DEGENERATE FOLIATIONS IN SASAKIAN SEMI-RIEMANNIAN MANIFOLDS

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Abstract: *In the Semi-Riemannian case we do not have the liability of the existence of such a metric being a difference from the Riemannian case. A Semi-Riemannian manifold provided with a normal contact metric structure is called Sasakian manifold.*

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In the Semi-Riemannian case we do not have the liability of the existence of such a metric being a difference from the Riemannian case.

Definition Let $M=M^{2n+1}$ a connected Semi-Riemannian manifold and φ, ξ, η tensorial fields on M of types (1,1), (1,0) and (0,1) respectively. We say that (φ,ξ,η) define an almost contact structure on M if:

(1) $$\varphi^2 Y=-Y+\eta(Y)\xi \quad \forall Y\in \mathcal{X}(M)$$

(2) $$\eta(\xi)=1$$

In these conditions we have the relations:

(3) $$\eta(\varphi Y)=0 \quad \forall Y\in \mathcal{X}(M)$$

(4) $$\varphi\xi=0$$

We define the contact distribution like $D=\{Y\in \mathcal{X}(M) \mid \eta(Y)=0\}$.

Definition A metric g on M for which:

(5) $$g(\varphi X,\varphi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y) \quad \forall X,Y\in \mathcal{X}(M)$$

where $\varepsilon\in\{-1,1\}$ is called compatible metric with the almost contact structure.

In the Semi-Riemannian case we do not have the liability of the existence of such a metric being a difference from the Riemannian case.

Definition If on a Semi-Riemannian manifold M^{2n+1} we have an almost contact structure (φ,ξ,η) and a compatible metric with the structure we shall say that (φ,ξ,η,g) define an almost contact metric structure.

For $Y=\xi$ in (1) we have with (2) and (4):

(6) $$g(X,\xi)=\varepsilon\eta(X)\ \forall X\in \mathcal{X}(M)$$

and if in (6) $Y=\xi$:

(7) $$g(\xi,\xi)=\varepsilon$$

Remark If we shall consider in the upper $g'=-g$, $\xi'=-\xi$, $\eta'=-\eta$ and $\varphi'=\varphi$ we have together with $\varepsilon'=-\varepsilon$ also an almost contact metric structure. We can further suppose that $\varepsilon=1$.

Definition: Let $M=M^{2n+1}$ a connected Semi-Riemannian manifold provided with an almost contact metric structure (φ,ξ,η,g). We shall say that the structure is contact metric if:

(8) $$(d\eta)(X,Y)=g(\varphi X,Y)\ \forall X,Y\in \mathcal{X}(M)$$

Definition: An almost contact metric structure (φ,ξ,η,g) is normal if:

(9) $$(\nabla_X\varphi)Y= g(X,Y)\xi-\eta(Y)X\ \forall X,Y\in \mathcal{X}(M)$$

where ∇ is the Levi-Civita connection for the metric g.

Definition A Semi-Riemannian manifold provided with a normal contact metric structure is called Sasakian manifold.

On a Sasakian manifold the following relations hold:

(10) $$\nabla_X\xi=-\varphi X\ \forall X\in \mathcal{X}(M)$$

(11) $$N^1(X,Y)=N_\varphi(X,Y)+2d\eta(X,Y)\xi=0\ \forall X,Y\in \mathcal{X}(M)$$

(12) $$N^2(X,Y)=(L_{\varphi X}\eta)Y-(L_{\varphi Y}\eta)X=0\ \forall X,Y\in \mathcal{X}(M)$$

(13) $$N^3(X)=(L_\xi\varphi)X=0\ \forall X\in \mathcal{X}(M)$$

(14) $$N^4(X)=(L_\xi\eta)X=0\ \forall X\in \mathcal{X}(M)$$

where $N_\varphi(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+\varphi^2[X,Y]$.

Let now a φ-basis of TM that is an orthonormal basis of the form: $\{\xi, Y_1,...,Y_n,\varphi Y_1,...,\varphi Y_n\}$ where $Y_i\in D$, $i=1,...,n$. We denote by

$$\varepsilon_i=g(Y_i,Y_i)=g(\varphi Y_i,\varphi Y_i),\ i=1,...,n.$$

We define now the 1-differential forms:

(15) $$\omega_i(Z)=g(Y_i,Z)\ \forall Z\in \mathcal{X}(M)\ \forall i=1,...,n$$

and from (15) we have:

(16) $$g(\varphi Y_i,Z)=-\omega_i(\varphi Z)\ \forall Z\in \mathcal{X}(M)\ \forall i=1,...,n$$

Let now two arbitrary vector fields:

(17) $$X=\eta(X)\xi+\sum_{i=1}^{n}\varepsilon_i\omega_i(X)Y_i-\sum_{i=1}^{n}\varepsilon_i\omega_i(\varphi X)\varphi Y_i$$

(18) $$Z=\eta(Z)\xi+\sum_{j=1}^{n}\varepsilon_j\omega_j(Z)Y_j-\sum_{j=1}^{n}\varepsilon_j\omega_j(\varphi Z)\varphi Y_j$$

We have now:

(19) $$[X,Z]=\{\eta(X)\nabla_\xi\eta(Z)-\eta(Z)\nabla_\xi\eta(X)+\sum_{j=1}^{n}\varepsilon_j\omega_j(X)\nabla_{Y_j}\eta(Z)-$$
$$\sum_{j=1}^{n}\varepsilon_j\omega_j(Z)\nabla_{Y_j}\eta(X)+\sum_{j=1}^{n}\varepsilon_j\omega_j(\varphi Z)\nabla_{\varphi Y_j}\eta(X)-\sum_{j=1}^{n}\varepsilon_j\omega_j(\varphi X)\nabla_{\varphi Y_j}\eta(Z)\}\xi+$$
$$\sum_{j=1}^{n}\varepsilon_j\{\eta(X)\nabla_\xi\omega_j(Z)-\eta(Z)\nabla_\xi\omega_j(X)+\sum_{i=1}^{n}\varepsilon_i\omega_i(X)\nabla_{Y_i}\omega_j(Z)-$$
$$\sum_{i=1}^{n}\varepsilon_i\omega_i(Z)\nabla_{Y_i}\omega_j(X)+\sum_{i=1}^{n}\varepsilon_i\omega_i(\varphi Z)\nabla_{\varphi Y_i}\omega_j(X)-$$
$$\sum_{i=1}^{n}\varepsilon_i\omega_i(\varphi X)\nabla_{\varphi Y_i}\omega_j(Z)\}Y_j+\sum_{j=1}^{n}\varepsilon_j\{\eta(Z)\nabla_\xi\omega_j(\varphi Z)-\eta(X)\nabla_\xi\omega_j(\varphi Z)+\sum_{i=1}^{n}\varepsilon_i\omega_i(Z)\nabla_{Y_i}\omega_j(\varphi X)-$$
$$\sum_{i=1}^{n}\varepsilon_i\omega_i(X)\nabla_{Y_i}\omega_j(\varphi Z)+\sum_{i=1}^{n}\varepsilon_i\omega_i(\varphi X)\nabla_{\varphi Y_i}\omega_j(\varphi Z)-\sum_{i=1}^{n}\varepsilon_i\omega_i(\varphi Z)\nabla_{\varphi Y_i}\omega_j(\varphi X)\}\varphi Y_j+$$
$$\sum_{j=1}^{n}\varepsilon_j\{\eta(X)\omega_j(Z)-\eta(Z)\omega_j(X)\}[\xi,Y_j]+\sum_{j=1}^{n}\varepsilon_j\{\eta(Z)\omega_j(\varphi X)-\eta(X)\omega_j(\varphi Z)\}[\xi,\varphi Y_j]-$$
$$\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\,\omega_i(Z)\omega_j(X)[Y_i,Y_j]+\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\{\omega_i(Z)\omega_j(\varphi X)-\omega_i(X)\omega_j(\varphi Z)\}[Y_i,\varphi Y_j]-$$
$$\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\,\omega_i(\varphi Z)\omega_j(\varphi X)[\varphi Y_i,\varphi Y_j].$$

If we define now:

(20) $$Z_j=\omega_j(Z)X-\omega_j(X)Z\ \forall j=1,...,n$$

(21) $$W_j=\omega_j(\varphi Z)X-\omega_j(\varphi X)Z\ \forall j=1,...,n$$

(22) $$T=\eta(X)Z-\eta(Z)X$$

we obtain:

(23) $$[X,Z]=(\mathrm{div}\,Z)X-(\mathrm{div}\,X)Z-\{\mathrm{div}\,T-2\sum_{j=1}^{n}\varepsilon_j\omega_j(W_j)\}\xi+\sum_{j=1}^{n}\varepsilon_j(\mathrm{div}\,Z_j)Y_j-$$
$$\sum_{j=1}^{n}\varepsilon_j(\mathrm{div}\,W_j)\varphi Y_j+\sum_{j=1}^{n}\varepsilon_j\eta(Z_j)[\xi,Y_j]-\sum_{j=1}^{n}\varepsilon_j\eta(W_j)\,\varphi[\xi,Y_j]-\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\,\omega_j(W_i)[\varphi Y_i,Y_j]-$$
$$\sum_{\substack{i,j=1\\ i<j}}^{n}\varepsilon_i\varepsilon_j\,\omega_j(\varphi W_i)\varphi[\varphi Y_i,Y_j]+\sum_{\substack{i,j=1\\ i>j}}^{n}\varepsilon_i\varepsilon_j\,\omega_i(\varphi W_j)\varphi[\varphi Y_i,Y_j]+\sum_{\substack{i,j=1\\ i<j}}^{n}\varepsilon_i\varepsilon_j\{\omega_i(Z_j)-\omega_j(\varphi W_i)\}[Y_i,Y_j].$$

In order that the two vector fields define a foliation we must have:

(24) $$-\{\mathrm{div}\,T-2\sum_{j=1}^{n}\varepsilon_j\omega_j(W_j)\}\xi+\sum_{j=1}^{n}\varepsilon_j(\mathrm{div}\,Z_j)Y_j-\sum_{j=1}^{n}\varepsilon_j(\mathrm{div}\,W_j)\varphi Y_j+$$

$$\sum_{j=1}^{n}\varepsilon_j\eta(Z_j)[\xi,Y_j]-\sum_{j=1}^{n}\varepsilon_j\eta(W_j)\,\varphi[\xi,Y_j]-\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\,\omega_j(W_i)[\varphi Y_i,Y_j]-\sum_{\substack{i,j=1\\ i<j}}^{n}\varepsilon_i\varepsilon_j\,\omega_j(\varphi W_i)\varphi[\varphi Y_i,Y_j]+$$

$$\sum_{\substack{i,j=1\\ i>j}}^{n}\varepsilon_i\varepsilon_j\,\omega_i(\varphi W_j)\varphi[\varphi Y_i,Y_j]+\sum_{\substack{i,j=1\\ i<j}}^{n}\varepsilon_i\varepsilon_j\,\{\omega_i(Z_j)-\omega_j(\varphi W_i)\}[Y_i,Y_j]\in \mathrm{Span}(X,Z).$$

Example

Let on $\mathbf{R}_v^{2n+1}$ with the coordinates $(x^1,...,x^n,y^1,...,y^n,z)$ the usual contact structure defined by:

(25) $$\eta=dz-\sum_{i=1}^{n}y^i dx^i$$

(26) $$\xi=\frac{\partial}{\partial z}$$

(27) $$\varphi=\begin{pmatrix}0 & I & 0\\ -I & 0 & 0\\ 0 & y^t & 0\end{pmatrix}$$

where $y^t=(y^1,...,y^n)$ and I the identity.

The contact distribution D is generated by $\left\{\frac{\partial}{\partial y^i},\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial z}\right\}_{i=1,...,n}$.

The compatible metric is:

(28) $$g=\sum_{i,j=1}^{n}(\varepsilon_i\delta_{ij}+y^iy^j)dx^idx^j-2\sum_{i=1}^{n}y^idx^idz+\sum_{i=1}^{n}\varepsilon_i\left(dy^i\right)^2+dz^2$$

Let

(29) $$X=\sum_{i=1}^{n}a_i\frac{\partial}{\partial x^i}+\sum_{i=1}^{n}b_i\frac{\partial}{\partial y^i}+c\frac{\partial}{\partial z}$$

(30) $$Z=\sum_{i=1}^{n}d_i\frac{\partial}{\partial x^i}+\sum_{i=1}^{n}e_i\frac{\partial}{\partial y^i}+f\frac{\partial}{\partial z}$$

where $a_i,b_i,c,d_i,e_i,f\in F\,(M)$.

We have: $\omega_i=\varepsilon_i dy^i\ \forall i=1,...,n$. It is easy to see that:

(6.31) $\omega_i(X)=\varepsilon_i b_i$, $\omega_i(\varphi X)=-\varepsilon_i a_i$, $\omega_i(Z)=\varepsilon_i e_i$, $\omega_i(\varphi Z)=-\varepsilon_i d_i$, $\eta(X)=c-\sum_{i=1}^{n}a_iy^i$,

$$\eta(Z)=f-\sum_{i=1}^{n}d_iy^i$$

From (20)-(22) we have:

(32) $$Z_j=\varepsilon_j\left[\sum_{i=1}^{n}(a_ie_j-d_ib_j)\frac{\partial}{\partial x^i}+\sum_{i=1}^{n}(b_ie_j-e_ib_j)\frac{\partial}{\partial y^i}+(ce_j-fb_j)\frac{\partial}{\partial z}\right]$$

(33) $$W_j=\varepsilon_j\left[\sum_{i=1}^{n}(d_i a_j - a_i d_j)\frac{\partial}{\partial x^i}+\sum_{i=1}^{n}(e_i a_j - b_i d_j)\frac{\partial}{\partial y^i}+(fa_j - cd_j)\frac{\partial}{\partial z}\right]$$

(34) $$T=\sum_{i=1}^{n}\left[(d_i c-a_i f)+\sum_{j=1}^{n}(a_i d_j - a_j d_i)y^j\right]\frac{\partial}{\partial x^i}+\sum_{i=1}^{n}\left[(e_i c-b_i f)+\sum_{j=1}^{n}(b_i d_j - a_j e_i)y^j\right]\frac{\partial}{\partial y^i}+\sum_{i=1}^{n}(cd_i - fa_i)\frac{\partial}{\partial z}$$

For a field $U=\sum_{i=1}^{n}U^i\frac{\partial}{\partial x^i}+\sum_{i=1}^{n}U^{n+i}\frac{\partial}{\partial y^i}+U^{2n+1}\frac{\partial}{\partial z}$ we have

$$\text{div } U=\sum_{i=1}^{n}\frac{\partial U^i}{\partial x^i}+\sum_{i=1}^{n}\frac{\partial U^{n+i}}{\partial y^i}+\frac{\partial U^{2n+1}}{\partial z}$$

therefore:

(35) $$\text{div } Z_j=\varepsilon_j\left[\sum_{i=1}^{n}\frac{\partial(a_i e_j - d_i b_j)}{\partial x^i}+\sum_{i=1}^{n}\frac{\partial(b_i e_j - e_i b_j)}{\partial y^i}+\frac{\partial(ce_j - fb_j)}{\partial z}\right]$$

(36) $$\text{div } W_j=\varepsilon_j\left[\sum_{i=1}^{n}\frac{\partial(d_i a_j - a_i d_j)}{\partial x^i}+\sum_{i=1}^{n}\frac{\partial(e_i a_j - b_i d_j)}{\partial y^i}+\frac{\partial(fa_j - cd_j)}{\partial z}\right]$$

(6.37) $$\text{div } T=\sum_{i=1}^{n}\left[\frac{\partial(d_i c-a_i f)}{\partial x^i}+\sum_{j=1}^{n}\frac{\partial(a_i d_j - a_j d_i)}{\partial x^i}y^j\right]+\sum_{i=1}^{n}\left[\frac{\partial(e_i c-b_i f)}{\partial y^i}+\sum_{j=1}^{n}\frac{\partial(b_i d_j - a_j e_i)}{\partial y^i}y^j+(b_i d_i - a_i e_i)\right]+\sum_{i=1}^{n}\frac{\partial(cd_i - fa_i)}{\partial z}y^i$$

(38) $\omega_j(W_j)=a_j e_j-b_j d_j$

The condition of integrability is:

(39) $$\left[\sum_{j=1}^{n}(2\varepsilon_j+1)\omega_j(W_j)-\text{div } T\right]\xi+\sum_{j=1}^{n}\varepsilon_j(\text{div } Z_j)Y_j-\sum_{j=1}^{n}\varepsilon_j(\text{div } W_j)\varphi Y_j\in \text{Span}(X,Z)$$

and with (35)-(38) becomes:

(6.40) $$-\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i}+\sum_{i=1}^{n}\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i}+\frac{\partial(fa_j - cd_j)}{\partial z}\right]\frac{\partial}{\partial x^j}+\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\frac{\partial(a_i e_j - d_i b_j)}{\partial x^i}+\sum_{i=1}^{n}\frac{\partial(b_i e_j - e_i b_j)}{\partial y^i}+\frac{\partial(ce_j - fb_j)}{\partial z}\right]\frac{\partial}{\partial y^j}+\left[\sum_{j=1}^{n}2(\varepsilon_j+1)(e_j a_j - b_j d_j)+\sum_{j=1}^{n}\frac{\partial(fa_j - cd_j)}{\partial x^j}-\sum_{j=1}^{n}\frac{\partial(ce_j - fb_j)}{\partial y^j}\right]\frac{\partial}{\partial z}\in \text{Span}(X,Z)$$

The conditions that $g(X,X)=0$, $g(Z,Z)\neq 0$, $g(X,Z)=0$ become:

$$(41)\quad \begin{cases} \left(\sum_{i=1}^{n} a_i y^i - c\right)^2 + \sum_{i=1}^{n} \varepsilon_i a_i^2 + \sum_{i=1}^{n} \varepsilon_i b_i^2 = 0 \\ \left(\sum_{i=1}^{n} d_i y^i - f\right)^2 + \sum_{i=1}^{n} \varepsilon_i d_i^2 + \sum_{i=1}^{n} \varepsilon_i e_i^2 \neq 0 \\ \left(\sum_{i=1}^{n} a_i y^i - c\right)\left(\sum_{j=1}^{n} d_j y^j - f\right) + \sum_{i=1}^{n} \varepsilon_i (a_i d_i + b_i e_i) = 0 \end{cases}$$

From (40) we have:

$$\mathrm{rank}\begin{pmatrix} a_j & d_j & -\left[\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^{n}\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial(f a_j - c d_j)}{\partial z}\right] \\ b_k & e_k & \left[\sum_{i=1}^{n}\frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \sum_{i=1}^{n}\frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \frac{\partial(c e_k - f b_k)}{\partial z}\right] \\ c & f & \left[\sum_{p=1}^{n} 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \sum_{p=1}^{n}\frac{\partial(f a_p - c d_p)}{\partial x^p} - \sum_{p=1}^{n}\frac{\partial(c e_p - f b_p)}{\partial y^p}\right] \end{pmatrix} = 2$$

where j, k=1,...,n.

We have now:

$$\begin{vmatrix} a_j & d_j & -\left[\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^{n}\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial(f a_j - c d_j)}{\partial z}\right] \\ b_k & e_k & \left[\sum_{i=1}^{n}\frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \sum_{i=1}^{n}\frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \frac{\partial(c e_k - f b_k)}{\partial z}\right] \\ c & f & \left[\sum_{p=1}^{n} 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \sum_{p=1}^{n}\frac{\partial(f a_p - c d_p)}{\partial x^p} - \sum_{p=1}^{n}\frac{\partial(c e_p - f b_p)}{\partial y^p}\right] \end{vmatrix} = 0$$

therefore:

$$\left[\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + \sum_{i=1}^{n}\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} + \frac{\partial(f a_j - c d_j)}{\partial z}\right](c e_k - f b_k) +$$

$$\left[\sum_{i=1}^{n}\frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} + \sum_{i=1}^{n}\frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + \frac{\partial(c e_k - f b_k)}{\partial z}\right](c d_j - f a_j) +$$

$$\left[\sum_{p=1}^{n} 2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + \sum_{p=1}^{n}\frac{\partial(f a_p - c d_p)}{\partial x^p} - \sum_{p=1}^{n}\frac{\partial(c e_p - f b_p)}{\partial y^p}\right](a_j e_k - b_k d_j) = 0$$

$$(ce_k - fb_k)\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + (ce_k - fb_k)\sum_{i=1}^{n}\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} +$$
$$(ce_k - fb_k)\frac{\partial(fa_j - cd_j)}{\partial z} + (cd_j - fa_j)\sum_{i=1}^{n}\frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} +$$
$$(cd_j - fa_j)\sum_{i=1}^{n}\frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} + (cd_j - fa_j)\frac{\partial(ce_k - fb_k)}{\partial z} +$$
$$(a_j e_k - b_k d_j)\sum_{p=1}^{n}2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + (a_j e_k - b_k d_j)\sum_{p=1}^{n}\frac{\partial(fa_p - cd_p)}{\partial x^p} -$$
$$(a_j e_k - b_k d_j)\sum_{p=1}^{n}\frac{\partial(ce_p - fb_p)}{\partial y^p} = 0$$

$$(ce_k - fb_k)\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + (ce_k - fb_k)\sum_{i=1}^{n}\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} +$$
$$+ (ce_k - fb_k)^2\frac{\partial\frac{fa_j - cd_j}{ce_k - fb_k}}{\partial z} - (fa_j - cd_j)\sum_{i=1}^{n}\frac{\partial(a_i e_k - d_i b_k)}{\partial x^i} +$$
$$(cd_j - fa_j)\sum_{i=1}^{n}\frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} +$$
$$(a_j e_k - b_k d_j)\sum_{p=1}^{n}2(\varepsilon_p + 1)(e_p a_p - b_p d_p) + (a_j e_k - b_k d_j)\sum_{p=1}^{n}\frac{\partial(fa_p - cd_p)}{\partial x^p} -$$
$$(a_j e_k - b_k d_j)\sum_{p=1}^{n}\frac{\partial(ce_p - fb_p)}{\partial y^p} = 0$$

$$(ce_k - fb_k)\sum_{i=1}^{n}\frac{\partial(a_j d_i - a_i d_j)}{\partial x^i} + (ce_k - fb_k)^2\frac{\partial\frac{fa_j - cd_j}{ce_k - fb_k}}{\partial z} + (cd_j - fa_j)\sum_{i=1}^{n}\frac{\partial(b_i e_k - e_i b_k)}{\partial y^i} +$$
$$(a_j e_k - b_k d_j)\sum_{p=1}^{n}2(\varepsilon_p + 1)(e_p a_p - b_p d_p) +$$
$$\sum_{i,p=1}^{n}\delta_{ij}(a_i e_k - b_k d_i)\frac{\partial(fa_p - cd_p)}{\partial x^p} - \delta_{jp}(fa_p - cd_p)\frac{\partial(a_i e_k - d_i b_k)}{\partial x^i}$$
$$+ \sum_{i,p=1}^{n}\delta_{kp}(ce_p - fb_p)\frac{\partial(a_j e_i - b_i d_j)}{\partial y^i} - \delta_{ij}(a_i e_k - b_k d_i)\frac{\partial(ce_p - fb_p)}{\partial y^p} = 0$$

From (38) we have:

(39) $$\frac{\partial\frac{cd - af}{bd - ae}}{\partial x^1}(bd - ae)^2 + \frac{\partial\frac{bd - ae}{bf - ce}}{\partial y^1}(bf - ce)^2 + \frac{\partial\frac{cd - af}{bf - ce}}{\partial z}(bf - ce)^2 = 0$$

From (39), (40) we have:

$$(41)\quad \frac{bd-ae}{bf-ce}=\frac{ay^1-c}{(ay^1-c)y^1-a}$$

$$(42)\quad \frac{cd-af}{bd-ae}=-\frac{1}{ay^1-c}$$

$$(43)\quad \frac{cd-af}{bf-ce}=\frac{-1}{(ay^1-c)y^1-a}$$

If we replace (41)-(43) in (39) we have:

$$(44)\quad \left[\frac{\partial a}{\partial z}-a^2\right](y^1)^2+\left[\frac{\partial a}{\partial x^1}-\frac{\partial c}{\partial z}+2ac\right]y^1+\left[-\frac{\partial c}{\partial x^1}+a\frac{\partial c}{\partial y^1}-c\frac{\partial a}{\partial y^1}-\frac{\partial a}{\partial z}-a^2-c^2\right]=0$$

If we note $A=ay^1-c$ we have:

$$(45)\quad \frac{\partial A}{\partial x^1}+\left(A\frac{\partial a}{\partial y^1}-a\frac{\partial A}{\partial y^1}-A^2\right)+\left(y^1\frac{\partial A}{\partial z}-\frac{\partial a}{\partial z}\right)=0$$

We have.

$$\text{div } X=\sum_{i=1}^{n}\left(\frac{\partial X^i}{\partial x^i}+\sum_{j=1}^{2n+1}\left|\begin{matrix} i \\ i\ j\end{matrix}\right|X^j\right)+\sum_{a=1}^{n}\left(\frac{\partial X^{n+a}}{\partial y^a}+\sum_{j=1}^{2n+1}\left|\begin{matrix} n+a \\ n+a\ j\end{matrix}\right|X^j\right)+\left(\frac{\partial X^{2n+1}}{\partial z}+\sum_{j=1}^{2n+1}\left|\begin{matrix} 2n+1 \\ 2n+1\ j\end{matrix}\right|X^j\right)$$

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