On Homogeneous Functions

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Abstract: The paper investigates some aspects of the behavior of homogeneous functions. After determining the degree of homogeneity of partial derivatives of a homogeneous function, it is determined their general form in the case of integer degree of homogeneity and they are defined in 0. It also generalizes the Euler relation for homogeneous functions to the higher order partial derivatives. Finally, it is determined a necessary condition for concavity of these functions.

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1. Introduction

The production functions are fundamental in the theory of producer behavior. One of the basic requirements that needs to be satisfied is that of homogeneity, meaning that an increase in inputs will result in an increase in the below sense of its production. We will propose in what following, to determine some of their fundamental properties, many of them very useful for economic research, but not only.

2. Some Facts about Homogenous Functions

Definition 2.1 A non-constant function $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ is said to be homogenous of degree $\alpha\neq0$ if $\Psi(\lambda x)=\lambda^\alpha\Psi(x)$ for any $\lambda>0$, $\lambda\neq1$, $x\in D$ such that $\lambda x\in D$.

Theorem 2.1 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a homogenous function of degree $\alpha\in\mathbf{R}$. If $0\in D$ then $\Psi(0)=0$.

Proof. For x=0 we have: $\Psi(0)=\lambda^\alpha\Psi(0)$ therefore $(\lambda^\alpha-1)\Psi(0)=0$. Because $\lambda\neq1$, $\alpha\neq0$, we have $\Psi(0)=0$.

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Remark 2.1 If the relationship $\Psi(\lambda x)=\lambda^{\alpha}\Psi(x)$ holds for $\alpha=0$, and the function Ψ is continous in $0\in D$, then: $\Psi(\lambda x)=\Psi(x)$ $\forall x\in D$, hence, by passing to limit after $\lambda\to 0$ we obtain: $\Psi(x)=\Psi(0)$=constant.

Lemma 2.1 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a homogenous function of degree α. Then $\frac{\partial\Psi}{\partial x_i}$ is homogenous of degree α-1 $\forall i=\overline{1,n}$.

Proof. We have in an arbitrary point $\left(x_1^0,...,x_n^0\right)\in D$:

$$\frac{\partial\Psi}{\partial x_i}\left(x_1^0,...,x_n^0\right)=\lim_{t\to 0}\frac{\Psi\left(x_1^0,...,x_i^0+t,...,x_n^0\right)-\Psi\left(x_1^0,...,x_i^0,...,x_n^0\right)}{t}$$

But:

$$\frac{\partial\Psi}{\partial x_i}\left(\lambda x_1^0,...,\lambda x_n^0\right)=\lim_{t\to 0}\frac{\Psi\left(\lambda x_1^0,...,\lambda x_i^0+t,...,\lambda x_n^0\right)-\Psi\left(\lambda x_1^0,...,\lambda x_i^0,...,\lambda x_n^0\right)}{t}=$$

$$\lim_{t\to 0}\frac{\Psi\left(\lambda x_1^0,...,\lambda x_i^0+\lambda t,...,\lambda x_n^0\right)-\Psi\left(\lambda x_1^0,...,\lambda x_i^0,...,\lambda x_n^0\right)}{\lambda t}=$$

$$\lim_{t\to 0}\frac{\lambda^{\alpha}\Psi\left(x_1^0,...,x_i^0+t,...,x_n^0\right)-\lambda^{\alpha}\Psi\left(x_1^0,...,x_i^0,...,x_n^0\right)}{\lambda t}=$$

$$\lambda^{\alpha-1}\lim_{t\to 0}\frac{\Psi\left(x_1^0,...,x_i^0+t,...,x_n^0\right)-\Psi\left(x_1^0,...,x_i^0,...,x_n^0\right)}{t}=\lambda^{\alpha-1}\frac{\partial\Psi}{\partial x_i}\left(x_1^0,...,x_n^0\right).$$

Corollary 2.1 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a homogenous function of degree α and of class $C^k(D)$. Then $\frac{\partial^k\Psi}{\partial x_{i_1}...\partial x_{i_k}}$ is homogenous of degree α-k $\forall i_1,...,i_k=\overline{1,n}$, $k\geq 1$.

Theorem 2.2 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a non-constant function, homogenous of degree $\alpha\notin\mathbf{Z}$, $0\in D$, $\Psi\in C^{\infty}(D^*)$. Then $\exists k\geq 1$ $\exists i_1,...,i_k=\overline{1,n}$ such that $\frac{\partial^k\Psi}{\partial x_{i_1}...\partial x_{i_k}}$ is not defined in 0.

Proof. Suppose that all the partial derivatives of Ψ are defined in 0. Because Ψ is homogenous of degree α, follows that $\frac{\partial^k\Psi}{\partial x_{i_1}...\partial x_{i_k}}$ is homogenous of degree α-k

$\forall i_1,...,i_k=\overline{1,n}$, $k\geq 1$ and therefore, from the theorem 1 follows: $\frac{\partial^k \Psi}{\partial x_{i_1}...\partial x_{i_k}}(0)=0$.

Developing in Taylor series around to $0\in D$, follows:

$\Psi(x)=\Psi(0)+\sum_{k=1}^{\infty}\sum_{i_1,...,i_j=1}^{n}\frac{\partial^k \Psi}{\partial x_{i_1}...\partial x_{i_k}}(0)x_{i_1}...x_{i_k}=0$ – contradiction.

Theorem 2.3 Let $\Psi:D\subset \mathbf{R}^n\to\mathbf{R}$ a non-constant function, homogenous of degree $\alpha\in\mathbf{Z}$, $0\in D$, $\Psi\in C^{\infty}(D)$. Then $\Psi=\sum_{\substack{\beta_1,...,\beta_n=0\\ \beta_1+...+\beta_n=\alpha}}^{\alpha} c_{\beta_1...\beta_n}x_1^{\beta_1}...x_n^{\beta_n}$.

Proof. From statement, follows that all the partial derivatives of Ψ are defined in 0. Because Ψ is homogenous of degree α follows that $\frac{\partial^k \Psi}{\partial x_{i_1}...\partial x_{i_k}}$ is homogenous of degree α-k $\forall i_1,...,i_k=\overline{1,n}$, $k\geq 1$. For $k=\alpha$ we have from remark 2.1: $\frac{\partial^\alpha \Psi}{\partial x_{i_1}...\partial x_{i_\alpha}}=C_{i_1...i_\alpha}$ =constant $\forall i_1,...,i_\alpha=\overline{1,n}$. For $i_1=...=i_\alpha=p=\overline{1,n}$ we get: $\frac{\partial^\alpha \Psi}{\partial x_p^\alpha}=C_p$ where, by successive integrations with brespect to x_p follows: $\Psi=\sum_{u=0}^{\alpha}A_{u,p}\left(x_1,...,\hat{x}_p,...,x_n\right)x_p^u$ where $A_{u,p}$ are arbitrary functions. For $i_\alpha\neq i_1=...=i_{\alpha-1}=p=\overline{1,n}$ follows: $\frac{\partial^\alpha \Psi}{\partial x_p^{\alpha-1}\partial x_{i_\alpha}}=C_{p,i_\alpha}$. We obtain now:

$$\alpha!\frac{\partial A_{\alpha,p}\left(x_1,...,\hat{x}_p,...,x_n\right)}{\partial x_{i_\alpha}}x_p+(\alpha-1)!\frac{\partial A_{\alpha-1,p}\left(x_1,...,\hat{x}_p,...,x_n\right)}{\partial x_{i_\alpha}}=C_{p,i_\alpha}$$

therefore: $\frac{\partial A_{\alpha,p}\left(x_1,...,\hat{x}_p,...,x_n\right)}{\partial x_{i_\alpha}}=0$ $\forall i_\alpha\neq p$ therefore: $A_{\alpha,p}\left(x_1,...,\hat{x}_p,...,x_n\right)=a_p$=const.

From $(\alpha-1)!\frac{\partial A_{\alpha-1,p}\left(x_1,...,\hat{x}_p,...,x_n\right)}{\partial x_{i_\alpha}}=C_{p,i_\alpha}$ follows, analogously:

$A_{\alpha-1,p}\left(x_1,...,\hat{x}_p,...,x_n\right)=\sum_{\substack{i=1\\ i\neq p}}^{n} b_{ip}x_i$. If we continue:

$A_{\alpha-2,p}\left(x_1,...,\hat{x}_p,...,x_n\right)=\sum_{\substack{i,j=1\\ i,j\neq p}}^{n} c_{ip}x_i x_j$ etc.

Finally: $\Psi=\sum_{\substack{\beta_1,...\beta_n=0\\ \beta_1+...+\beta_n=\alpha}}^{\alpha} c_{\beta_1..\beta_n} x_1^{\beta_1}...x_n^{\beta_n}$.

Lemma 2.2 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}_+^*$ a homogenous function of degree α and $\Phi=\ln\Psi$. Then $\frac{\partial\Phi}{\partial x_i}$ is homogenous of degree -1.

Proof. We have: $\Phi(\lambda x_1,...,\lambda x_n)=\ln\Psi(\lambda x_1,...,\lambda x_n)=\alpha\ln\lambda+\ln\Psi(x_1,...,x_n)=\alpha\ln\lambda+\Phi(x_1,...,x_n)$

The partial derivatives in a point $\left(x_1^0,...,x_n^0\right)\in D$ are:

$$\frac{\partial\Phi}{\partial x_i}\left(x_1^0,...,x_n^0\right)=\lim_{t\to 0}\frac{\Phi\left(x_1^0,...,x_i^0+t,...,x_n^0\right)-\Phi\left(x_1^0,...,x_i^0,...,x_n^0\right)}{t}$$

$$\frac{\partial\Phi}{\partial x_i}\left(\lambda x_1^0,...,\lambda x_n^0\right)=\lim_{t\to 0}\frac{\Phi\left(\lambda x_1^0,...,\lambda x_i^0+t,...,\lambda x_n^0\right)-\Phi\left(\lambda x_1^0,...,\lambda x_i^0,...,\lambda x_n^0\right)}{t}=$$

$$\lim_{t\to 0}\frac{\Phi\left(\lambda x_1^0,...,\lambda x_i^0+\lambda t,...,\lambda x_n^0\right)-\Phi\left(\lambda x_1^0,...,\lambda x_i^0,...,\lambda x_n^0\right)}{\lambda t}=$$

$$\lim_{t\to 0}\frac{\alpha\ln\lambda+\Phi\left(x_1^0,...,x_i^0+t,...,x_n^0\right)-\alpha\ln\lambda-\Phi\left(x_1^0,...,x_i^0,...,x_n^0\right)}{\lambda t}=$$

$$\frac{1}{\lambda}\lim_{t\to 0}\frac{\Phi\left(x_1^0,...,x_i^0+t,...,x_n^0\right)-\Phi\left(x_1^0,...,x_i^0,...,x_n^0\right)}{t}=\frac{1}{\lambda}\frac{\partial\Phi}{\partial x_i}\left(x_1^0,...,x_n^0\right).$$

Theorem 2.4 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a C^2-differentiable function on an open subset D, $\Psi(0)=0$. The following statements are equivalent:

1. $\exists f:I\to\mathbf{R}$, $I\subset\mathbf{R}$, $f(\lambda)\neq 0$ $\forall\lambda\in I$ such that: $\Psi(\lambda x)=f(\lambda)\Psi(x)$ $\forall x\in D$ $\forall\lambda\in I$;
2. $\exists\alpha\in\mathbf{R}$ such that: $\sum_{j=1}^{n} x_j\frac{\partial\Psi}{\partial x_j}=\alpha\Psi$;

3. $\exists\alpha\in\mathbf{R}$ such that: $\sum_{i=1}^{n} x_i \frac{\partial^2\Psi}{\partial x_i \partial x_j} = (\alpha-1)\frac{\partial\Psi}{\partial x_j}$, $j=\overline{1,n}$;

4. $\exists\alpha\in\mathbf{R}$ $\exists k\geq 1$ such that: $\frac{\partial^k\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}}(0)=0$ and $\sum_{i=1}^{n} x_i \frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} =$ $(\alpha-k)\frac{\partial^k\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}}$, $i_1,\ldots,i_k=\overline{1,n}$.

Proof. <u>1)⇒2)</u> If $\Psi(\lambda x)=f(\lambda)\Psi(x)$ $\forall x\in D$ $\forall\lambda\in I$ then, differentiating with λ: $\sum_{j=1}^{n} x_j \frac{\partial\Psi}{\partial x_j}(\lambda x) = f'(\lambda)\Psi(x)$ $\forall x\in D$ $\forall\lambda\in I$. For $\lambda=1$ we have: $\sum_{j=1}^{n} x_j \frac{\partial\Psi}{\partial x_j}(x) =$ $f'(1)\Psi(x)$ and 2) follows for $\alpha= f'(1)$.

<u>2)⇒1)</u> From $\sum_{j=1}^{n} x_j \frac{\partial\Psi}{\partial x_j}=\alpha\Psi$ we have: $\sum_{j=1}^{n} \lambda x_j \frac{\partial\Psi}{\partial x_j}(\lambda x)=\alpha\Psi(\lambda x)$ or: $\lambda\frac{d\Psi(\lambda x)}{d\lambda}$ $=\alpha\Psi(\lambda x)$. But this is equivalent with: $\frac{d\Psi(\lambda x)}{\Psi(\lambda x)}=\alpha\frac{d\lambda}{\lambda}$ from where: $\Psi(\lambda x)=\lambda^{\alpha}C(x)$. For $\lambda=1$ we have: $\Psi(x)=C(x)$ therefore: $\Psi(\lambda x)=\lambda^{\alpha}\Psi(x)$ and 1) follows for $f(\lambda)=\lambda^{\alpha}$.

<u>2)⇒3)</u> If $\sum_{j=1}^{n} x_j \frac{\partial\Psi}{\partial x_j}=\alpha\Psi$ then differentiating with x_i, we have: $\sum_{j=1}^{n} x_j \frac{\partial^2\Psi}{\partial x_i\partial x_j} + \frac{\partial\Psi}{\partial x_i} = \alpha\frac{\partial\Psi}{\partial x_i}$ from where: $\sum_{j=1}^{n} x_j \frac{\partial^2\Psi}{\partial x_i\partial x_j} = (\alpha-1)\frac{\partial\Psi}{\partial x_i}$.

<u>3)⇒2)</u> From $\sum_{i=1}^{n} x_i \frac{\partial^2\Psi}{\partial x_i\partial x_j} = (\alpha-1)\frac{\partial\Psi}{\partial x_j}$ integrating with respect to x_j we have:

$$\sum_{\substack{i=1\\ i\neq j}}^{n} x_i\left(\frac{\partial\Psi}{\partial x_i} + \Phi_i(x_1,\ldots,\hat{x}_j,\ldots,x_n)\right) + \int x_j \frac{\partial\left(\frac{\partial\Psi}{\partial x_j} + \Phi_j(x_1,\ldots,\hat{x}_j,\ldots,x_n)\right)}{\partial x_j} dx_j =$$
$$(\alpha-1)\left[\Psi + \Lambda(x_1,\ldots,\hat{x}_j,\ldots,x_n)\right]$$

where Φ_i and Λ are arbitrary functions.

Integrating through parts:

$$\sum_{\substack{i=1\\ i\neq j}}^{n} x_i\left(\frac{\partial\Psi}{\partial x_i}+\Phi_i(x_1,...,\widehat{x}_j,...,x_n)\right)+x_j\left(\frac{\partial\Psi}{\partial x_j}+\Phi_j(x_1,...,\widehat{x}_j,...,x_n)\right)-$$

$$\int\left(\frac{\partial\Psi}{\partial x_j}+\Phi_j(x_1,...,\widehat{x}_j,...,x_n)\right)dx_j=(\alpha-1)\left(\Psi+\Lambda(x_1,...,\widehat{x}_j,...,x_n)\right)$$

Further:

$$\sum_{\substack{i=1\\ i\neq j}}^{n} x_i\left(\frac{\partial\Psi}{\partial x_i}+\Phi_i(x_1,...,\widehat{x}_j,...,x_n)\right)+x_j\left(\frac{\partial\Psi}{\partial x_j}+\Phi_j(x_1,...,\widehat{x}_j,...,x_n)\right)-$$

$$\Psi-\Theta(x_1,...,\widehat{x}_j,...,x_n)-\Phi_j(x_1,...,\widehat{x}_j,...,x_n)x_j=(\alpha-1)\left(\Psi+\Lambda(x_1,...,\widehat{x}_j,...,x_n)\right)$$

where Θ is an arbitrary function. We have therefore, finally:

$$\sum_{i=1}^{n} x_i\frac{\partial\Psi}{\partial x_i}-\alpha\Psi=(\alpha-1)\Lambda(x_1,...,\widehat{x}_j,...,x_n)+\Theta(x_1,...,\widehat{x}_j,...,x_n)-\sum_{\substack{i=1\\ i\neq j}}^{n} x_i\Phi_i(x_1,...,\widehat{x}_j,...,x_n)$$

Because the right side does not depend from x_j and x_j was arbitrary chosen, we have that: $\sum_{i=1}^{n} x_i\frac{\partial\Psi}{\partial x_i}-\alpha\Psi=\beta$=constant or: $\sum_{i=1}^{n} x_i\frac{\partial\Psi}{\partial x_i}=\alpha\Psi+\beta$. Because $\Psi(0)=0$ we have: $\beta=0$ and the relation becomes: $\sum_{i=1}^{n} x_i\frac{\partial\Psi}{\partial x_i}=\alpha\Psi$.

3)⇒4) Let P(k): $\exists\alpha\in\mathbf{R}$ such that: $\sum_{i=1}^{n} x_i\frac{\partial^{k+1}\Psi}{\partial x_i\partial x_{i_1}..\partial x_{i_k}}=(\alpha-k)\frac{\partial^{k}\Psi}{\partial x_{i_1}..\partial x_{i_k}}$, $i_1,...,i_k=\overline{1,n}$, $k\geq1$. From 3) follows that P(1) is true. Suppose that P(k) is true.

Differentiating with respect to $x_{i_{k+1}}$ we have:

$$\sum_{i=1}^{n} x_i\frac{\partial^{k+2}\Psi}{\partial x_i\partial x_{i_1}..\partial x_{i_k}\partial x_{i_{k+1}}}+\frac{\partial^{k+1}\Psi}{\partial x_{i_1}..\partial x_{i_k}\partial x_{i_{k+1}}}=(\alpha-k)\frac{\partial^{k+1}\Psi}{\partial x_{i_1}..\partial x_{i_k}\partial x_{i_{k+1}}}$$

of where:

$$\sum_{i=1}^{n} x_i\frac{\partial^{k+2}\Psi}{\partial x_i\partial x_{i_1}..\partial x_{i_k}\partial x_{i_{k+1}}}=(\alpha-k-1)\frac{\partial^{k+1}\Psi}{\partial x_{i_1}..\partial x_{i_k}\partial x_{i_{k+1}}}$$

therefore P(k+1) is true.

4)⇒3) Suppose that: $\sum_{i=1}^{n} x_i \frac{\partial^{k+2}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_{k+1}}} = (\alpha - k - 1)\frac{\partial^{k+1}\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}\partial x_{i_{k+1}}}$, $i_1,\ldots,i_{k+1} = \overline{1,n}$, k≥1. Integrating with respect to $x_{i_{k+1}}$ we have:

$$\sum_{\substack{i=1\\ i\neq i_{k+1}}}^{n} x_i\left(\frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} + \Phi_i(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) +$$

$$\int x_{i_{k+1}} \frac{\partial\left(\frac{\partial^{k+1}\Psi}{\partial x_{i_1}\ldots\partial x_{i_{k+1}}} + \Phi_{i_{k+1}}(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right)}{\partial x_{i_{k+1}}} dx_{i_{k+1}} = (\alpha - k - 1)\left(\frac{\partial^{k}\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} + \Lambda(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right)$$

where $\Phi_{i_{k+1}}$ and Λ has arbitrary functions. Integrating through parts:

$$\sum_{\substack{i=1\\ i\neq i_{k+1}}}^{n} x_i\left(\frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} + \Phi_i(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) + x_{i_{k+1}}\left(\frac{\partial^{k+1}\Psi}{\partial x_{i_1}\ldots\partial x_{i_{k+1}}} + \Phi_{i_{k+1}}(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) -$$

$$\int\left(\frac{\partial^{k+1}\Psi}{\partial x_{i_1}\ldots\partial x_{i_{k+1}}} + \Phi_{i_{k+1}}(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) dx_{i_{k+1}} = (\alpha - k - 1)\left(\frac{\partial^{k}\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} + \Lambda(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right)$$

Further:

$$\sum_{\substack{i=1\\ i\neq i_{k+1}}}^{n} x_i\left(\frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} + \Phi_i(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) + x_{i_{k+1}}\left(\frac{\partial^{k+1}\Psi}{\partial x_{i_1}\ldots\partial x_{i_{k+1}}} + \Phi_{i_{k+1}}(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) -$$

$$\left(\frac{\partial^{k}\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} + \Theta(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right) - \Phi_{i_{k+1}}(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)x_{i_{k+1}} =$$

$$(\alpha - k - 1)\left(\frac{\partial^{k}\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} + \Lambda(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right)$$

where Θ has an arbitrary function. We have therefore, finally:

$$\sum_{i=1}^{n} x_i \frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} - (\alpha - k)\frac{\partial^{k}\Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} =$$

$$(\alpha - k - 1)\Lambda(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n) + \Theta(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n) - \sum_{\substack{i=1\\ i\neq i_{k+1}}}^{n} x_i\left(\Phi_i(x_1,\ldots,\hat{x}_{i_{k+1}},\ldots,x_n)\right)$$

Because the right side does not depend from $x_{i_{k+1}}$ and $x_{i_{k+1}}$ was arbitrary chosen, we have that: $\sum_{i=1}^{n} x_i \frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} - (\alpha - k)\frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}}$ =β=constant or:

$\sum_{i=1}^{n} x_i \frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} = (\alpha - k)\frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} + \beta$. Because $\frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}}(0)=0$ we have: β=0 and the relation becomes: $\sum_{i=1}^{n} x_i \frac{\partial^{k+1}\Psi}{\partial x_i \partial x_{i_1}\ldots\partial x_{i_k}} = (\alpha - k)\frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}}$. From induction after k we will find that 3) holds. **Q.E.D.**

Corollary 2.2 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a homogenous function of degree α, of class $C^k(D)$, Ψ(0)=0. If ∃k≥1 such that: $\frac{\partial^{k-1}\Psi}{\partial x_{i_1}\ldots\partial x_{i_{k-1}}}(0)=0$ then:

$$\sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} x_{i_1}\ldots x_{i_k} = (\alpha-k+1)(\alpha-k+2)\ldots\alpha U$$

Proof. From Theorem 2.4 $\sum_{i_1=1}^{n} x_{i_1} \frac{\partial^k \Psi}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_k}} = (\alpha-k+1)\frac{\partial^{k-1}\Psi}{\partial x_{i_2}\ldots\partial x_{i_k}}$, $i_2,\ldots,i_k=\overline{1,n}$

Multiplying with $x_{i_2},\ldots,x_{i_k}$ and summing after $x_{i_2},\ldots,x_{i_k}$ we obtain:

$$\sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} x_{i_1}\ldots x_{i_k} = (\alpha-k+1)\sum_{i_2,\ldots,i_k=1}^{n} \frac{\partial^{k-1}\Psi}{\partial x_{i_2}\ldots\partial x_{i_k}} x_{i_2}\ldots x_{i_k}$$

Through induction, follows:

$$\sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k \Psi}{\partial x_{i_1}\ldots\partial x_{i_k}} x_{i_1}\ldots x_{i_k} = (\alpha-k+1)(\alpha-k+2)\ldots\alpha U.$$

Corollary 2.3 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ a homogenous function of degree 1 and of class $C^k(D)$, Ψ(0)=0. Then: $\det\left(\frac{\partial^2\Psi}{\partial x_i \partial x_j}\right)$=0.

Proof. From the theorem, we have that for a homogenous function of degree 1 for which in addition Ψ(0)=0 we have that $\sum_{j=1}^{n} x_j \frac{\partial \Psi}{\partial x_j} = \Psi$ and $\sum_{i=1}^{n} x_i \frac{\partial^2 \Psi}{\partial x_i \partial x_j} = 0$, $j=\overline{1,n}$.

Because the equality holds for any $x_1,...,x_n$ we find that: $\det\left(\frac{\partial^2 \Psi}{\partial x_i \partial x_j}\right)=0$.

3. The Concavity of the Functions

We will present in this section some of the remarkable results of concavity of functions.

Definition 3.1 A subset $D\subset\mathbf{R}^n$ is called convex if $\forall x,y\in D$ $\forall\lambda\in[0,1]\Rightarrow\lambda x+(1-\lambda)y\in D$.

Definition 3.2 A function $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ is called **convex** if $\forall x,y\in D$ $\forall\lambda\in[0,1]$ follows
$\Psi(\lambda x+(1-\lambda)y)\le\lambda\Psi(x)+(1-\lambda)\Psi(y)$.

Definition 3.3 A function $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}$ is called **concave** if $\forall x,y\in D$ $\forall\lambda\in[0,1]$ follows
$\Psi(\lambda x+(1-\lambda)y)\ge\lambda\Psi(x)+(1-\lambda)\Psi(y)$.

Definition 3.4 A function $f:D\subset\mathbf{R}^n\to\mathbf{R}$ is called **strictly convex** if $\forall x,y\in D$ $\forall\lambda\in(0,1)$ follows $f(\lambda x+(1-\lambda)y)<\lambda f(x)+(1-\lambda)f(y)$.

Definition 3.5 A function $f:D\subset\mathbf{R}^n\to\mathbf{R}$ is called **strictly concave** if $\forall x,y\in D$ $\forall\lambda\in(0,1)$ follows $f(\lambda x+(1-\lambda)y)>\lambda f(x)+(1-\lambda)f(y)$.

Suppose now that $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}_+$, $0\in D$, $\Psi(0)=0$, is homogenous of degree α and convex. From the definition 3.2, for y=0 follows: $\Psi(\lambda x)\le\lambda\Psi(x)$ $\forall x\in D$. From the homogenity of the function, follows: $(\lambda^\alpha-\lambda)\Psi(x)\le0$ $\forall\lambda\in(0,1)$ $\forall x\in D$. Since $\Psi(x)\ge0$ we obtain: $\lambda^\alpha\le\lambda$ $\forall\lambda\in(0,1)$. The function $g(t)=\lambda^t$ being decreasing we obtain $\alpha\ge1$. Analogously, if the function Ψ is concave then: $\alpha\le1$. In the case of strictly convexity we will have, analogously: $\alpha>1$, and of strictly concavity: $\alpha<1$.

Theorem 3.1 Let $\Psi:D\subset\mathbf{R}^n\to\mathbf{R}_+$, $0\in D$, $\Psi(0)=0$, homogenous of degree α. Then:

1. If $\alpha<1$ then the function Ψ cannot be convex;
2. If $\alpha>1$ then the function Ψ cannot be concave;
3. If $\alpha\le1$ then the function Ψ cannot be strictly convex;
4. If $\alpha\ge1$ then the function Ψ cannot be strictly concave.

4. References

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