Macroeconomics and Monetary Economics

The Equilibrium Analysis of a Closed Economy Model with Government and Money Market Sector - II

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Abstract: In this paper, we will continue the study of the dynamic equilibrium solutions in the purpose of investigating the dependence limits (potential output and interest rate limit). We find also an interesting linear relation between the potential output and interest rate limit.

Keywords: equilibrium; demand; income

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1 Introduction

The purpose of this paper is to continue the analysis of a closed economy model when the net exports are zero. After a correction of some errors founded in [2] we have analyzed the dependence of the potential output and the interest rate limit on the depending variables presented in the model.

2 The Model Equations ([5])

For the beginning let remind the model equations:

(1) $D=C+I+G$

(2) $C=c_YV+C_0$, $C_0>0$, $c_Y\in(0, 1)$

(3) $V=Y+TR-TI$, $TR>0$

(4) $TI=ri_YY+T_0$, $ri_Y\in(0, 1)$, $T_0\in\mathbf{R}$

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(5) $I=in_Y Y+i_r r+I_0$, $in_Y \in (0, 1)$, $i_r<0$, $I_0>0$

(6) $G=\overline{G}$

(7) $D=Y$

(8) $MD=md_Y Y+m_r r+M_0 \leq M_0$, $md_Y>0$, $m_r<0$, $M_0>0$

(9) $MD=M$

(10) $\frac{dY}{dt}=\alpha(D-Y)$, $\alpha>0$

(11) $\frac{dr}{dt}=\beta(MD-M)$, $\beta>0$

where:

- D – the aggregate demand;
- C – the consumer demand (a concave function of V);
- I – the investment demand;
- G – the government spending;
- V – the disposable income;
- Y – the aggregate supply (national income);
- TR – the government transfers;
- TI – taxes;
- c_Y – the marginal propensity to consume, $c=\frac{dC}{dV} \in (0, 1)$, $\frac{d^2c}{dV^2} \leq 0$;
- ri_Y – the tax rate, $ri_Y \in (0, 1)$;
- in_Y –the rate of investments, $in_Y \in (0, 1)$;
- i_r – a factor of influence on the investment rate, $i_r<0$;
- r – the interest rate;
- MD – the money demand in the economy;
- md_Y – the rate of money demand in the economy;

- m_r – a factor of influencing the demand for currency from the interest rate, $m_r<0$;
- M – themoney supply.

3 The Static Equilibrium

Let note for the beginning, the autonomous component:

(12) $E=c_Y(TR-T_0)+C_0+I_0+\overline{G}>0$

In order to have the equilibrium, that is D=Y, from (1)-(6) we obtain:

$$(13)\ \begin{cases} r=\dfrac{(M-M_0)(1-c_Y(1-ri_Y)-in_Y)-md_YE}{i_rmd_Y+m_r(1-c_Y(1-ri_Y)-in_Y)} \\ Y=\dfrac{m_rE+i_r(M-M_0)}{i_rmd_Y+m_r(1-c_Y(1-ri_Y)-in_Y)} \end{cases}$$

We will note below, for simplification:

(14) $\Omega=m_rE-i_r(M_0-M)$

4 A result on the stability of solutions of a system of differential equations of first order, linear, with constant coefficients satisfying some conditions

Lemma

Let the system of differential equations:

$$\begin{pmatrix} \dfrac{dX}{dt} \\ \dfrac{dY}{dt} \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}+\begin{pmatrix} e \\ f \end{pmatrix},\ a, b, c, d, e, f\in\mathbf{R},\ a, b, d<0,\ c, e, f>0,\ X(0)=X_0,$$

$Y(0)=Y_0$.

Then $\lim_{t\to\infty}X(t)=\tilde{X}$, $\lim_{t\to\infty}Y(t)=\tilde{Y}$, $\tilde{X},\tilde{Y}\in\mathbf{R}$ if and only if:

1. $(a-d)^2+4bc=0$ with the solution:

$$\begin{cases} X=\left(\dfrac{a-d}{2}X_0+bY_0+\dfrac{2bf+e(a-d)}{a+d}\right)te^{\frac{a+d}{2}t}+\left(X_0+4\dfrac{de-bf}{(a+d)^2}\right)e^{\frac{a+d}{2}t}+4\dfrac{bf-de}{(a+d)^2} \\ Y=\left[Y_0+4\dfrac{af-ce}{(a+d)^2}\right]e^{\frac{a+d}{2}t}-\left(\dfrac{a-d}{2}X_0+bY_0+\dfrac{2bf+e(a-d)}{a+d}\right)\dfrac{a-d}{2b}te^{\frac{a+d}{2}t}+4\dfrac{ce-af}{(a+d)^2} \end{cases}$$

2. $(a-d)^2+4bc>0$ and $\lambda_1 \neq \lambda_2$ are roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: $e, f \in \mathbf{R}$ with the solution:

$$\begin{cases} X=k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t}-\dfrac{de-bf}{ad-bc} \\ Y=\dfrac{\lambda_1-a}{b}k_1e^{\lambda_1 t}+\dfrac{\lambda_2-a}{b}k_2e^{\lambda_2 t}+\dfrac{ce-af}{ad-bc} \end{cases}$$

where:

$$k_1=\frac{(\lambda_2-a)X_0+(\lambda_2-a)\dfrac{de-bf}{ad-bc}-bY_0+b\dfrac{ce-af}{ad-bc}}{\lambda_2-\lambda_1}$$

$$k_2=-\frac{(\lambda_1-a)X_0+(\lambda_1-a)\dfrac{de-bf}{ad-bc}-bY_0+b\dfrac{ce-af}{ad-bc}}{\lambda_2-\lambda_1}$$

3. $(a-d)^2+4bc<0$ and $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$, $\beta \neq 0$ are the roots of the equation: $\lambda^2-(a+d)\lambda+(ad-bc)=0$: $e, f \in \mathbf{R}$ with the solution:

$$\begin{cases} X=\left(X_0+\dfrac{de-bf}{ad-bc}\right)e^{\alpha t}\cos\beta t+\dfrac{1}{\beta}\left(bY_0-\dfrac{d-a}{2}X_0+\dfrac{(d-a)(bf-de)+2b(af-ce)}{2(ad-bc)}\right)e^{\alpha t}\sin\beta t+\dfrac{bf-de}{ad-bc} \\ Y=\left(Y_0+\dfrac{af-ce}{ad-bc}\right)e^{\alpha t}\cos\beta t+\dfrac{1}{\beta}\left(cX_0+\dfrac{d-a}{2}Y_0+\dfrac{(d-a)(af-ce)+2c(de-bf)}{2(ad-bc)}\right)e^{\alpha t}\sin\beta t+\dfrac{ec-af}{ad-bc} \end{cases}$$

5 The Dynamic Equilibrium

Let the system of first order differential equations:

$$(15)\ \begin{cases} \dfrac{dY}{dt}=\alpha(D-Y) \\ \dfrac{dr}{dt}=\beta(MD-M) \end{cases},\ \alpha,\ \beta>0$$

which becomes after (1)-(9):

$$(16)\quad \begin{pmatrix} \dfrac{dY}{dt} \\ \dfrac{dr}{dt} \end{pmatrix} = \begin{pmatrix} -\alpha\chi_Y & \alpha i_r \\ \beta md_Y & \beta m_r \end{pmatrix}\begin{pmatrix} Y \\ r \end{pmatrix} + \begin{pmatrix} \alpha E \\ \beta(M_0 - M) \end{pmatrix}$$

where we note $\chi_Y=1-c_Y(1-ri_Y)-in_Y>0$

Using the above lemma, it follows that: $\lim_{t\to\infty} Y(t) = \tilde{Y}$, $\lim_{t\to\infty} r(t) = \tilde{r}$, $\tilde{Y}, \tilde{r} \in \mathbf{R}_+$ if and only if:

1. $(\alpha\chi_Y+\beta m_r)^2+4\alpha\beta i_r md_Y=0$ then:

$$\begin{cases} Y = \left(-\dfrac{\alpha\chi_Y+\beta m_r}{2}Y_0 + \alpha i_r r_0 - \alpha\dfrac{2i_r\beta(M_0-M)-E(\alpha\chi_Y+\beta m_r)}{\alpha\chi_Y-\beta m_r}\right)te^{\frac{-\alpha\chi_Y+\beta m_r}{2}t} + \\ \left(Y_0 + 4\alpha\beta\dfrac{m_rE-i_r(M_0-M)}{(\alpha\chi_Y-\beta m_r)^2}\right)e^{\frac{-\alpha\chi_Y+\beta m_r}{2}t} + 4\alpha\beta\dfrac{i_r(M_0-M)-m_rE}{(\alpha\chi_Y-\beta m_r)^2} \\ r = \left[r_0 - 4\alpha\beta\dfrac{\chi_Y(M_0-M)+md_YE}{(\alpha\chi_Y-\beta m_r)^2}\right]e^{\frac{-\alpha\chi_Y+\beta m_r}{2}t} - \\ \left(\dfrac{\alpha\chi_Y+\beta m_r}{2}Y_0 - \alpha i_r r_0 + \alpha\dfrac{2i_r\beta(M_0-M)-E(\alpha\chi_Y+\beta m_r)}{\alpha\chi_Y-\beta m_r}\right)\dfrac{\alpha\chi_Y+\beta m_r}{2\alpha i_r}te^{\frac{-\alpha\chi_Y+\beta m_r}{2}t} + \\ \quad 4\alpha\beta\dfrac{md_YE+\chi_Y(M_0-M)}{(\alpha\chi_Y-\beta m_r)^2} \end{cases}$$

and: $$\begin{cases} \tilde{Y} = \dfrac{m_rE-i_r(M_0-M)}{\chi_Y m_r + i_r md_Y} \\ \tilde{r} = -\dfrac{md_YE+\chi_Y(M_0-M)}{\chi_Y m_r + i_r md_Y} \end{cases}$$

2. $(\alpha\chi_Y+\beta m_r)^2+4\alpha\beta i_r md_Y>0$ and $\lambda_1\neq\lambda_2$are roots of the equation: $\lambda^2+(\alpha\chi_Y - \beta m_r)\lambda-\alpha\beta(\chi_Y m_r+i_r md_Y)=0$ then:

$$\begin{cases} Y = k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t} + \dfrac{m_rE-i_r(M_0-M)}{\chi_Y m_r + i_r md_Y} \\ r = \dfrac{\lambda_1+\alpha\chi_Y}{\alpha i_r}k_1e^{\lambda_1 t} + \dfrac{\lambda_2+\alpha\chi_Y}{\alpha i_r}k_2e^{\lambda_2 t} - \dfrac{md_YE+\chi_Y(M_0-M)}{\chi_Y m_r + i_r md_Y} \end{cases}$$

where:

$$k_1 = \frac{(\lambda_2 + \alpha\chi_Y)Y_0 - (\lambda_2 + \alpha\chi_Y)\dfrac{m_r E - i_r(M_0 - M)}{\chi_Y m_r + i_r md_Y} - \alpha i_r r_0 - \alpha i_r \dfrac{md_Y E + \chi_Y(M_0 - M)}{\chi_Y m_r + i_r md_Y}}{\lambda_2 - \lambda_1}$$

$$k_2 = \frac{\alpha i_r r_0 + \alpha i_r \dfrac{md_Y E + \chi_Y(M_0 - M)}{\chi_Y m_r + i_r md_Y} - (\lambda_1 + \alpha\chi_Y)Y_0 + (\lambda_1 + \alpha\chi_Y)\dfrac{m_r E - i_r(M_0 - M)}{\chi_Y m_r + i_r md_Y}}{\lambda_2 - \lambda_1}$$

and: $$\begin{cases} \tilde{Y} = \dfrac{m_r E - i_r(M_0 - M)}{\chi_Y m_r + i_r md_Y} \\ \tilde{r} = -\dfrac{md_Y E + \chi_Y(M_0 - M)}{\chi_Y m_r + i_r md_Y} \end{cases}$$

3. $(\alpha\chi_Y+\beta m_r)^2+4\alpha\beta i_r md_Y<0$ and $\lambda_1=\mu+i\nu$, $\lambda_2=\mu-i\nu$, $\nu\neq 0$ are roots of the equation: $\lambda^2+(\alpha\chi_Y-\beta m_r)\lambda-\alpha\beta(\chi_Y m_r+i_r md_Y)=0$ then:

$$\begin{cases} Y = \left(Y_0 - \dfrac{m_r E - i_r(M_0 - M)}{\chi_Y m_r + i_r md_Y} \right) e^{\mu t} \cos \nu t + \\ \dfrac{1}{\nu}\left(br_0 - \dfrac{\beta m_r + \alpha\chi_Y}{2} Y_0 - \dfrac{(\beta m_r + \alpha\chi_Y)(i_r(M_0 - M) - m_r E) - 2\alpha i_r(\chi_Y(M_0 - M) + md_Y E)}{2(\chi_Y m_r + i_r md_Y)} \right) e^{\mu t} \sin \nu t - \\ \dfrac{i_r(M_0 - M) - m_r E}{\chi_Y m_r + i_r md_Y} \\ r = \left(r_0 + \dfrac{\chi_Y(M_0 - M) + md_Y E}{\chi_Y m_r + i_r md_Y} \right) e^{\mu t} \cos \nu t + \\ \dfrac{1}{\nu}\left(cY_0 + \dfrac{\beta m_r + \alpha\chi_Y}{2} r_0 + \dfrac{(\beta m_r + \alpha\chi_Y)(\chi_Y(M_0 - M) + md_Y E) - 2\beta md_Y(m_r E - i_r(M_0 - M))}{2(\chi_Y m_r + i_r md_Y)} \right) e^{\mu t} \sin \nu t - \\ \dfrac{md_Y E + \chi_Y(M_0 - M)}{\chi_Y m_r + i_r md_Y} \end{cases}$$

and: $$\begin{cases} \tilde{Y} = \dfrac{m_r E - i_r(M_0 - M)}{\chi_Y m_r + i_r md_Y} \\ \tilde{r} = -\dfrac{md_Y E + \chi_Y(M_0 - M)}{\chi_Y m_r + i_r md_Y} \end{cases}$$

We will call $\tilde{Y}$ - the potential output and $\tilde{r}$ - the interest rate limit.

6 The Analysis of Variation Limits

Therefore again:

$$(17)\ \begin{cases} \tilde{Y} = \dfrac{m_r E - i_r (M_0 - M)}{\chi_Y m_r + i_r md_Y} \\ \tilde{r} = -\dfrac{md_Y E + \chi_Y (M_0 - M)}{\chi_Y m_r + i_r md_Y} \end{cases}$$

From the above relations, we obtain:

$$(18)\ \chi_Y \tilde{Y} - i_r \tilde{r} = E$$

or, in original terms:

$$(19)\ (1 - c_Y(1 - ri_Y) - in_Y)\tilde{Y} - i_r \tilde{r} = c_Y(TR\text{-}T_0) + C_0 + I_0 + \overline{G}$$

We can easily write the relation (18) as:

$$(20)\ \tilde{Y} = \frac{i_r}{\chi_Y}\tilde{r} + \frac{E}{\chi_Y}$$

Because $\dfrac{i_r}{\chi_Y}<0$ it follows that the dependence of the output potential of the interest rate limit is inverse.

Because $\chi_Y m_r + i_r md_Y <0$ in order to have $\tilde{Y}>0$ it must that $\Omega<0$ that is:

$$(21)\ m_r E - i_r (M_0 - M) < 0$$

The first partial derivatives of $\tilde{Y}$ are:

$$(22)\ \frac{\partial \tilde{Y}}{\partial m_r} = \frac{i_r (md_Y E + \chi_Y (M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} < 0$$

$$(23)\ \frac{\partial \tilde{Y}}{\partial E} = \frac{m_r}{\chi_Y m_r + i_r md_Y} > 0$$

$$(24)\ \frac{\partial \tilde{Y}}{\partial i_r} = \frac{-m_r (md_Y E + (M_0 - M)\chi_Y)}{(\chi_Y m_r + i_r md_Y)^2} > 0$$

$$(25)\ \frac{\partial \tilde{Y}}{\partial \chi_Y} = -\frac{m_r (m_r E - i_r (M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = -\frac{m_r \Omega}{(\chi_Y m_r + i_r md_Y)^2} < 0$$

But χ_Y=1-c_Y(1-ri_Y)-in_Y imply: $\frac{\partial \chi_Y}{\partial c_Y} = -1$, $\frac{\partial \chi_Y}{\partial ri_Y} = c_Y$, $\frac{\partial \chi_Y}{\partial in_Y} = -1$, from where:

$$(26)\ \frac{\partial \tilde{Y}}{\partial c_Y} = \frac{\partial \tilde{Y}}{\partial \chi_Y}\frac{\partial \chi_Y}{\partial c_Y} = \frac{m_r(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = \frac{m_r \Omega}{(\chi_Y m_r + i_r md_Y)^2} > 0$$

$$(27)\ \frac{\partial \tilde{Y}}{\partial ri_Y} = \frac{\partial \tilde{Y}}{\partial \chi_Y}\frac{\partial \chi_Y}{\partial ri_Y} = -\frac{c_Y m_r(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = -\frac{c_Y m_r \Omega}{(\chi_Y m_r + i_r md_Y)^2} < 0$$

$$(28)\ \frac{\partial \tilde{Y}}{\partial in_Y} = \frac{\partial \tilde{Y}}{\partial \chi_Y}\frac{\partial \chi_Y}{\partial in_Y} = \frac{m_r(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = \frac{m_r \Omega}{(\chi_Y m_r + i_r md_Y)^2} > 0$$

$$(29)\ \frac{\partial \tilde{Y}}{\partial (M_0 - M)} = -\frac{i_r}{\chi_Y m_r + i_r md_Y} > 0$$

$$(30)\ \frac{\partial \tilde{Y}}{\partial md_Y} = -\frac{i_r(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = -\frac{i_r \Omega}{(\chi_Y m_r + i_r md_Y)^2} < 0$$

Also, for $\tilde{r} = -\frac{md_Y E + \chi_Y(M_0 - M)}{\chi_Y m_r + i_r md_Y}$ >0 we have:

$$(31)\ \frac{\partial \tilde{r}}{\partial m_r} = \frac{\chi_Y(md_Y E + \chi_Y(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} > 0$$

$$(32)\ \frac{\partial \tilde{r}}{\partial E} = -\frac{md_Y}{\chi_Y m_r + i_r md_Y} > 0$$

$$(33)\ \frac{\partial \tilde{r}}{\partial i_r} = \frac{md_Y(md_Y E + \chi_Y(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} > 0$$

$$(34)\ \frac{\partial \tilde{r}}{\partial \chi_Y} = \frac{md_Y(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = \frac{md_Y \Omega}{(\chi_Y m_r + i_r md_Y)^2} < 0 \text{ from where:}$$

$$(35)\ \frac{\partial \tilde{r}}{\partial c_Y} = \frac{\partial \tilde{r}}{\partial \chi_Y}\frac{\partial \chi_Y}{\partial c_Y} = -\frac{md_Y(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = -\frac{md_Y \Omega}{(\chi_Y m_r + i_r md_Y)^2} > 0$$

$$(36)\ \frac{\partial \tilde{r}}{\partial ri_Y} = \frac{\partial \tilde{r}}{\partial \chi_Y}\frac{\partial \chi_Y}{\partial ri_Y} = \frac{md_Y c_Y(m_r E - i_r(M_0 - M))}{(\chi_Y m_r + i_r md_Y)^2} = \frac{md_Y c_Y \Omega}{(\chi_Y m_r + i_r md_Y)^2} < 0$$

(37) $$\frac{\partial\tilde{r}}{\partial in_Y} = \frac{\partial\tilde{r}}{\partial\chi_Y}\frac{\partial\chi_Y}{\partial in_Y} = -\frac{md_Y\left(m_r E - i_r\left(M_0 - M\right)\right)}{\left(\chi_Y m_r + i_r md_Y\right)^2} = -\frac{md_Y\Omega}{\left(\chi_Y m_r + i_r md_Y\right)^2} > 0$$

(38) $$\frac{\partial\tilde{r}}{\partial\left(M_0 - M\right)} = -\frac{\chi_Y}{\chi_Y m_r + i_r md_Y} > 0$$

(39) $$\frac{\partial\tilde{r}}{\partial md_Y} = -\frac{\chi_Y\left(m_r E - i_r\left(M_0 - M\right)\right)}{\left(\chi_Y m_r + i_r md_Y\right)^2} = -\frac{\chi_Y\Omega}{\left(\chi_Y m_r + i_r md_Y\right)^2} > 0$$

7 Conclusions

As a general conclusion, we obtain the following:

- the increase in absolute value of m_r – the factor of influencing the demand for currency from the interest rate imply the increasing of the potential output and the decreasing of the interest rate limit;
- the potential output and the interest rate limit are increasing at an increase of the autonomous component;
- the increase in absolute value of i_r – the factor of influence on the investment rate imply the decreasing of the potential output and the decreasing of the interest rate limit;
- the increase of c_Y – the marginal propensity to consumeimply the increasing of the potential output and the increasing of the interest rate limit;
- the increase of ri_Y – the tax rate imply the decreasing of the potential output and the decreasing of the interest rate limit;
- the increase of in_Y – the rate of investments imply the increasing of the potential output and the increasing of the interest rate limit;
- the increase of M – the money supply imply the decreasing of the potential output and the decreasing of the interest rate limit;
- the increase of md_Y – the rate of money demand in the economy imply the decreasing of the potential output and the increasing of the interest rate limit.

8 References

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