Mathematical and Quantative Methods

The Reduction of Quadratic Forms to the Normal Form with Applications for Production Functions

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Abstract: The article treats the reduction of quadratic forms to the normal form by Gauss's method taking in discussing various determinants whose behavior will determine its nature. Applications of this method are illustrated for the most common production functions: Cobb-Douglas and CES.

Keywords: quadratic form; production function; Cobb-Douglas; CES

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1. Introduction

Let consider the quadratic form: $H:\mathbf{R}^n\to\mathbf{R}$, $H(x)=\sum_{i,j=1}^{n} a_{ij}x_ix_j$ $\forall x=(x_1,...,x_n)\in\mathbf{R}^n$.

The quadratic form H is called positive definite if $H(x)>0$ $\forall x\neq 0$, negative definite if $H(x)<0$ $\forall x\neq 0$, positive semi-definite if $H(x)\geq 0$ $\forall x\in R^n$ and $\exists x_0\in R^n-\{0\}$ such that $H(x_0)=0$, negative semi-definite if $H(x)\leq 0$ $\forall x\in R^n$ and $\exists x_0\in R^n-\{0\}$ such that $H(x_0)=0$, semi-definite if $\exists x_1,x_2\in R^n$ such that $H(x_1)H(x_2)<0$.

It is known that if a quadratic form is positive (negative) (semi) definite in a basis then it retains that character in any other basis.

We say that the quadratic form H has the normal form if there is a basis B of $\mathbf{R}^n$ where $H(x)=\sum_{i=1}^{m} b_iy_i^2$ $\forall x_B=(y_1,...,y_n)$, $m\leq n$.

It follows from the above that, being given the normal form of H (whatever the process by which this is achieved), H is positive definite if and only if $b_i>0$, $\forall i=$

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$\overline{1,n}$, negative definite if and only if $b_i<0$, $\forall i=\overline{1,n}$, positive semi-definite if and only if m<n and $b_i>0$, $\forall i=\overline{1,m}$, negative semi-definite if and only if m<n and $b_i<0$, $\forall i=\overline{1,m}$, semi-definite if and only if $\exists i\neq j=\overline{1,n}$ such that $b_i>0$, $b_j<0$.

Relative to bringing a quadratic form to the normal expression are essentially three big methods.

Jacobi's Method

Considering the matrix associated to the quadratic form $[H]=\left(a_{ij}\right)_{i,j=\overline{1,n}} \in M_n(\mathbf{R})$, let

$\Delta_i=\begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ii} \end{vmatrix}$, $i=\overline{1,n}$ - the principal diagonal determinants.

If $\Delta_i\neq 0$ $\forall i=\overline{1,n}$ then there is a basis $B=\{f_1,...,f_n\}$ obtained from the canonical basis through a triangular matrix, such that the normal expression of H is: $H(x)=\frac{1}{\Delta_1}y_1^2+\frac{\Delta_1}{\Delta_2}y_2^2+...+\frac{\Delta_{n-1}}{\Delta_n}y_n^2$.

The method is limited by the fact that the determinants obtained from the first i rows and columns must be non-zero. If $\exists i=\overline{1,n}$ such that $\Delta_k\neq 0$ $\forall k=\overline{1,i-1}$ (considering $\Delta_0=1$ we can assume that the condition is always satisfied) and $\Delta_i=0$, then it will investigate all determinants of the form: $\Delta_{i-1,p}=\begin{vmatrix} a_{11} & \dots & a_{1i-1} & a_{1p} \\ \dots & \dots & \dots & \dots \\ a_{i-11} & \dots & a_{i-1i-1} & a_{i-1p} \\ a_{p1} & \dots & a_{pi-1} & a_{pp} \end{vmatrix}$ with p>i. If such a determinant is non-null, then after the change of variables (which is basically a renumbering of variables): $\begin{cases} y_k=x_k,\ k=\overline{1,n},\ k\neq i,p \\ y_i=x_p \\ y_p=x_i \end{cases}$ the condition that $\Delta_i\neq 0$ is satisfied.

From the Jacobi's method is obtained that if $\Delta_i>0$ $\forall i=\overline{1,n}$ then H is positive definite, and if $(-1)^i\Delta_i>0$ $\forall i=\overline{1,n}$ then H is negative definite.

The essential drawback of Jacobi's method is that all determinants Δ_i must be non-null (regardless of any renumbering). The method also does not specify the nature of quadratic form when $\exists i=\overline{1,n}$ such that $\Delta_k=0$ $\forall k=\overline{1,n}$, $k\geq i$ (obviously after possible renumbering).

The Eigenvalues Method

Considering the associated matrix of H, let the characteristic polynomial $P(\lambda)=\det(A-\lambda I_n)$. It is shown that: $P(\lambda)=(-1)^n(\lambda^n-\delta_1\lambda^{n-1}+\delta_2\lambda^{n-2}-...+(-1)^n\delta_n)$ where δ_k is the sum of diagonal minors of order k of the matrix A.

Considering the characteristic equation: $P(\lambda)=0$, its roots are called the eigenvalues of the matrix A. For an eigenvalue λ, the vector $v\in\mathbf{R}^n$ such that: $Av=\lambda v$ is called eigenvector corresponding to λ.

It is shown that the eigenvalues of a symmetric matrix are real. Considering the basis B consisting of eigenvectors corresponding to the eigenvalues $\lambda_1,...,\lambda_n$ we get:

$$[H]_B=\begin{pmatrix} \lambda_1 & ... & 0 \\ ... & ... & ... \\ 0 & ... & \lambda_n \end{pmatrix} \text{ from where: } H(x)=\lambda_1 y_1^2+\lambda_2 y_2^2+...+\lambda_n y_n^2.$$

The eigenvalues method appears, at first sight, much better to determine the nature of quadratic form in the sense that H is positive definite if and only if $\lambda_i>0$, $\forall i=\overline{1,n}$, negative definite if and only if $\lambda_i<0$, $\forall i=\overline{1,n}$, positive semi-definite if and only if $m<n$ and $\lambda_i>0$, $\forall i=\overline{1,m}$ (therefore there are also null eigenvalues, but those non-null are strictly positive), negative semi-definite if and only if $m<n$ and $b_i<0$, $\forall i=\overline{1,m}$ (therefore there are also null eigenvalues, but those non-null are strictly negative), semi-definite if and only if $\exists i\neq j=\overline{1,n}$ such that $\lambda_i>0$, $\lambda_j<0$ (so there are at least two eigenvalues of sign contrary).

This method presents also a key deficiency, consisting in the difficulty of solving the characteristic equation (of n-th degree).

The Gauss Method

The Gauss method identifies the terms of the form $a_{ii}x_i^2$ and builds a perfect square that contains all occurrences of the variable x_i. The process is continued on the remaining quadratic form. If there is no term of the form $a_{ii}x_i^2$, then it identifies a mixed term: $a_{ij}x_ix_j$ with $a_{ij}\neq 0$. If no such term appears, the process ends. If so, it is considered a change of variable of the form: $x_i=y_i+y_j$, $x_j=y_i-y_j$ obtaining new square terms and the process continues as above.

2. Preliminary Results

Be a square symmetric matrix $A=\begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix} \in M_n(\mathbf{R})$.

Let note, as above, $\Delta_k=\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \cdots & \cdots & \cdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}$, $k=\overline{1,n}$ - the prinicipal diagonal determinants and define the appropriate determination of Δ_k board with the row i and the column j as: $\Delta_{k,ij}=\begin{vmatrix} a_{11} & ... & a_{1k} & a_{1j} \\ ... & ... & ... & ... \\ a_{k1} & ... & a_{kk} & a_{kj} \\ a_{i1} & ... & a_{ik} & a_{ij} \end{vmatrix}$.

It is noted that, due to the symmetry of the matrix A, we have: $\Delta_{k,ij}=\Delta_{k,ji}$. We also consider that: $\Delta_{0,ij}=a_{ij}$. We will note below:

- $\Delta_{k,\alpha\beta\gamma}=\begin{vmatrix} a_{11} & ... & a_{1k} & \alpha_1 \\ ... & ... & ... & ... \\ a_{k1} & ... & a_{kk} & \alpha_k \\ \beta_1 & ... & \beta_k & \gamma \end{vmatrix}$ where $\alpha=(\alpha_1,...,\alpha_k)^t$, $\beta=(\beta_1,...,\beta_k)^t \in \mathbf{R}^k$, $\gamma\in\mathbf{R}$
- $\Delta_{k,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=\begin{vmatrix} a_{11} & ... & a_{1k} & \alpha_1 & \delta_1 \\ ... & ... & ... & ... & ... \\ a_{k1} & ... & a_{kk} & \alpha_k & \delta_k \\ \beta_1 & ... & \beta_k & \gamma & \mu \\ \varepsilon_1 & ... & \varepsilon_k & \eta & \lambda \end{vmatrix}$ where $\alpha=(\alpha_1,...,\alpha_k)^t$, $\delta=(\delta_1,...,\delta_k)^t$, $\beta=(\beta_1,...,\beta_k)^t$, $\varepsilon=(\varepsilon_1,...,\varepsilon_k)^t \in \mathbf{R}^k$, $\gamma,\mu,\eta,\lambda\in\mathbf{R}$
- Γ_{pq} the minor of a_{pq} from the matrix $\begin{pmatrix} a_{11} & ... & a_{1k} \\ ... & ... & ... \\ a_{k1} & ... & a_{kk} \end{pmatrix}$
- $\Gamma_{pq,rs}$ the determinant of the matrix $\begin{pmatrix} a_{11} & ... & a_{1k} \\ ... & ... & ... \\ a_{k1} & ... & a_{kk} \end{pmatrix}$ obtained by deleting the rows p and q and the columns r and s. For k=2 we define $\Gamma_{pq,rs}=1$.

It is noted that due to symmetry, we have: $\Delta_{k,\alpha\beta\gamma}=\Delta_{k,\beta\alpha\gamma}$ and $\Delta_{k,\beta\varepsilon\alpha\delta\gamma\eta\mu\lambda}$.
The proofs of the following two lemmas are absolutely trivial, following the Laplace development of appropriate determinants.

Lemma 2.1

$$\Delta_{k,\alpha\beta\gamma}=\sum_{r,s=1}^{k}(-1)^{r+s+1}\alpha_r\beta_s\Gamma_{rs}+\gamma\Delta_k$$

Lemma 2.2

$$\Delta_{k,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=\sum_{\substack{p,q,r,s=1\\ p<q\\ r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Gamma_{pq,rs}+\sum_{p,s=1}^{k}(-1)^{p+s}\left(\varepsilon_s\begin{vmatrix}\alpha_p & \delta_p\\ \gamma & \mu\end{vmatrix}-\beta_s\begin{vmatrix}\alpha_p & \delta_p\\ \eta & \lambda\end{vmatrix}\right)\Gamma_{ps}+\begin{vmatrix}\gamma & \mu\\ \eta & \lambda\end{vmatrix}\Delta_k$$

Lemma 2.3

Be the vectors $\alpha=(\alpha_1,...,\alpha_k)^t$, $\beta=(\beta_1,...,\beta_k)^t$, $\delta=(\delta_1,...,\delta_k)^t$, $\varepsilon=(\varepsilon_1,...,\varepsilon_k)^t\in\mathbf{R}^k$ and $\gamma,\mu,\eta,\lambda\in\mathbf{R}$. Then:

$$\Delta_{k,\alpha\beta\gamma}\Delta_{k,\delta\varepsilon\lambda}-\Delta_{k,\alpha\varepsilon\mu}\Delta_{k,\beta\delta\eta}-\Delta_k\,\Delta_{k,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=$$

$$\sum_{\substack{p,q,r,s=1\\ p<q\\ r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\left(\Gamma_{pr}\Gamma_{qs}-\Gamma_{ps}\Gamma_{qr}-\Gamma_{pq,rs}\Delta_k\right)$$

Proof

From lemmas 2.1 and 2.2 it follows:

$$\Delta_{k,\alpha\beta\gamma}\Delta_{k,\delta\varepsilon\lambda}-\Delta_{k,\alpha\varepsilon\mu}\Delta_{k,\beta\delta\eta}-\Delta_k\,\Delta_{k,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=$$

$$\left(\sum_{u,v=1}^{k}(-1)^{u+v+1}\alpha_u\beta_v\Gamma_{uv}+\gamma\Delta_k\right)\left(\sum_{r,s=1}^{k}(-1)^{r+s+1}\delta_r\varepsilon_s\Gamma_{rs}+\lambda\Delta_k\right)-$$

$$\left(\sum_{u,v=1}^{k}(-1)^{u+r+1}\alpha_u\varepsilon_v\Gamma_{uv}+\mu\Delta_k\right)\left(\sum_{r,s=1}^{k}(-1)^{r+s+1}\beta_s\delta_r\Gamma_{rs}+\eta\Delta_k\right)-$$

$$\Delta_k\left(\sum_{\substack{p,q,r,s=1\\ p<q\\ r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Gamma_{pq,rs}+\sum_{p,s=1}^{k}(-1)^{p+s}\left(\varepsilon_s\begin{vmatrix}\alpha_p & \delta_p\\ \gamma & \mu\end{vmatrix}-\beta_s\begin{vmatrix}\alpha_p & \delta_p\\ \eta & \lambda\end{vmatrix}\right)\Gamma_{ps}+\begin{vmatrix}\gamma & \mu\\ \eta & \lambda\end{vmatrix}\Delta_k\right)=$$

$$\sum_{r,s,u,v=1}^{k}(-1)^{r+s+u+v}\alpha_u\delta_r(\beta_v\varepsilon_s-\beta_s\varepsilon_v)\Gamma_{rs}\Gamma_{uv}+\Delta_k\sum_{r,s=1}^{k}(-1)^{r+s+1}(\gamma\delta_r\varepsilon_s+\lambda\alpha_r\beta_s-\mu\beta_r\delta_s-\eta\alpha_s\varepsilon_r)\Gamma_{rs}-$$

$$\sum_{\substack{p,q,r,s=1\\ p<q\\ r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Delta_k\Gamma_{pq,rs}-\sum_{p,s=1}^{k}(-1)^{p+s}\varepsilon_s\begin{vmatrix}\alpha_p & \delta_p\\ \gamma & \mu\end{vmatrix}\Delta_k\Gamma_{ps}+\sum_{p,s=1}^{k}(-1)^{p+s}\beta_s\begin{vmatrix}\alpha_p & \delta_p\\ \eta & \lambda\end{vmatrix}\Delta_k\Gamma_{ps}=$$

$$\sum_{r,s,u,v=1}^{k}(-1)^{r+s+u+v}\alpha_u\delta_r(\beta_v\varepsilon_s-\beta_s\varepsilon_v)\Gamma_{rs}\Gamma_{uv}-\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p&\delta_p\\\alpha_q&\delta_q\end{vmatrix}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k+$$

$$\Delta_k\sum_{p,s=1}^{k}(-1)^{p+s}\left(\beta_s\begin{vmatrix}\alpha_p&\delta_p\\\eta&\lambda\end{vmatrix}-\varepsilon_s\begin{vmatrix}\alpha_p&\delta_p\\\gamma&\mu\end{vmatrix}-\gamma\delta_p\varepsilon_s-\lambda\alpha_p\beta_s+\mu\beta_s\delta_p+\eta\alpha_p\varepsilon_s\right)\Gamma_{ps}=$$

$$\sum_{p,q,r,s=1}^{k}(-1)^{p+q+r+s}\alpha_p\delta_q\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{qs}-\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p&\delta_p\\\alpha_q&\delta_q\end{vmatrix}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q}}^{k}(-1)^{p+q+r+s}\alpha_p\delta_q\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{qs}+\sum_{\substack{p,q,r,s=1\\p>q}}^{k}(-1)^{p+q+r+s}\alpha_p\delta_q\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{qs}+$$

$$\sum_{p,r,s=1}^{k}(-1)^{r+s}\alpha_p\delta_p\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{ps}-\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p&\delta_p\\\alpha_q&\delta_q\end{vmatrix}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q}}^{k}(-1)^{p+q+r+s}\alpha_p\delta_q\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{qs}+\sum_{\substack{p,q,r,s=1\\q>p}}^{k}(-1)^{p+q+r+s}\alpha_q\delta_p\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{qr}\Gamma_{ps}+$$

$$\sum_{\substack{p,r,s=1\\r<s}}^{k}(-1)^{r+s}\alpha_p\delta_p\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{ps}+\sum_{\substack{p,r,s=1\\r>s}}^{k}(-1)^{r+s}\alpha_p\delta_p\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{ps}-$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p&\delta_p\\\alpha_q&\delta_q\end{vmatrix}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{pr}\Gamma_{qs}+\alpha_q\delta_p\Gamma_{qr}\Gamma_{ps}\right)+\sum_{\substack{p,r,s=1\\r<s}}^{k}(-1)^{r+s}\alpha_p\delta_p\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pr}\Gamma_{ps}+$$

$$\sum_{\substack{p,r,s=1\\s>r}}^{k}(-1)^{r+s}\alpha_p\delta_p\begin{vmatrix}\beta_s&\beta_r\\\varepsilon_s&\varepsilon_r\end{vmatrix}\Gamma_{ps}\Gamma_{pr}-\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p&\delta_p\\\alpha_q&\delta_q\end{vmatrix}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{pr}\Gamma_{qs}+\alpha_q\delta_p\Gamma_{qr}\Gamma_{ps}\right)-\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p&\delta_p\\\alpha_q&\delta_q\end{vmatrix}\begin{vmatrix}\beta_r&\beta_s\\\varepsilon_r&\varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{pr}\Gamma_{qs}+\alpha_q\delta_p\Gamma_{qr}\Gamma_{ps}\right)+\sum_{\substack{p,q,r,s=1\\p<q\\r>s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{pr}\Gamma_{qs}+\alpha_q\delta_p\Gamma_{qr}\Gamma_{ps}\right)-$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{pr}\Gamma_{qs}+\alpha_q\delta_p\Gamma_{qr}\Gamma_{ps}\right)+\sum_{\substack{p,q,r,s=1\\p<q\\s>r}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_s & \beta_r\\ \varepsilon_s & \varepsilon_r\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{ps}\Gamma_{qr}+\alpha_q\delta_p\Gamma_{qs}\Gamma_{pr}\right)-$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\left(\alpha_p\delta_q\Gamma_{pr}\Gamma_{qs}+\alpha_q\delta_p\Gamma_{qr}\Gamma_{ps}-\alpha_p\delta_q\Gamma_{ps}\Gamma_{qr}-\alpha_q\delta_p\Gamma_{qs}\Gamma_{pr}\right)-$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\left(\Gamma_{pr}\Gamma_{qs}-\Gamma_{ps}\Gamma_{qr}\right)-\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\Gamma_{pq,rs}\Delta_k=$$

$$\sum_{\substack{p,q,r,s=1\\p<q\\r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\left(\Gamma_{pr}\Gamma_{qs}-\Gamma_{ps}\Gamma_{qr}-\Gamma_{pq,rs}\Delta_k\right)$$

Corollary 2.1

Be the vectors $\alpha=\left(\alpha_1,...,\alpha_k\right)^t$, $\delta=\left(\delta_1,...,\delta_k\right)^t$, $\varepsilon=\left(\varepsilon_1,...,\varepsilon_k\right)^t\in\mathbf{R}^k$ and $\gamma,\mu,\eta,\lambda\in\mathbf{R}$. Then:

$$\Delta_{k,\delta\varepsilon\lambda}\Delta_{k,\alpha\alpha\gamma}-\Delta_{k,\alpha\varepsilon\eta}\Delta_{k,\delta\alpha\mu}-\Delta_k\,\Delta_{k,\alpha\delta\alpha\varepsilon\gamma\mu\eta\lambda}=$$

$$\sum_{\substack{p,s,q,r=1\\p<q\\r<s}}^{k}(-1)^{p+s+q+r}\begin{vmatrix}\alpha_r & \alpha_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\begin{vmatrix}\alpha_p & \alpha_q\\ \delta_p & \delta_q\end{vmatrix}\left(\Gamma_{qs}\Gamma_{pr}-\Gamma_{ps}\Gamma_{qr}-\Gamma_{pq,rs}\Delta_k\right)$$

Proof

It follows from Lemma 2.3 for $\beta=\alpha$.

Lemma 2.4

$\Gamma_{qs}\Gamma_{pr} - \Gamma_{ps}\Gamma_{qr} - \Gamma_{pq,rs}\Delta_k = 0$, p<q, r<s, p,q,r,s=$\overline{1,k}$, $\forall k \geq 2$.

Proof

Let P(k): $\Gamma_{qs}\Gamma_{pr} - \Gamma_{ps}\Gamma_{qr} - \Gamma_{pq,rs}\Delta_k = 0$, p<q, r<s, p,q,r,s=$\overline{1,k}$.

For k=2, let $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ - symmetrical. We have: $\Gamma_{22}\Gamma_{11} - \Gamma_{12}\Gamma_{21} - \Gamma_{12,12}\Delta_2$ $= a_{11}a_{22} - a_{12}^2 -$
$(a_{11}a_{22} - a_{12}^2) = 0$.

For k=3, let $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ - symmetrical. We have 9 variants:

- p=1, q=2, r=1, s=2;
- p=1, q=3, r=1, s=2;
- p=2, q=3, r=1, s=2;
- p=1, q=2, r=1, s=3;
- p=1, q=3, r=1, s=3;
- p=2, q=3, r=1, s=3;
- p=1, q=2, r=2, s=3;
- p=1, q=3, r=2, s=3;
- p=2, q=3, r=2, s=3

We will proof the equality for on variant, for the others doing analogously.

Let, for example: p=1, q=2, r=1, s=3. Then:

$$\Gamma_{23}\Gamma_{11} - \Gamma_{13}\Gamma_{21} - \Gamma_{12,13}\Delta_3 =$$

$$(a_{11}a_{23} - a_{12}a_{13})(a_{22}a_{33} - a_{23}^2) - (a_{12}a_{23} - a_{22}a_{13})(a_{12}a_{33} - a_{13}a_{23}) -$$
$$a_{23}(a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{13}^2a_{22} - a_{23}^2a_{11} - a_{12}^2a_{33}) =$$

$$a_{11}a_{22}a_{23}a_{33} - a_{11}a_{23}^3 - a_{12}a_{13}a_{22}a_{33} + a_{12}a_{13}a_{23}^2 - a_{12}^2a_{23}a_{33} + a_{12}a_{13}a_{23}^2 + a_{12}a_{13}a_{22}a_{33} - a_{13}^2a_{22}a_{23} -$$
$$a_{11}a_{22}a_{23}a_{33} - 2a_{12}a_{13}a_{23}^2 + a_{13}^2a_{22}a_{23} + a_{11}a_{23}^3 + a_{12}^2a_{23}a_{33} = 0$$

Therefore, P(3) is true. Assuming P(i) true $\forall i=\overline{3,k}$, let P(k+1): $\Gamma_{qs}\Gamma_{pr}-\Gamma_{ps}\Gamma_{qr}-\Gamma_{pq,rs}\Delta_{k+1}=0$, p<q, r<s, p,q,r,s=$\overline{1,k+1}$.

Let $\delta_{k-1}=\Gamma_{pq,rs}$ obtained from Δ_{k+1} by removing the rows p and q and the columns r and s.

Considering in Lemma 2.3: α=the column r, δ= the column s, β=the row p, ε= the row q of Δ_{k+1}, $\gamma=a_{pr}$, $\mu=a_{ps}$, $\eta=a_{qr}$, $\lambda=a_{qs}$ it follows first:

- $\delta_{k-1,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=(-1)^{r+s+p+q}\Delta_{k+1}$
- $\delta_{k-1,\alpha\beta\gamma}=(-1)^{r+p}\Gamma_{sq}$
- $\delta_{k-1,\delta\varepsilon\lambda}=(-1)^{s+q}\Gamma_{rp}$
- $\delta_{k-1,\alpha\varepsilon\mu}=(-1)^{r+q}\Gamma_{ps}$
- $\delta_{k-1,\beta\delta\eta}=(-1)^{p+s}\Gamma_{qr}$

From Lemma 2.3, it follows:

$(-1)^{r+s+p+q}\left(\Gamma_{qs}\Gamma_{pr}-\Gamma_{ps}\Gamma_{qr}-\Gamma_{pq,rs}\Delta_{k+1}\right)=\delta_{k-1,\alpha\beta\gamma}\delta_{k-1,\delta\varepsilon\lambda}-\delta_{k-1,\alpha\varepsilon\mu}\delta_{k-1,\beta\delta\eta}-\delta_{k-1}\,\delta_{k-1,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=$

$\sum_{\substack{p,q,r,s=1\\ p<q\\ r<s}}^{k}(-1)^{p+q+r+s}\begin{vmatrix}\beta_r & \beta_s\\ \varepsilon_r & \varepsilon_s\end{vmatrix}\begin{vmatrix}\alpha_p & \delta_p\\ \alpha_q & \delta_q\end{vmatrix}\left(\gamma_{pr}\gamma_{qs}-\gamma_{ps}\gamma_{qr}-\gamma_{pq,rs}\delta_{k-1}\right)=0$ from the induction hypothesis (where γ_{ij} are appropriates minors of δ_{k-1}.

Corollary 2.2

Be the vectors $\alpha=\left(\alpha_1,...,\alpha_k\right)^t$, $\beta=\left(\beta_1,...,\beta_k\right)^t$, $\delta=\left(\delta_1,...,\delta_k\right)^t$, $\varepsilon=\left(\varepsilon_1,...,\varepsilon_k\right)^t\in\mathbf{R}^k$ and $\gamma,\mu,\eta,\lambda\in\mathbf{R}$, k≥2. Then: $\Delta_{k,\alpha\beta\gamma}\Delta_{k,\delta\varepsilon\lambda}-\Delta_{k,\alpha\varepsilon\mu}\Delta_{k,\beta\delta\eta}-\Delta_k\,\Delta_{k,\alpha\delta\beta\varepsilon\gamma\mu\eta\lambda}=0$.

Proof

It follows from lemmas 2.3 and 2.4.

Lemma 2.5

$\Delta_{k,ij}\Delta_{k+1}-\Delta_{k,k+1\,i}\Delta_{k,k+1\,j}=\Delta_k\Delta_{k+1,ij}$ $\forall i,j\geq k+2$ $\forall k\geq 2$.

Proof

For k=1 we will prove directly.

We have therefore:

$$\Delta_{k,ij}\Delta_{k+1} - \Delta_{k,k+1i}\Delta_{k,k+1j} - \Delta_k\Delta_{k+1,ij} =$$

$$\begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix}\begin{vmatrix} a_{11} & a_{1j} \\ a_{21} & a_{2j} \end{vmatrix} - a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{1j} \\ a_{21} & a_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix} =$$

$$(a_{11}a_{ij} - a_{1i}a_{1j})(a_{11}a_{22} - a_{12}^2) - (a_{11}a_{2i} - a_{1i}a_{12})(a_{11}a_{2j} - a_{1j}a_{12}) -$$
$$a_{11}(a_{11}a_{22}a_{ij} + a_{12}a_{2i}a_{1j} + a_{12}a_{1i}a_{2j} - a_{1j}a_{22}a_{1i} - a_{11}a_{2i}a_{2j} - a_{12}^2a_{ij}) =$$

$$a_{11}^2a_{22}a_{ij} - a_{11}a_{12}^2a_{ij} - a_{11}a_{22}a_{1i}a_{1j} + a_{12}^2a_{1i}a_{1j} - a_{11}^2a_{2i}a_{2j} + a_{11}a_{12}a_{2i}a_{1j} + a_{11}a_{12}a_{2j}a_{1i} - a_{12}^2a_{1i}a_{1j} -$$
$$a_{11}^2a_{22}a_{ij} - a_{11}a_{12}a_{2i}a_{1j} - a_{11}a_{12}a_{1i}a_{2j} + a_{11}a_{1j}a_{22}a_{1i} + a_{11}^2a_{2i}a_{2j} + a_{11}a_{12}^2a_{ij} = 0$$

From corollary 2.1, for $\alpha=(a_{1k+1},...,\alpha_{k\,k+1})^t$, $\delta=(a_{1j},...,a_{kj})^t$, $\varepsilon=(a_{i1},...,a_{ik})^t \in \mathbf{R}^k$, $\gamma=a_{k+1\,k+1}$, $\mu=a_{k+1\,j}$,

$\eta=a_{i\,k+1}$, $\lambda=a_{ij}$ we have:

$$\Delta_{k,ij}\Delta_{k+1} - \Delta_{k,k+1i}\Delta_{k,k+1j} - \Delta_k\Delta_{k+1,ij} =$$
$$\sum_{\substack{p,s,q,r=1 \\ p<q \\ r<s}}^{k} (-1)^{p+s+q+r}\begin{vmatrix} a_{r\,k+1} & a_{s\,k+1} \\ a_{ir} & a_{is} \end{vmatrix}\begin{vmatrix} a_{p\,k+1} & a_{q\,k+1} \\ a_{pj} & a_{qj} \end{vmatrix}\left(\Gamma_{qs}\Gamma_{pr} - \Gamma_{ps}\Gamma_{qr} - \Gamma_{pq,rs}\Delta_k\right).$$

From Lemma 2.4, we have $\Gamma_{qs}\Gamma_{pr} - \Gamma_{ps}\Gamma_{qr} - \Gamma_{pq,rs}\Delta_k=0$, $p<q$, $r<s$, $p,q,r,s=\overline{1,k}$, $\forall k\geq 2$.

It follows therefore: $\Delta_{k,ij}\Delta_{k+1} - \Delta_{k,k+1i}\Delta_{k,k+1j} - \Delta_k\Delta_{k+1,ij}=0$.

3. A New Approach of the Gauss Method

Suppose, first, that (after a possible renumbering) $a_{11}\neq 0$. We have:

$$H(x)= a_{11}x_1^2 + 2x_1\sum_{j=2}^{n} a_{1j}x_j + \sum_{i,j=2}^{n} a_{ij}x_ix_j =$$

$$a_{11}\left[x_1^2 + 2x_1\sum_{j=2}^{n}\frac{a_{1j}}{a_{11}}x_j + \left(\sum_{j=2}^{n}\frac{a_{1j}}{a_{11}}x_j\right)^2\right] - a_{11}\left(\sum_{j=2}^{n}\frac{a_{1j}}{a_{11}}x_j\right)^2 + \sum_{i,j=2}^{n} a_{ij}x_ix_j =$$

$$\frac{1}{a_{11}}\left(a_{11}x_1 + \sum_{j=2}^{n} a_{1j}x_j\right)^2 - \frac{1}{a_{11}}\sum_{i,j=2}^{n} a_{1i}a_{1j}x_ix_j + \sum_{i,j=2}^{n} a_{ij}x_ix_j =$$

$$\frac{1}{a_{11}}y_1^2 + \frac{1}{a_{11}}\sum_{i,j=2}^{n}\left(a_{11}a_{ij} - a_{1i}a_{1j}\right)x_ix_j$$

where we performed the change of variables: $y_1 = a_{11}x_1 + \sum_{j=2}^{n} a_{1j}x_j$ the others remaining the same.

From the above it follows that if $\Delta_1\neq 0$ then: $H(x)= \frac{1}{\Delta_1}y_1^2 + \frac{1}{\Delta_1}\sum_{i,j=2}^{n}\Delta_{1,ij}x_ix_j$.

Let P(k): $H(x)= \frac{1}{\Delta_1}y_1^2 + \frac{1}{\Delta_1\Delta_2}y_2^2 + \ldots + \frac{1}{\Delta_{k-1}\Delta_k}y_k^2 + \frac{1}{\Delta_k}\sum_{i,j=k+1}^{n}\Delta_{k,ij}x_ix_j$, $\Delta_i\neq 0\ \forall i=\overline{1,k}$

Since P(1) is true, suppose P(k) true. If $\exists i=\overline{k+1,n}$ such that: $\Delta_{k,ii}\neq 0$ then, after a possible renumbering, we assume: $\Delta_{k,k+1\ k+1}=\Delta_{k+1}\neq 0$.

Considering $H'(x) = \frac{1}{\Delta_k}\sum_{i,j=k+1}^{n}\Delta_{k,ij}x_ix_j$, we get:

$$H'(x) = \frac{1}{\Delta_k}\left(\Delta_{k+1}x_{k+1}^2 + 2x_{k+1}\sum_{i=k+2}^{n}\Delta_{k,k+1i}x_i + \sum_{i,j=k+2}^{n}\Delta_{k,ij}x_ix_j\right) =$$

$$\frac{\Delta_{k+1}}{\Delta_k}\left(x_{k+1}^2 + 2x_{k+1}\sum_{i=k+2}^{n}\frac{\Delta_{k,k+1i}}{\Delta_{k+1}}x_i + \sum_{i,j=k+2}^{n}\frac{\Delta_{k,ij}}{\Delta_{k+1}}x_ix_j\right) =$$

$$\frac{\Delta_{k+1}}{\Delta_k}\left(x_{k+1}^2 + 2x_{k+1}\sum_{i=k+2}^{n}\frac{\Delta_{k,k+1i}}{\Delta_{k+1}}x_i + \left(\sum_{i=k+2}^{n}\frac{\Delta_{k,k+1i}}{\Delta_{k+1}}x_i\right)^2 - \sum_{i,j=k+2}^{n}\frac{\Delta_{k,k+1i}}{\Delta_{k+1}}\frac{\Delta_{k,k+1j}}{\Delta_{k+1}}x_ix_j + \sum_{i,j=k+2}^{n}\frac{\Delta_{k,ij}}{\Delta_{k+1}}x_ix_j\right)$$
$=$

$$\frac{\Delta_{k+1}}{\Delta_k}\left(\left(x_{k+1}+\sum_{i=k+2}^{n}\frac{\Delta_{k,k+1i}}{\Delta_{k+1}}x_i\right)^2+\sum_{i,j=k+2}^{n}\left(\frac{\Delta_{k,ij}}{\Delta_{k+1}}-\frac{\Delta_{k,k+1i}}{\Delta_{k+1}}\frac{\Delta_{k,k+1j}}{\Delta_{k+1}}\right)x_ix_j\right)=$$

$$\frac{1}{\Delta_k\Delta_{k+1}}\left(\left(\Delta_{k+1}x_{k+1}+\sum_{i=k+2}^{n}\Delta_{k,k+1i}x_i\right)^2+\sum_{i,j=k+2}^{n}\left(\Delta_{k,ij}\Delta_{k+1}-\Delta_{k,k+1i}\Delta_{k,k+1j}\right)x_ix_j\right)=$$

$$\frac{1}{\Delta_k\Delta_{k+1}}y_{k+1}^2+\frac{1}{\Delta_{k+1}}\sum_{i,j=k+2}^{n}\frac{\Delta_{k,ij}\Delta_{k+1}-\Delta_{k,k+1i}\Delta_{k,k+1j}}{\Delta_k}x_ix_j$$

where we performed the change of variable: $y_{k+1}=\Delta_{k+1}x_{k+1}+\sum_{i=k+2}^{n}\Delta_{k,k+1i}x_i$.

From Lemma 5, $\Delta_{k,ij}\Delta_{k+1}-\Delta_{k,k+1i}\Delta_{k,k+1j}=\Delta_k\Delta_{k+1,ij}$ $\forall i,j\geq k+2$ $\forall k\geq 2$ therefore:

$$H'(x)=\frac{1}{\Delta_k\Delta_{k+1}}y_{k+1}^2+\frac{1}{\Delta_{k+1}}\sum_{i,j=k+2}^{n}\Delta_{k+1,ij}x_ix_j .$$

Therefore:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+\ldots+\frac{1}{\Delta_k\Delta_{k+1}}y_{k+1}^2+\frac{1}{\Delta_{k+1}}\sum_{i,j=k+2}^{n}\Delta_{k+1,ij}x_ix_j$$

and P(k+1) is also true.

Suppose now that $\Delta_1,\ldots,\Delta_k\neq 0$ and all $\Delta_{k,ii}=0$ $\forall i=\overline{k+1,n}$.

We have therefore: $H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+\ldots+\frac{1}{\Delta_{k-1}\Delta_k}y_k^2+\frac{1}{\Delta_k}\sum_{i,j=k+1}^{n}\Delta_{k,ij}x_ix_j$.

We have now two cases:

1. If $\Delta_{k,ij}=0$ $\forall i,j=\overline{k+1,n}$ then the algorithm ends and the normal form of H is:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+\ldots+\frac{1}{\Delta_{k-1}\Delta_k}y_k^2$$

Considering the matrix of changing of the canonical basis to a new basis: $M_{B_cB}=S^{-1}$ where:

$$S=\begin{pmatrix} \Delta_1 & \Delta_{0,12} & \dots & \Delta_{0,1k} & \Delta_{0,1\,k+1} & \dots & \Delta_{0,1n} \\ 0 & \Delta_2 & \dots & \Delta_{1,2k} & \Delta_{1,2\,k+1} & \dots & \Delta_{1,2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta_k & \Delta_{k-1,k\,k+1} & \dots & \Delta_{k-1,kn} \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

we have: $x=M_{B_cB}y$ from where $H(x)=x^t[H]_{B_c}x=y^tM^t_{B_cB}[H]_{B_c}M_{B_cB}y=y^t[H]_By$

In the new basis:

$$[H]_B=\begin{pmatrix} \frac{1}{\Delta_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\Delta_1\Delta_2} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\Delta_{k-1}\Delta_k} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

2. If $\exists p\neq q$ such that $\Delta_{k,pq}\neq 0, k\leq n-2$, let the transformation: $\begin{cases} z_p = \frac{2(\delta x_p+\beta x_q)}{\alpha\delta+\beta\gamma} \\ z_q = \frac{2(\gamma x_p-\alpha x_q)}{\alpha\delta+\beta\gamma} \end{cases}$ with $\alpha\delta+\beta\gamma\neq 0$, $\alpha\gamma\neq 0$, $\beta\delta\neq 0$. We have therefore: $x_p=\frac{\alpha z_p+\beta z_q}{2}$ and $x_q=\frac{\gamma z_p-\delta z_q}{2}$.

After replacing, the term $\Delta_{k,pq}x_px_q$ becomes:

$\Delta_{k,pq}\frac{\alpha z_p+\beta z_q}{2}\frac{\gamma z_p-\delta z_q}{2}=\frac{\Delta_{k,pq}}{4}\left(\alpha\gamma z_p^2-\beta\delta z_q^2-(\alpha\delta-\beta\gamma)z_pz_q\right)$ and thus proceed as above.

We assume (again after a possible renumbering) that: $\Delta_{k,k+1\ k+2}\neq 0$ therefore: $x_{k+1}=\frac{\alpha z_{k+1}+\beta z_{k+2}}{2}$ and $x_{k+2}=\frac{\gamma z_{k+1}-\delta z_{k+2}}{2}$. For the form: $H'(x)=\sum_{i,j=k+1}^{n}\Delta_{k,ij}x_ix_j$ it follows:

$$H'(x) = 2\Delta_{k,k+1k+2}x_{k+1}x_{k+2} + 2\sum_{j=k+3}^{n}\Delta_{k,k+1\,j}x_{k+1}x_j + 2\sum_{j=k+3}^{n}\Delta_{k,k+2\,j}x_{k+2}x_j + \sum_{i,j=k+3}^{n}\Delta_{k,ij}x_i x_j =$$

$$2\frac{\Delta_{k,k+1k+2}}{4}\left(\alpha\gamma z_{k+1}^2 - \beta\delta z_{k+2}^2 - (\alpha\delta-\beta\gamma)z_{k+1}z_{k+2}\right) + 2\sum_{j=k+3}^{n}\Delta_{k,k+1\,j}\frac{\alpha z_{k+1}+\beta z_{k+2}}{2}x_j +$$
$$2\sum_{j=k+3}^{n}\Delta_{k,k+2\,j}\frac{\gamma z_{k+1}-\delta z_{k+2}}{2}x_j + \sum_{i,j=k+3}^{n}\Delta_{k,ij}x_i x_j =$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}z_{k+1}^2 - \beta\delta\frac{\Delta_{k,k+1k+2}}{2}z_{k+2}^2 - (\alpha\delta-\beta\gamma)\frac{\Delta_{k,k+1k+2}}{2}z_{k+1}z_{k+2} +$$
$$\sum_{j=k+3}^{n}\alpha\Delta_{k,k+1\,j}z_{k+1}x_j + \sum_{j=k+3}^{n}\beta\Delta_{k,k+1\,j}z_{k+2}x_j + \sum_{j=k+3}^{n}\gamma\Delta_{k,k+2\,j}z_{k+1}x_j - \sum_{j=k+3}^{n}\delta\Delta_{k,k+2\,j}z_{k+2}x_j + \sum_{i,j=k+3}^{n}\Delta_{k,ij}x_i x_j$$

Proceeding as above, it follows:

$$H'(x) = \alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}z_{k+1}^2 + z_{k+1}\left(-(\alpha\delta-\beta\gamma)\frac{\Delta_{k,k+1k+2}}{2}z_{k+2} + \sum_{j=k+3}^{n}\gamma\Delta_{k,k+2\,j}x_j + \sum_{j=k+3}^{n}\alpha\Delta_{k,k+1\,j}x_j\right) -$$
$$\beta\delta\frac{\Delta_{k,k+1k+2}}{2}z_{k+2}^2 + \sum_{j=k+3}^{n}\beta\Delta_{k,k+1\,j}z_{k+2}x_j - \sum_{j=k+3}^{n}\delta\Delta_{k,k+2\,j}z_{k+2}x_j + \sum_{i,j=k+3}^{n}\Delta_{k,ij}x_i x_j =$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}\left[z_{k+1} + \frac{1}{\alpha\gamma\Delta_{k,k+1k+2}}\left(-(\alpha\delta-\beta\gamma)\frac{\Delta_{k,k+1k+2}}{2}z_{k+2} + \sum_{j=k+3}^{n}\gamma\Delta_{k,k+2\,j}x_j + \sum_{j=k+3}^{n}\alpha\Delta_{k,k+1\,j}x_j\right)\right]^2 -$$
$$\frac{1}{2\alpha\gamma\Delta_{k,k+1k+2}}\left(-(\alpha\delta-\beta\gamma)\frac{\Delta_{k,k+1k+2}}{2}z_{k+2} + \sum_{j=k+3}^{n}\gamma\Delta_{k,k+2\,j}x_j + \sum_{j=k+3}^{n}\alpha\Delta_{k,k+1\,j}x_j\right)^2 - \beta\delta\frac{\Delta_{k,k+1k+2}}{2}z_{k+2}^2 +$$
$$\sum_{j=k+3}^{n}\beta\Delta_{k,k+1\,j}z_{k+2}x_j - \sum_{j=k+3}^{n}\delta\Delta_{k,k+2\,j}z_{k+2}x_j + \sum_{i,j=k+3}^{n}\Delta_{k,ij}x_i x_j$$

With the variable transformation:

$$y_{k+1} =$$
$$z_{k+1} + \frac{1}{\alpha\gamma\Delta_{k,k+1k+2}}\left(-(\alpha\delta-\beta\gamma)\frac{\Delta_{k,k+1k+2}}{2}z_{k+2} + \sum_{j=k+3}^{n}\gamma\Delta_{k,k+2\,j}x_j + \sum_{j=k+3}^{n}\alpha\Delta_{k,k+1\,j}x_j\right)$$

we get:

$$H'(x)=\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-\frac{1}{2\alpha\gamma\Delta_{k,k+1k+2}}\left(-(\alpha\delta-\beta\gamma)\frac{\Delta_{k,k+1k+2}}{2}z_{k+2}+\sum_{j=k+3}^{n}\gamma\Delta_{k,k+2j}x_j+\sum_{j=k+3}^{n}\alpha\Delta_{k,k+1j}x_j\right)^2-$$

$$\beta\delta\frac{\Delta_{k,k+1k+2}}{2}z_{k+2}^2+\sum_{j=k+3}^{n}\beta\Delta_{k,k+1j}z_{k+2}x_j-\sum_{j=k+3}^{n}\delta\Delta_{k,k+2j}z_{k+2}x_j+\sum_{i,j=k+3}^{n}\Delta_{k,ij}x_ix_j=$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta-\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}z_{k+2}^2-\frac{\gamma}{2\alpha\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}\Delta_{k,k+2i}\Delta_{k,k+2j}x_ix_j-$$

$$\frac{\alpha}{2\gamma\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}\Delta_{k,k+1i}\Delta_{k,k+1j}x_ix_j+(\alpha\delta-\beta\gamma)\frac{1}{2\alpha}\sum_{j=k+3}^{n}\Delta_{k,k+2j}x_jz_{k+2}+(\alpha\delta-\beta\gamma)\frac{1}{2\gamma}\sum_{j=k+3}^{n}\Delta_{k,k+1j}x_jz_{k+2}-$$

$$\frac{1}{\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}\Delta_{k,k+1i}\Delta_{k,k+2j}x_ix_j-\beta\delta\frac{\Delta_{k,k+1k+2}}{2}z_{k+2}^2+\sum_{j=k+3}^{n}\beta\Delta_{k,k+1j}z_{k+2}x_j-\sum_{j=k+3}^{n}\delta\Delta_{k,k+2j}z_{k+2}x_j+$$

$$\sum_{i,j=k+3}^{n}\Delta_{k,ij}x_ix_j=$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}z_{k+2}^2+\frac{\alpha\delta+\beta\gamma}{2\alpha\gamma}\sum_{j=k+3}^{n}(\alpha\Delta_{k,k+1j}-\gamma\Delta_{k,k+2j})x_jz_{k+2}+$$

$$\frac{1}{2\alpha\gamma\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}(2\alpha\gamma\Delta_{k,ij}\Delta_{k,k+1k+2}-\gamma^2\Delta_{k,k+2i}\Delta_{k,k+2j}-\alpha^2\Delta_{k,k+1i}\Delta_{k,k+1j}-2\alpha\gamma\Delta_{k,k+1i}\Delta_{k,k+2j})x_ix_j=$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}\left(z_{k+2}-\frac{2}{(\alpha\delta+\beta\gamma)\Delta_{k,k+1k+2}}\sum_{j=k+3}^{n}(\alpha\Delta_{k,k+1j}-\gamma\Delta_{k,k+2j})x_j\right)^2+$$

$$\frac{1}{2\alpha\gamma\Delta_{k,k+1k+2}}\left(\sum_{j=k+3}^{n}(\alpha\Delta_{k,k+1j}-\gamma\Delta_{k,k+2j})x_j\right)^2+$$

$$\frac{1}{2\alpha\gamma\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}(2\alpha\gamma\Delta_{k,ij}\Delta_{k,k+1k+2}-\gamma^2\Delta_{k,k+2i}\Delta_{k,k+2j}-\alpha^2\Delta_{k,k+1i}\Delta_{k,k+1j}-2\alpha\gamma\Delta_{k,k+1i}\Delta_{k,k+2j})x_ix_j$$

Also, with the variable transformation:

$$y_{k+2}=z_{k+2}-\frac{2}{(\alpha\delta+\beta\gamma)\Delta_{k,k+1k+2}}\sum_{j=k+3}^{n}(\alpha\Delta_{k,k+1j}-\gamma\Delta_{k,k+2j})x_j$$

it follows:

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}y_{k+2}^2+$$

$$\frac{1}{2\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}(2\Delta_{k,ij}\Delta_{k,k+1k+2}-3\Delta_{k,k+1i}\Delta_{k,k+2j}-\Delta_{k,k+2i}\Delta_{k,k+1j})x_ix_j=$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}y_{k+2}^2+\frac{1}{\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}(\Delta_{k,ij}\Delta_{k,k+1k+2}-2\Delta_{k,k+1i}\Delta_{k,k+2j})x_ix_j$$

$=$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}y_{k+2}^2+$$

$$\frac{2}{\Delta_{k,k+1k+2}}\left(\sum_{\substack{i,j=k+3\\ i<j}}^{n}\left(\Delta_{k,ij}\Delta_{k,k+1k+2}-\Delta_{k,k+1i}\Delta_{k,k+2j}-\Delta_{k,k+1j}\Delta_{k,k+2i}\right)x_ix_j-2\sum_{i=k+3}^{n}\Delta_{k,k+1i}\Delta_{k,k+2i}x_i^2\right)=$$

$$\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma}y_{k+2}^2+$$

$$\frac{2}{\Delta_{k,k+1k+2}}\sum_{\substack{i,j=k+3\\ i\le j}}^{n}\left(\Delta_{k,ij}\Delta_{k,k+1k+2}-\Delta_{k,k+1i}\Delta_{k,k+2j}-\Delta_{k,k+1j}\Delta_{k,k+2i}\right)x_ix_j$$

The form H becomes therefore:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+\ldots+\frac{1}{\Delta_{k-1}\Delta_k}y_k^2+\alpha\gamma\frac{\Delta_{k,k+1k+2}}{2\Delta_k}y_{k+1}^2-(\alpha\delta+\beta\gamma)^2\frac{\Delta_{k,k+1k+2}}{8\alpha\gamma\Delta_k}y_{k+2}^2+$$

$$\frac{1}{\Delta_k\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}\left(\Delta_{k,ij}\Delta_{k,k+1k+2}-2\Delta_{k,k+1i}\Delta_{k,k+2j}\right)x_ix_j$$

In particular, for $\alpha=\beta=\gamma=\delta=\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}$ it follows:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+\ldots+\frac{1}{\Delta_{k-1}\Delta_k}y_k^2+\operatorname{sign}(\Delta_k)\Delta_{k,k+1k+2}y_{k+1}^2-\operatorname{sign}(\Delta_k)\Delta_{k,k+1k+2}y_{k+2}^2+$$

$$\frac{1}{\Delta_k\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}\left(\Delta_{k,ij}\Delta_{k,k+1k+2}-2\Delta_{k,k+1i}\Delta_{k,k+2j}\right)x_ix_j$$

As:

$$\begin{cases} z_{k+1}=\dfrac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+1}+\dfrac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+2}\\ z_{k+2}=\dfrac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+1}-\dfrac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+2}\end{cases}$$

it follows:

$y_{k+1}=$

$$\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+1}+\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+2}+\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}\,\Delta_{k,k+1k+2}}\sum_{j=k+3}^{n}\left(\Delta_{k,k+2j}+\Delta_{k,k+1j}\right)x_j$$

$$y_{k+2}=\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+1}-\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}}x_{k+2}+\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}\,\Delta_{k,k+1k+2}}\sum_{j=k+3}^{n}\left(\Delta_{k,k+2j}-\Delta_{k,k+1j}\right)x_j$$

Considering the matrix of changing of the canonical basis to a new basis: $M_{B_cB}=S^{-1}$ where:

S=

$$\begin{pmatrix} \Delta_1 & \Delta_{0,12} & \dots & \Delta_{0,1k} & \Delta_{0,1k+1} & \Delta_{0,1k+2} & \Delta_{0,1k+3} & \dots & \Delta_{0,1n} \\ 0 & \Delta_2 & \dots & \Delta_{1,2k} & \Delta_{1,2k+1} & \Delta_{1,2k+2} & \Delta_{1,2k+3} & \dots & \Delta_{1,2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta_k & \Delta_{k-1,k\,k+1} & \Delta_{k-1,k\,k+2} & \Delta_{k-1,k\,k+3} & \dots & \Delta_{k-1,kn} \\ 0 & 0 & \dots & 0 & \frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}} & \frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}} & \frac{\Delta_{k,k+2\,k+3}+\Delta_{k,k+1\,k+3}}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}\Delta_{k,k+1\,k+2}} & \dots & \frac{\Delta_{k,k+2\,n}+\Delta_{k,k+1\,n}}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}\Delta_{k,k+1\,k+2}} \\ 0 & 0 & \dots & 0 & \frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}} & -\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}} & \frac{\Delta_{k,k+2\,k+3}-\Delta_{k,k+1\,k+3}}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}\Delta_{k,k+1\,k+2}} & \dots & \frac{\Delta_{k,k+2\,n}-\Delta_{k,k+1\,n}}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k}\Delta_{k,k+1\,k+2}} \\ 0 & 0 & \dots & 0 & 0 & 0 & b_{k+3\,k+3} & \dots & b_{k+3\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & b_{n\,k+3} & \dots & b_{nn} \end{pmatrix}$$

we have: $x=M_{B_cB}\,y$ from where $H(x)=x^t[H]_{B_c}\,x=y^tM^t_{B_cB}[H]_{B_c}M_{B_cB}y=y^t[H]_By$.

In the new basis:

$[H]_B=$

$$\begin{pmatrix} \frac{1}{\Delta_1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\Delta_1\Delta_2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\Delta_{k-1}\Delta_k} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \operatorname{sign}(\Delta_k)\Delta_{k,k+1\,k+2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -\operatorname{sign}(\Delta_k)\Delta_{k,k+1\,k+2} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & c_{k+3\,k+3} & \dots & c_{k+3\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & c_{n\,k+3} & \dots & c_{nn} \end{pmatrix}$$

where $c_{ij}=\frac{\Delta_{k,ij}\Delta_{k,k+1\,k+2}-2\Delta_{k,k+1\,i}\Delta_{k,k+2\,j}}{\Delta_k\Delta_{k,k+1\,k+2}}$, $i,j=\overline{k+3,n}$.

After these considerations, we can formulate the following theorem:

Theorem 3.1

Given the quadratic form $H:\mathbf{R}^n \to \mathbf{R}$, $H(x)=\sum_{i,j=1}^{n} a_{ij}x_i x_j$ $\forall x=(x_1,...,x_n)\in\mathbf{R}^n$ we have:

1. If $\Delta_1,...,\Delta_n \neq 0$ then the normal form of H is:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+...+\frac{1}{\Delta_{n-1}\Delta_n}y_n^2$$

where: $y_k=\Delta_k x_k+\sum_{i=k+1}^{n}\Delta_{k-1,ki}x_i$, $k=\overline{1,n}$, $\Delta_0=1$.

2. If $\Delta_1,...,\Delta_k \neq 0$, $\Delta_{k,ij}=0$ $\forall i,j=\overline{k+1,n}$ then the normal form of H is:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+...+\frac{1}{\Delta_{k-1}\Delta_k}y_k^2$$

where: $y_k=\Delta_k x_k+\sum_{i=k+1}^{n}\Delta_{k-1,ki}x_i$, $k=\overline{1,n}$, $\Delta_0=1$.

3. If $\Delta_1,...,\Delta_k \neq 0$, $\Delta_{k,ii}=0$ $\forall i=\overline{k+1,n}$ and $\exists i\neq j=\overline{k+1,n}$ such that $\Delta_{k,ij}\neq 0$ then the normal form of H is:

$$H(x)=\frac{1}{\Delta_1}y_1^2+\frac{1}{\Delta_1\Delta_2}y_2^2+...+\frac{1}{\Delta_{k-1}\Delta_k}y_k^2+\operatorname{sign}(\Delta_k)\Delta_{k,ij}y_i^2-\operatorname{sign}(\Delta_k)\Delta_{k,ij}y_j^2+H'_n(x')$$

$$,\ x'=(x_{k+3},...,x_n)$$

where $y_p=\Delta_p x_p+\sum_{i=p+1}^{n}\Delta_{p-1,pi}x_i$, $p=\overline{1,k}$, $\Delta_0=1$,

$$y_{k+1}=z_{k+1}+\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k\Delta_{k,k+1k+2}}}\sum_{j=k+3}^{n}\left(\Delta_{k,k+2\,j}+\Delta_{k,k+1\,j}\right)x_j\ ,$$

$$y_{k+2}=z_{k+2}+\frac{1}{\sqrt{2\operatorname{sign}(\Delta_k)\Delta_k\Delta_{k,k+1k+2}}}\sum_{j=k+3}^{n}\left(\Delta_{k,k+2\,j}-\Delta_{k,k+1\,j}\right)x_j$$

and H'_n is the normal form of $H'(x)=\frac{1}{\Delta_k\Delta_{k,k+1k+2}}\sum_{i,j=k+3}^{n}\left(\Delta_{k,ij}\Delta_{k,k+1k+2}-2\Delta_{k,k+1i}\Delta_{k,k+2\,j}\right)x_i x_j$.

Corollary 3.1

Given the quadratic form H:$\mathbf{R}^n \rightarrow \mathbf{R}$, H(x)= $\sum_{i,j=1}^{n} a_{ij}x_i x_j$ $\forall x=(x_1,...,x_n)\in\mathbf{R}^n$ it follows (after a possible renumbering):

1. The quadratic form is positive definite if and only if: $\Delta_1,...,\Delta_n>0$;
2. The quadratic form is negative definite if and only if: $(-1)^k\Delta_k>0$ $\forall k=\overline{1,n}$;
3. The quadratic form is positive semi-definite if and only if: $\exists k=\overline{1,n}$ such that: $\Delta_1,...,\Delta_k>0$ and $\Delta_{k,ij}=0$ $\forall i,j=\overline{k+1,n}$;
4. The quadratic form is negative semi-definite if and only if: $\exists k=\overline{1,n}$ such that: $(-1)^i\Delta_i>0$, $i=\overline{1,k}$ and $\Delta_{k,ij}=0$ $\forall i,j=\overline{k+1,n}$;
5. The quadratic form is semi-definite if and only if:

a. $\Delta_1,...,\Delta_n\neq0$, but do not meet 1 or 2;

b. $\exists k=\overline{1,n}$ such that $\Delta_1,...,\Delta_k\neq0$ not meeting 1 or 2 and $\forall i,j=\overline{k+1,n}$: $\Delta_{k,ij}=0$;

c. $\exists k=\overline{1,n}$ such that $\Delta_1,...,\Delta_k\neq0$, $\Delta_{k,ii}=0$ $\forall i=\overline{k+1,n}$ and $\exists i,j=\overline{k+1,n}$ such that $\Delta_{k,ij}\neq0$.

We ask now the question what happens to the coefficient C_{ij} of x_ix_j from H'_n.

We have $C_{ij}= \Delta_{k,ij}\Delta_{k,k+1\,k+2} - 2\Delta_{k,k+1\,i}\Delta_{k,k+2\,j}$.

How $C_{ij}=C_{ji}$ it follows: $\Delta_{k,k+1\,i}\Delta_{k,k+2\,j} = \Delta_{k,k+1\,j}\Delta_{k,k+2\,i}$.

Noting δ_{ij} the determinant obtaind by board Δ_k with the columns k+2 and j and the rows k+1 and i, from corollary 2, it follows:

(1) $\Delta_{k,ij}\Delta_{k,k+2\,k+1} - \Delta_{k,i\,k+1}\Delta_{k,j\,k+2} = \Delta_k\delta_{ij}$ $\forall i\geq k+2,\ j\geq k+3$

In particular, for i=j, we have:

(2) $\Delta_{k,ii}\Delta_{k,k+2\,k+1} - \Delta_{k,i\,k+1}\Delta_{k,i\,k+2} = \Delta_k\delta_{ii}$ $\forall i\geq k+3$

How $\Delta_{k,ii}=0$, it follows:

(3) $\Delta_{k,i\,k+1}\Delta_{k,i\,k+2} = -\Delta_k\delta_{ii}$

From Lemma 5:

(4) $\Delta_{k,ij}\Delta_{k+1} - \Delta_{k,k+1\,i}\Delta_{k,k+1\,j} = \Delta_k\Delta_{k+1,ij}$ $\forall$i,j≥k+2 $\forall$k≥2

How $\Delta_{k+1}=\Delta_{k,k+1\ k+1}=0$, we get from (4):

(5) $\Delta_{k,k+1\,i}\Delta_{k,k+1\,j} = -\Delta_k\Delta_{k+1,ij}$

In particular, for i=j:

(6) $\Delta^2_{k,k+1\,i} = -\Delta_k\Delta_{k+1,ii}$

For i=k+2 în (6):

(7) $\Delta^2_{k,k+1\,k+2} = -\Delta_k\Delta_{k+1,k+2\,k+2} = -\Delta_k\Delta_{k+2}$

therefore implicitly $\Delta_{k+2}\neq 0$.

From (5), for j=k+2:

(8) $\Delta_{k,k+1\,i}\Delta_{k,k+1\,k+2} = -\Delta_k\Delta_{k+1,k+2\,i}$

We have now, from (3) and (5):

(9) $\Delta_{k,k+1\,i}\Delta_{k,k+2\,j}\Delta^2_{k,k+1\,j} = \Delta^2_k\Delta_{k+1,ij}\delta_{jj}$

From (6) and (9):

$\Delta_{k,k+1\,i}\Delta_{k,k+2\,j}\Delta_{k+1,jj} = -\Delta_k\Delta_{k+1,ij}\delta_{jj}$

therefore:

(10) $\Delta_{k,k+1\,i}\Delta_{k,k+2\,j} = -\dfrac{\Delta_k\Delta_{k+1,ij}\delta_{jj}}{\Delta_{k+1,jj}}$ if $\Delta_{k+1,jj}\neq 0$

If $\Delta_{k+1,jj}=0$ then, from (6) it follows: $\Delta_{k,k+1\,j}=0$, and from (5) it follows: $\Delta_{k+1,ij}=0$. Also, from (3) it follows: $\delta_{jj}=0$.

Therefore, if $\Delta_{k+1,jj}\neq 0$ then:

$C_{ij}= \Delta_{k,ij}\Delta_{k,k+1\,k+2} - 2\Delta_{k,k+1\,i}\Delta_{k,k+2\,j} = \dfrac{\Delta_{k,ij}\Delta_{k+1,jj}\Delta_{k,k+1\,k+2} + 2\Delta_k\Delta_{k+1,ij}\delta_{jj}}{\Delta_{k+1,jj}}$ and, in particular: $C_{jj}= 2\Delta_k\delta_{jj}$

and if $\Delta_{k+1,jj}=0$ then: $C_{ij}= \Delta_{k,ij}\Delta_{k,k+1\,k+2}$ and, in particular: $C_{jj}=0$.

Therefore:

$$H'(x) = \frac{1}{\Delta_k \Delta_{k,k+1\,k+2}} \sum_{i,j=k+3}^{n} \left(\Delta_{k,ij}\Delta_{k,k+1\,k+2} - 2\Delta_{k,k+1\,i}\Delta_{k,k+2\,j}\right) x_i x_j =$$

$$\frac{1}{\Delta_k \Delta_{k,k+1\,k+2}} \left(\sum_{\substack{i,j=k+3 \\ \Delta_{k+1,jj}\neq 0}}^{n} \frac{\Delta_{k,ij}\Delta_{k+1,jj}\Delta_{k,k+1\,k+2} + 2\Delta_k \Delta_{k+1,ij}\delta_{jj}}{\Delta_{k+1,jj}} x_i x_j + \sum_{\substack{i,j=k+3 \\ \Delta_{k+1,jj}=0}}^{n} \Delta_{k,ij}\Delta_{k,k+1\,k+2} x_i x_j \right)$$

=

$$\frac{2}{\Delta_k \Delta_{k,k+1\,k+2}} \left(\sum_{i,j=k+3}^{n} \Delta_{k,ij}\Delta_{k,k+1\,k+2} x_i x_j + \sum_{\substack{i,j=k+3 \\ \Delta_{k+1,jj}\neq 0}}^{n} \frac{\Delta_k \Delta_{k+1,ij}\delta_{jj}}{\Delta_{k+1,jj}} x_i x_j \right) =$$

$$2\left(\frac{1}{\Delta_k} \sum_{i,j=k+3}^{n} \Delta_{k,ij} x_i x_j + \frac{1}{\Delta_{k,k+1\,k+2}} \sum_{\substack{i,j=k+3 \\ \Delta_{k+1,jj}\neq 0}}^{n} \frac{\Delta_{k+1,ij}\delta_{jj}}{\Delta_{k+1,jj}} x_i x_j \right)$$

4. Bordered Matrices

Let the bordered matrix: $H_B = \begin{pmatrix} 0 & b_1 & b_2 & \dots & b_n \\ b_1 & a_{11} & a_{12} & \dots & a_{1n} \\ b_2 & a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ b_n & a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ and $\Delta_k^B =$

$$\begin{pmatrix} 0 & b_1 & b_2 & \dots & b_k \\ b_1 & a_{11} & a_{12} & \dots & a_{1k} \\ b_2 & a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ b_k & a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}, k=\overline{1,n}.$$

From Lemma 2.1, we have: $\Delta_k^B = \sum_{r,s=1}^{k} (-1)^{r+s+1} b_r b_s \Gamma_{rs}$ where Γ_{rs} is the appropriate minor of a_{ij} from the matrix of $H(x)= \sum_{i,j=1}^{n} a_{ij} x_i x_j$.

Let consider the quadratic form: $H_k^B(b) = -\sum_{r,s=1}^{k}(-1)^{r+s}\Gamma_{rs}b_r b_s$ and $\Delta_k^{Bb} = \begin{vmatrix} \Gamma_{11} & \dots & (-1)^{k+1}\Gamma_{1k} \\ \dots & \dots & \dots \\ (-1)^{k+1}\Gamma_{k1} & \dots & \Gamma_{kk} \end{vmatrix}$.

Since $\begin{pmatrix} a_{11} & \dots & a_{1k} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}\begin{pmatrix} \Gamma_{11} & \dots & (-1)^{k+1}\Gamma_{1k} \\ \dots & \dots & \dots \\ (-1)^{k+1}\Gamma_{k1} & \dots & \Gamma_{kk} \end{pmatrix} = \Delta_k I_k$ it follows: $\Delta_k \Delta_k^{Bb} = \Delta_k^k$

from where: $\Delta_k^{Bb} = \Delta_k^{k-1}$ if $\Delta_k \neq 0$.

If $\Delta_1,\dots,\Delta_k \neq 0$ then $\exists P \in M_k(\mathbf{R})$, invertible, such that:

$$P^t \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} P = \begin{pmatrix} \frac{1}{\Delta_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\Delta_1\Delta_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\Delta_{k-1}\Delta_k} \end{pmatrix}$$

We obtain after few computations:

$$P^{-1}\begin{pmatrix} \Gamma_{11} & \dots & (-1)^{k+1}\Gamma_{1k} \\ \dots & \dots & \dots \\ (-1)^{k+1}\Gamma_{k1} & \dots & \Gamma_{kk} \end{pmatrix}\left(P^{-1}\right)^t = \Delta_k \begin{pmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_1\Delta_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta_{k-1}\Delta_k \end{pmatrix}$$

Therefore, for $\Delta_1,\dots,\Delta_k \neq 0$ the normal form of $H_k^B(b)$ is:

$$H_k^B(b) = -\Delta_k\left(\Delta_1 c_1^2 + \Delta_1\Delta_2 c_2^2 + \dots + \Delta_{k-1}\Delta_k c_k^2\right)$$

where $c=(c_1,\dots,c_k)$ are the coordinates of b in the new basis.

As a result of these relations, it follows that if: $\Delta_1,\dots,\Delta_k>0$ then: $\Delta_k^B<0$. If $(-1)^i\Delta_i>0$ $\forall i=\overline{1,k}$ then: $\text{sign}(\Delta_k^B)=\text{sign}(\Delta_k)$ therefore $(-1)^k\Delta_k^B>0$.

Analogously the things happen if $\Delta_1,...,\Delta_p \neq 0$ and $\Delta_{p,ij}=0$ $\forall i,j=\overline{p+1,k}$. In this case, the normal form of $H_k^B(b)$ is:

$$H^B(b) = -\Delta_p\left(\Delta_1 c_1^2 + \Delta_1\Delta_2 c_2^2 + ... + \Delta_{p-1}\Delta_p c_p^2\right)$$

where $c=(c_1,...,c_k)$ are the coordinates of b in the new basis.

As a result of these relations, it follows that if: $\Delta_1,...,\Delta_p>0$ and $\Delta_{p,ij}=0$ $\forall i,j=\overline{p+1,k}$ then: $\Delta_k^B \leq 0$. If $(-1)^i\Delta_i>0$ $\forall i=\overline{1,p}$ then: sign(Δ_k^B)=sign(Δ_k) therefore $(-1)^k\Delta_k^B \geq 0$.

If H is semi-definite, the problem is more complicated. So, in the case of expression $\Delta_k^B = \sum_{r,s=1}^{k}(-1)^{r+s+1} b_r b_s \Gamma_{rs}$ if Δ_k^B is semi-definite then $\exists$b', b"$\in \mathbf{R}^k$ such that $\Delta_k^B(b')>0$, $\Delta_k^B(b'')<0$. Difficult issue arises where for another determinant Δ_s^B, $s\neq k$, which signs depending on the values of b is not strictly determined, the values b', b"$\in \mathbf{R}^k$ are not necessarily the same as in the case of Δ_k^B.

5. The Convexity of the Functions

We present in this section some of the remarkable results of concavity or quasi-concavity of functions.

Definition 5.1 A subset $D \subset \mathbf{R}^n$ is called convex if $\forall x,y \in D$ $\forall \lambda \in [0,1] \Rightarrow \lambda x+(1-\lambda)y \in D$.

From definition, it follows that D is convex if and only if for any two points $x,y \in D$, the segment $[x,y] \subset D$.

Definition 5.2 A function $f:D \subset \mathbf{R}^n \to \mathbf{R}$ is called **convex** if $\forall x,y \in D$ $\forall \lambda \in [0,1]$ follows
$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$.

Definition 5.3 A function $f:D \subset \mathbf{R}^n \to \mathbf{R}$ is called **concave** if $\forall x,y \in D$ $\forall \lambda \in [0,1]$ follows
$f(\lambda x+(1-\lambda)y) \geq \lambda f(x)+(1-\lambda)f(y)$.

From the definitions, it follows that a function is convex (concave) if and only if for any segment $[x,y] \subset D$ the values of the restriction function is under (above) or on the chord determined by the values of the function on the extremities of its.

Definition 5.4 A function $f:D \subset \mathbf{R}^n \to \mathbf{R}$ is called **strictly convex** if $\forall x,y \in D$ $\forall \lambda \in (0,1)$ follows

$f(\lambda x+(1-\lambda)y)<\lambda f(x)+(1-\lambda)f(y)$.

Definition 5.5 A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$ is called **strictly concave** if $\forall x,y\in D$ $\forall\lambda\in(0,1)$ follows

$f(\lambda x+(1-\lambda)y)>\lambda f(x)+(1-\lambda)f(y)$.

From these definitions, it follows that a function is strictly convex (concave) if and only if for any segment $[x,y]\subset D$ the values of the restriction function is under (above) the chord determined by the values of the function on the extremities of its.

Definition 5.6 A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is called **quasiconvex** if $\forall x,y\in D$ $\forall\lambda\in[0,1]$ then: $f(\lambda x+(1-\lambda)y)\le \max(f(x),f(y))$.

Definition 5.7 A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is called **quasiconcave** if $\forall x,y\in D$ $\forall\lambda\in[0,1]$ then: $f(\lambda x+(1-\lambda)y)\ge \min(f(x),f(y))$.

From the definitions, it follows that a function is quasiconvex (quasiconcave) if and only if for any segment $[x,y]\subset D$ the values of the restriction function is under (above) the maximum (minimum) level registered by the function at the ends.

Definition 5.8 A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is called **strictly quasiconvex** if $\forall x\neq y\in D$ $\forall\lambda\in(0,1)$ then: $f(\lambda x+(1-\lambda)y)<\max(f(x),f(y))$.

Definition 5.9 A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is called **strictly quasiconcave** if $\forall x\neq y\in D$ $\forall\lambda\in(0,1)$ then: $f(\lambda x+(1-\lambda)y)>\min(f(x),f(y))$.

From the definitions, it follows that a function is strictly quasiconvex (quasiconcave) if and only if for any segment $[x,y]\subset D$ the values of the restriction function is strictly under (above) the maximum (minimum) level registered by the function at the ends.

Theorem 5.1 If A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is quasiconvex (quasiconcave, convex, concave) then $-f$ is quasiconcave (quasiconvex, concave, convex).

After this theorem, where not explicitly stated, we state the results only for concave functions, ie quasi-concave.

Theorem 5.2 A function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is quasiconcave (quasiconvex) if and only if

$f^{-1}[a,\infty)$ $(f^{-1}(-\infty,a])$ is convex $\forall a\in\mathbf{R}$.

Theorem 5.3 If a function $f:D\subset \mathbf{R}^n\to\mathbf{R}$, D – convex, is concave then it is quasiconcave.

Remark 5.1 A function quasiconcave is not necessarily concave.

Theorem 5.4 If a function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is quasiconcave then αf is quasiconcave $\forall\alpha\geq 0$.

Theorem 5.5 If the functions $f_k:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, $k=\overline{1,m}$, are quasiconvex then $\forall p_i\geq 0$, $i=\overline{1,m}$ the function $f=\max(p_1f_1,...,p_mf_m)$ is also quasiconvex.

Theorem 5.6 If the functions $f_k:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, $k=\overline{1,m}$, are quasiconcave then $\forall p_i\geq 0$, $i=\overline{1,m}$ the function $f=\min(p_1f_1,...,p_mf_m)$ is also quasiconcave.

Theorem 5.7 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is quasiconvex (quasiconcave), and $g:\mathbf{R}\to\mathbf{R}$ is increasing, the function $g\circ f:D\to\mathbf{R}$ is quasiconvex (quasiconcave).

Theorem 5.8 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^1(D)$ then it is concave (strictly concave) if and only if:

$$f(x)-f(y)\leq(<)\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y)(x_i-y_i)\ \forall x,y\in D$$

Theorem 5.9 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^1(D)$ then it is convex (strictly convex) if and only if:

$$f(x)-f(y)\geq(>)\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y)(x_i-y_i)\ \forall x,y\in D$$

Theorem 5.10 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^1(D)$ then it is quasiconcave (strictly quasiconcave) if and only if:

$$f(x)\geq f(y)\Rightarrow\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y)(x_i-y_i)\geq(>)0\ \ \forall x,y\in D$$

Theorem 5.11 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^1(D)$ then it is quasiconvex (strictly quasiconvex) if and only if:

$$f(x)\geq f(y)\Rightarrow\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x)(x_i-y_i)\geq(>)0\ \ \forall x,y\in D$$

Definition 5.6 A function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, $f\in C^1(D)$ is called **pseudoconvex** if it is quasiconcave and $f(x)>f(y)$ $\Rightarrow\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y)(x_i-y_i)>0$ $\forall x,y\in D$.

Definition 5.7 A function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, $f\in C^1(D)$ is called **pseudo-concave** if it is quasiconvex and $f(x)>f(y) \Rightarrow \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x)(x_i - y_i) > 0$ $\forall x,y\in D$.

Suppose, in what follows, that $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^2(D)$.

Let $x_0\in D$. From Taylor series expansion:

$$f(x) = f(x_0) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_0)(x_i - x_{0i}) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + \alpha(x - x_0))(x_i - x_{0i})(x_j - x_{0j})$$

, $\alpha\in(0,1)$

or otherwise, for $x=x_0+h$:

$$f(x_0 + h) = f(x_0) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_0)h_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + \alpha h)h_i h_j,\ \alpha\in(0,1)$$

We can write this:

$$f(x_0 + h) - f(x_0) - \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_0)h_i = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + \alpha h)h_i h_j,\ \alpha\in(0,1)$$

From Theorem 5.8 it follows that f is concave (strictly concave) if and only if $\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + \alpha h)h_i h_j \le 0$ (<0).

Like a conclusion, if d^2f is negative semi-definite then f is concave.

Conversely, if f is concave, suppose that d^2f is not negative-semi-definite. In this case, $\exists x'\in D$ such that: $\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x')h_i h_j > 0$. Because the function f is of class $C^2(D)$ it follows that $\exists V\in \mathcal{V}(x')$ such that: $\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x)h_i h_j > 0$ $\forall x\in V$. Let $r>0$ such that the n-sphere of center x' and radius r: $B(x',r)=\{x\in\mathbf{R}^n \mid \|x - x'\| < r\}\subset V$. Let now $x\in B(x',r)$ and $h=x-x'$. We have:

$$f(x) - f(x') - \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x')h_i = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x'+\alpha h)h_i h_j,\ \alpha\in(0,1)$$

Because $\|x'+\alpha h - x'\| = \|\alpha h\| = |\alpha|\|h\| < \|h\| = \|x - x'\| < r$ it follows $\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x'+\alpha h)h_i h_j > 0$ therefore: $f(x) - f(x') - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x')h_i > 0$ which contradicts the fact that the function is concave.

The proof is analogous in the case of convexity. Therefore:

Theorem 5.12 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^2(D)$ then it is concave (convex) if and only if d^2f is negative (positive) semi-definite.

Suppose now that d^2f is negative definite. In this case, we have: $\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + \alpha h)h_i h_j < 0,\ \alpha\in(0,1)$ therefore: $f(x_0 + h) - f(x_0) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)h_i$ <0. Therefore, the function is strictly concave. Analogously is shown for strictly convex functions.

Theorem 5.13 If the function $f:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex, is of class $C^2(D)$ then if d^2f is negative (positive) definite, the function is strictly concave (strictly convex).

The reciprocal question is: if f is strictly concave then d^2f is defined negatively? The answer is unfortunately negative, meaning that d^2f is negative semi-definite.

Now consider a function $f:D\subset\mathbf{R}_+^n\to\mathbf{R}$, D – convex, $f\in C^2(D)$, the bordered hessian matrix:

$$H^B(f) = \begin{pmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

and the bordered principal diagonal determinants:

$$\Delta_k^B = \begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \frac{\partial^2 f}{\partial x_k \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{vmatrix}, \; k=\overline{1,n}$$

Theorem 5.14 If the function $f:D\subset \mathbf{R}_+^n \to \mathbf{R}$, D – convex, $f\in C^2(D)$ is quasiconcave then: $\Delta_1^B \le 0$, $\Delta_2^B \ge 0$, $\Delta_3^B \le 0$,... (*the determinants signs being alternate*).

Theorem 5.15 In order that the function $f:D\subset \mathbf{R}_+^n \to \mathbf{R}$, D – convex, $f\in C^2(D)$ be quasiconcave is sufficient that: $\Delta_1^B < 0$, $\Delta_2^B > 0$, $\Delta_3^B < 0$,... (*the determinants signs being alternate*).

Theorem 5.16 If the function $f:D\subset \mathbf{R}_+^n \to \mathbf{R}$, D – convex, $f\in C^2(D)$ is quasiconvex then: $\Delta_1^B \le 0$, $\Delta_2^B \le 0$, $\Delta_3^B \le 0$,..., $\Delta_n^B \le 0$.

Theorem 5.17 In order that the function $f:D\subset \mathbf{R}_+^n \to \mathbf{R}$, D – convex, $f\in C^2(D)$ be quasiconvex is sufficient that: $\Delta_1^B < 0$, $\Delta_2^B < 0$, $\Delta_3^B < 0$,..., $\Delta_n^B < 0$.

Remark 5.2 From Section 3, we have seen that if f is concave (convex, strictly concave, strictly convex) then $(-1)^k \Delta_k \ge 0$ ($\Delta_k \ge 0$, $(-1)^k \Delta_k > 0$, $\Delta_k > 0$). From Section 4, it follows that the function is quasiconcave (quasiconvex, strictly quasiconcave, strictly quasiconvex).

6. The Convexity Analysis of Production Functions

6.1. The Cobb-Douglas Function

The Cobb-Douglas function has the following expression:

$$f:D\subset \mathbf{R}_+^n - \{0\} \to \mathbf{R}_+,\ (x_1,...,x_n) \to f(x_1,...,x_n) = A x_1^{\alpha_1} ... x_n^{\alpha_n} \in \mathbf{R}_+\ \forall (x_1,...,x_n)\in D,\ A\in \mathbf{R}_+,\ \alpha_1,...,\alpha_n > 0$$

Computing the partial derivatives of first and second order, we get:

$$f'_{x_i} = \alpha_i A x_1^{\alpha_1} ... x_i^{\alpha_i - 1} ... x_n^{\alpha_n} = \frac{\alpha_i f}{x_i} \quad \forall i=\overline{1,n}$$

$$f''_{x_i x_j} = \alpha_i \alpha_j A x_1^{\alpha_1} ... x_i^{\alpha_i - 1} ... x_j^{\alpha_j - 1} ... x_n^{\alpha_n} = \frac{\alpha_i \alpha_j f}{x_i x_j} \quad \forall i \neq j = \overline{1, n}$$

$$f''_{x_i x_i} = \alpha_i (\alpha_i - 1) A x_1^{\alpha_1} ... x_i^{\alpha_i - 2} ... x_n^{\alpha_n} = \frac{\alpha_i (\alpha_i - 1) f}{x_i^2} \quad \forall i = \overline{1, n}$$

The Hessian matrix is:

$$H_f = \begin{pmatrix} \frac{\alpha_1(\alpha_1 - 1)f}{x_1^2} & ... & \frac{\alpha_1 \alpha_n f}{x_1 x_n} \\ ... & ... & ... \\ \frac{\alpha_1 \alpha_n f}{x_1 x_n} & ... & \frac{\alpha_n(\alpha_n - 1)f}{x_n^2} \end{pmatrix}$$

We have now:

$$\Delta_k = (-1)^k A^k x_1^{k\alpha_1 - 2} ... x_k^{k\alpha_k - 2} \prod_{i=1}^{k} \alpha_i \left(1 - \sum_{i=1}^{k} \alpha_i\right), \ k = \overline{1, n}.$$

$$\Delta_{k,ij} = (-1)^k \alpha_i \alpha_j \left(\prod_{i=1}^{k} \alpha_i\right) A^{k+1} x_1^{(k+1)\alpha_1 - 2} ... x_i^{(k+1)\alpha_i - 3} ... x_j^{(k+1)\alpha_j - 3} ... x_k^{(k+1)\alpha_k - 2}, \quad k = \overline{1, n}, \quad i \neq j,$$

$i,j \geq k+1$

We note first that $\Delta_{k,ij} \neq 0$, $k = \overline{1, n}$, $i \neq j$, $i,j \geq k+1$.

Because $\alpha_i > 0$, $i = \overline{1, n}$ it follows: $\text{sign}(\Delta_k) = \text{sign}(-1)^k \left(1 - \sum_{i=1}^{k} \alpha_i\right)$.

We get therefore that:

- $\text{sign}(-1)^k \left(1 - \sum_{i=1}^{k} \alpha_i\right) > 0$, $k = \overline{1, n}$ implies that f is strictly convex. We have however, for k=1: $1-\alpha_1<0$, and for k=2: $1-\alpha_1-\alpha_2>0$ therefore: $\alpha_1>1$, $\alpha_1+\alpha_2<1$ which conflicts with $\alpha_i>0$, $i = \overline{1, n}$. Therefore, the Cobb-Douglas function cannot be strictly convex.

- $\text{sign}\left(1 - \sum_{i=1}^{k} \alpha_i\right) > 0$, $k = \overline{1, n}$ implies that f is strictly concave;

• $\exists k=\overline{1,n}$ (after a possible renumbering), k=even such that: $1-\sum_{i=1}^{k}\alpha_i<0$ or $\exists k,p=\overline{1,n}$, k,p=odd such that $1-\sum_{i=1}^{k}\alpha_i<0$ and $1-\sum_{i=1}^{p}\alpha_i>0$ then f has a saddle point;

• $\exists k=\overline{1,n}$ (after a possible renumbering) such that: $1-\sum_{i=1}^{p}\alpha_i\neq 0$ $\forall p=\overline{1,k}$, but $1-\sum_{i=1}^{s}\alpha_i=0$ $\forall s=\overline{k+1,n}$ (this thing, because the fact that $\alpha_i>0$ cannot occur only for $1-\sum_{i=1}^{n}\alpha_i=0$) then:

○if $\text{sign}(-1)^k\left(1-\sum_{i=1}^{k}\alpha_i\right)>0$, $k=\overline{1,n-1}$ implies the fact that f is convex. In this case, for k=1: $1-\alpha_1<0$ therefore $\alpha_1>1$, the equality $\sum_{i=1}^{n}\alpha_i=1$ cannot occur;

○if $\text{sign}\left(1-\sum_{i=1}^{k}\alpha_i\right)>0$, $k=\overline{1,n-1}$ implies the fact that f is concave.

In particular, for the Cobb-Douglas function: $f(x_1,...,x_n)=Ax_1^{\alpha_1}x_2^{\alpha_2}$, $\alpha_1,\alpha_2>0$, we have:

• $\alpha_1+\alpha_2<1$implies the fact that f is strictly concave;

• $\alpha_1+\alpha_2>1$ implies the fact that f has saddle points, therefore it is not convex and not concave;

• $\alpha_1+\alpha_2=1$ implies the fact that f is concave.

6.2. The CES Function

The CES function has the following expression:

$$f:D\subset \mathbf{R}_+^n-\{0\}\rightarrow\mathbf{R}_+,\ (x_1,...,x_n)\rightarrow f(x_1,...,x_n)=\alpha\left(\sum_{i=1}^{n}\beta_i x_i^{\rho}\right)^{\frac{1}{\rho}}\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D,$$

$$\alpha,\beta_1...,\beta_n>0,\ \rho\neq 0,1,\ \sum_{i=1}^{n}\beta_i=1$$

Computing the partial derivatives of first and second order, we get:

$$f'_{x_i} = \alpha\beta_i x_i^{\rho-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\rho}\right)^{\frac{1}{\rho}-1} = \frac{\beta_i x_i^{\rho-1} f}{\sum_{k=1}^{n}\beta_k x_k^{\rho}} \quad \forall i=\overline{1,n}$$

$$f''_{x_i x_j} = \alpha\beta_i\beta_j(1-\rho)x_i^{\rho-1}x_j^{\rho-1}\left(\sum_{k=1}^{n}\beta_k x_k^{\rho}\right)^{\frac{1}{\rho}-2} = -\frac{\beta_i\beta_j(\rho-1)x_i^{\rho-1}x_j^{\rho-1} f}{\left(\sum_{k=1}^{n}\beta_k x_k^{\rho}\right)^2} \quad \forall i\neq j=\overline{1,n}$$

$$f''_{x_i x_i} = \alpha\beta_i(\rho-1)x_i^{\rho-2}\left(\sum_{k=1}^{n}\beta_k x_k^{\rho}\right)^{\frac{1}{\rho}-2}\left(\sum_{k=1}^{n}\beta_k x_k^{\rho} - \beta_i x_i^{\rho}\right) = \frac{\beta_i(\rho-1)x_i^{\rho-2}\left(\sum_{k=1}^{n}\beta_k x_k^{\rho} - \beta_i x_i^{\rho}\right) f}{\left(\sum_{k=1}^{n}\beta_k x_k^{\rho}\right)^2}$$

$\forall i=\overline{1,n}$

The Hessian matrix is:

$$H_f=$$

$$\begin{pmatrix} \frac{\beta_1(\rho-1)x_1^{\rho-2}\left(\sum_{s=1}^{n}\beta_s x_s^{\rho} - \beta_1 x_1^{\rho}\right) f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} & -\frac{\beta_1\beta_2(\rho-1)x_1^{\rho-1}x_2^{\rho-1} f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} & \dots & -\frac{\beta_1\beta_n(\rho-1)x_1^{\rho-1}x_n^{\rho-1} f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} \\ -\frac{\beta_1\beta_2(\rho-1)x_1^{\rho-1}x_2^{\rho-1} f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} & \frac{\beta_2(\rho-1)x_2^{\rho-2}\left(\sum_{s=1}^{n}\beta_s x_s^{\rho} - \beta_2 x_2^{\rho}\right) f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} & \dots & -\frac{\beta_2\beta_n(\rho-1)x_2^{\rho-1}x_n^{\rho-1} f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} \\ \dots & \dots & \dots & \dots \\ -\frac{\beta_1\beta_n(\rho-1)x_1^{\rho-1}x_n^{\rho-1} f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} & -\frac{\beta_2\beta_n(\rho-1)x_2^{\rho-1}x_n^{\rho-1} f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} & \dots & \frac{\beta_n(\rho-1)x_n^{\rho-2}\left(\sum_{s=1}^{n}\beta_s x_s^{\rho} - \beta_n x_n^{\rho}\right) f}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^2} \end{pmatrix}$$

We have now:

$\Delta_k = \frac{f^k(\rho-1)^k\prod_{s=1}^{k}\beta_s\prod_{s=1}^{k}x_s^{\rho-2}\sum_{s=k+1}^{n}\beta_s x_s^{\rho}}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^{k+1}}$, $k=\overline{1,n}$ (where the last sum in the numerator is 0 for k=n)

$$\Delta_{k,ij} = -\frac{\beta_i\beta_j\prod_{s=1}^{k}\beta_s(\rho-1)^{k+1}\left(\prod_{s=1}^{k}x_i\right)^{\rho-2} x_i^{\rho-1}x_j^{\rho-1}f^{\,k+1}}{\left(\sum_{s=1}^{n}\beta_s x_s^{\rho}\right)^{k+2}} \neq 0,\ k=\overline{1,n},\ i\neq j,\ i,j\geq k+1$$

If $\rho>1$ then $\Delta_k>0$, $k=\overline{1,n-1}$ and $\Delta_n=0$. In this case, the function is convex (non strictly).

If $\rho<1$ then $(-1)^k\Delta_k>0$ and $\Delta_n=0$, the function being concave (non strictly).

7. The Analysis of Quasi-Concavity of Production Functions

7.1. The Cobb-Douglas Function

Let the bordered Hessian matrix:

$$H^B(f)=\begin{pmatrix} 0 & \frac{\alpha_1 f}{x_1} & \frac{\alpha_2 f}{x_2} & \dots & \frac{\alpha_n f}{x_n} \\ \frac{\alpha_1 f}{x_1} & \frac{\alpha_1(\alpha_1-1)f}{x_1^2} & \frac{\alpha_1\alpha_2 f}{x_1x_2} & \dots & \frac{\alpha_1\alpha_n f}{x_1x_n} \\ \frac{\alpha_2 f}{x_2} & \frac{\alpha_1\alpha_2 f}{x_1x_2} & \frac{\alpha_2(\alpha_2-1)f}{x_2^2} & \dots & \frac{\alpha_2\alpha_n f}{x_2x_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\alpha_n f}{x_n} & \frac{\alpha_1\alpha_n f}{x_1x_n} & \frac{\alpha_2\alpha_n f}{x_2x_n} & \dots & \frac{\alpha_n(\alpha_n-1)f}{x_n^2} \end{pmatrix}$$

We have: $\Delta_k^B=(-1)^k f^{k+1}\frac{\prod_{i=1}^k \alpha_i \sum_{i=1}^k \alpha_i}{\left(\prod_{i=1}^k x_i\right)^2}$, $k=\overline{1,n}$. Because $(-1)^k\text{sign}(\Delta_k^B)>0$ it follows that the function is strictly quasiconcave. From Section 6, we have seen that for some values of α_i, $i=\overline{1,n}$ the function has allow saddle points, therefore it was not concave and not convex. On the other hand, we have seen now that the function is strictly quasiconcave.

7.2. The CES function

Let the bordered Hessian matrix:

$$H^B(f)=\begin{pmatrix} 0 & \frac{\beta_1 x_1^{\rho-1}}{\sum_{k=1}^n \beta_k x_k^\rho} & \frac{\beta_2 x_2^{\rho-1}}{\sum_{k=1}^n \beta_k x_k^\rho} & \dots & \frac{\beta_n x_n^{\rho-1}}{\sum_{k=1}^n \beta_k x_k^\rho} \\ \frac{\beta_1 x_1^{\rho-1}}{\sum_{k=1}^n \beta_k x_k^\rho} & \frac{\beta_1(\rho-1)x_1^{\rho-2}\left(\sum_{k=1}^n \beta_k x_k^\rho-\beta_1 x_1^\rho\right)}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} & -\frac{\beta_1\beta_2(\rho-1)x_1^{\rho-1}x_2^{\rho-1}}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} & \dots & -\frac{\beta_1\beta_n(\rho-1)x_1^{\rho-1}x_n^{\rho-1}}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} \\ \frac{\beta_2 x_2^{\rho-1}}{\sum_{k=1}^n \beta_k x_k^\rho} & -\frac{\beta_1\beta_2(\rho-1)x_1^{\rho-1}x_2^{\rho-1}}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} & \frac{\beta_2(\rho-1)x_2^{\rho-2}\left(\sum_{k=1}^n \beta_k x_k^\rho-\beta_2 x_2^\rho\right)}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} & \dots & -\frac{\beta_2\beta_n(\rho-1)x_2^{\rho-1}x_n^{\rho-1}}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\beta_n x_n^{\rho-1}}{\sum_{k=1}^n \beta_k x_k^\rho} & -\frac{\beta_1\beta_n(\rho-1)x_1^{\rho-1}x_n^{\rho-1}}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} & -\frac{\beta_2\beta_n(\rho-1)x_2^{\rho-1}x_n^{\rho-1}}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} & \dots & \frac{\beta_n(\rho-1)x_n^{\rho-2}\left(\sum_{k=1}^n \beta_k x_k^\rho-\beta_n x_n^\rho\right)}{\left(\sum_{k=1}^n \beta_k x_k^\rho\right)^2} \end{pmatrix}$$

We have: $\Delta_k^B = -\frac{(\rho-1)^{k-1}\left(\prod_{i=1}^{k}\beta_i\right)\prod_{i=1}^{k}x_i^{\rho-2}f^{k+1}}{E^{k+1}}$, k=$\overline{1,n}$.

If ρ<1 then $(-1)^k$sign(Δ_k^B)>0 therefore the function is quasiconcave.

If ρ>1 then sign(Δ_k^B)<0 therefore the function is quasiconvex.

7. Conclusions

The above analysis reveals several aspects. On the one hand, the Gauss method of reducing a quadratic form to the normal form using determinants (unlike classical algorithm) gives clues to its nature without necessarily having to actually obtain the normal form. On the other hand, makes a strong link with the theory of bordered matrices, the matrix remaining after removal behavior board determines, in some cases, the behavior of the original matrix. The production function analysis reveals that, despite the fact that they are not necessarily concave or strictly concave, they are quasi-concave and so satisfying the uniqueness of extreme points with linear constraints.

8. References

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