta_i}} (\mu_i - \sigma\beta_i)}{A_i^{\frac{1}{\alpha_i+\beta_i}} \alpha_i^{\frac{\alpha_i}{\alpha_i+\beta_i}} \beta_i^{\frac{\beta_i}{\alpha_i+\beta_i}}} = 0 \end{cases}$$

Substituting in the second equation follows:

(20)
$$\sum_{i=1}^{n} \frac{\gamma_i (\mu_i - \sigma\beta_i)}{\mu_i} = 0$$

or, taking into account that $\gamma_1+...+\gamma_n=1$:

(21)
$$\sum_{i=1}^{n} \frac{\gamma_i \beta_i}{\mu_i} = \frac{1}{\sigma} > 1$$

Let now the expression $\Omega_i = \mu_i - \sigma\beta_i$, $i=\overline{1,n}$. Beacuse γ_i, $\mu_i>0$, from the relation (20) we get that if $\Omega_i \geq 0$, $i=\overline{1,n}$: $\sum_{i=1}^{n} \frac{\gamma_i (\mu_i - \sigma\beta_i)}{\mu_i} \geq 0$ therefore the equality holds only for $\Omega_i=0$ that is: $\mu_i=\sigma\beta_i$, $i=\overline{1,n}$.

Following this result, we obtain from the assumption that consumption of a high enough rate of production of a particular good, it will be, inevitably, upper limited by the product between the share of income allocated to consumption and the corresponding elasticity of the good's labor. On the other hand, in terms of fixed capital and a constant number of workers, the elasticity depends (for the Cobb-Douglas function) directly from the logarithm of labor productivity. Therefore, a higher share of consumption (and therefore a higher share of production achieved) can be obtained only under labor productivity growth.

Analogously for $\Omega_i \le 0$, i= $\overline{1,n}$. In this case, assuming a rate below a certain level of consumption of all goods, will push consumption to equal the product between the share of income allocated to consumption and the corresponding elasticity of the good's labor.

If $\exists$ i= $\overline{1,n}$ such that $\Omega_i \neq 0$ therefore $\mu_i \neq \sigma\beta_i$ then $\exists k \neq p = \overline{1,n}$ such that: $\mu_k > \sigma\beta_k$ şi $\mu_p < \sigma\beta_p$.

In other words, for a given elasticity of labor, it exist in this case two products for which the share of consumption is lower limited by the product of the share of income allocated to purchase the n products and the elasticity of labor in the corresponding production, and upper limited respectively.

For the good G_k, an increase of the labor elasticity (under the same part of the income allocation) will push up the rate of consumption of G_k. Similarly, for the good G_p, a reduction of the elasticity of labor (under the same part of the income allocation) will push down the rate of consumption of G_p.

Returning at the equation (21), we have for two goods: $\frac{\gamma_1\beta_1}{\mu_1} + \frac{\gamma_2\beta_2}{\mu_2} = \frac{1}{\sigma}$ with $\gamma_1 + \gamma_2 = 1$ from where:

(22) $$\gamma_1 = \frac{\mu_1\mu_2 - \sigma\mu_1\beta_2}{\sigma(\beta_1\mu_2 - \beta_2\mu_1)}, \ \gamma_2 = 1 - \gamma_1$$

The Jensen's inequality says that for every convex (concave) function $f: D \subset \mathbf{R} \to \mathbf{R}$, the following inequality holds: $f\left(\frac{\sum_{i=1}^{n}\xi_i x_i}{\sum_{i=1}^{n}\xi_i}\right) \le (\ge) \frac{\sum_{i=1}^{n}\xi_i f(x_i)}{\sum_{i=1}^{n}\xi_i}$ $\forall x_i \in D$, $\forall \xi_i > 0$. The equality becomes effective if and only if: $x_1 = ... = x_n$.

In particular, for $\sum_{i=1}^{n}\xi_i = 1$ şi f(x)=ln x follows:

(23) $$\sum_{i=1}^{n}\xi_i x_i \ge \prod_{i=1}^{n} x_i^{\xi_i}$$

For $x_i \to \frac{1}{x_i}$ we have: $\prod_{i=1}^{n} x_i^{\xi_i} \geq \frac{1}{\sum_{i=1}^{n} \frac{\xi_i}{x_i}}$ therefore:

$$(24) \qquad \sum_{i=1}^{n} \xi_i x_i \geq \frac{1}{\sum_{i=1}^{n} \frac{\xi_i}{x_i}}$$

Again, in particular, for $\xi_i = \frac{\beta_i}{\mu_i}$ we obtain: $\sum_{i=1}^{n} \gamma_i \frac{\beta_i}{\mu_i} \geq \frac{1}{\sum_{i=1}^{n} \gamma_i \frac{\mu_i}{\beta_i}}$ therefore, from (21):

$$(25) \qquad \frac{1}{\sum_{i=1}^{n} \gamma_i \frac{\mu_i}{\beta_i}} \leq \frac{1}{\sigma}$$

For $\beta_i=\beta$, $i=\overline{1,n}$, we have:

$$(26) \qquad \sum_{i=1}^{n} \gamma_i \mu_i \geq \sigma\beta$$

Let us note, in relation to formula (26), that $\sum_{i=1}^{n} \gamma_i \mu_i$ is the weighted average of the consumer utility rates with the utility elasticities in relation to each product and is lower bounded by the product of the allocated income share and the labor elasticity.

If, in addition, $\mu_i=\mu$, $i=\overline{1,n}$ then the equation (26) becomes equality and we have:

$$(27) \qquad \mu=\sigma\beta$$

Therefore, at the same elasticity of labor and consumption the same share of each product, the share of consumption will be equal to the product between the part of the income allocated and the labor elasticity. An increasing of the share can be achieved, in this case, either by increasing σ, or by increasing the elasticity of labor.

3 Conclusions

The above analysis reveals, through the formula (21), that the share of consumption goods relative to production is dependent on both the elasticity of production and that of the utility function in relation to consumption of each good, but also from the share of income allocated to purchase those goods.

In the Romanian conditions, where the labor elasticity is 0.51 and assuming that the share of consumption relative to production of goods is the same, we see that this is about half (μ=0.51σ) from the disposable income share of earnings workers allocated to purchase goods. The output gap (whose rate is 0.49 σ) will address to the rest of the population.

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