Microeconomics

Generalized Cobb-Douglas Function for Two Inputs and Linear Elasticity

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Abstract: The article deals with a production function of two factors with constant scale return where the elasticity of one of the factors is a function of first degree. After the examination of parameters conditions according to the axioms of the production functions, there are computed the main indicators. Also, the combination of factors is determined in order to maximize the total output under a given cost.

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1. General Aspects of the Production Functions

In any economic activity, obtaining a result of it implies, by default, there is a certain number of resources, supposedly indivisible needed for the proper functioning of the production process.

We therefore define on $\mathbf{R}^2$ – the **production space** for two resources: K – capital and L - labor as $SP=\{(K,L) \mid K,L\geq 0\}$ where $x\in SP$, $x=(K,L)$ is an **ordered set of resources**.

Because in a production process, depending on the nature of applied technology, but also its specificity, not any amount of resources is possible, we restrict the production area to a subset $D_p\subset SP$ called **domain of production**.

In a context of the existence of the domain of production, we put the question of determining its output depending on the level of inputs of D_p.

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It is called **production function** an application $Q:D_p\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$.

For an efficient and complex mathematical analysis of a production function, we impose a number of axioms both its definition and its scope.

FP1. The domain of production is convex;

FP2. $Q(0,0)=0$;

FP3. The production function is of class C^2 on D_p that is it admits partial derivatives of order 2 and they are continuous;

FP4. The production function is monotonically increasing in each variable;

FP5. The production function is quasiconcave that is: $Q(\lambda x+(1-\lambda)y)\geq\min(Q(x),Q(y))$ $\forall\lambda\in[0,1]$ $\forall x,y\in R_p$.

From a geometric point of view, a quasiconcave function having the property of being above the lowest value recorded at the end of a certain segment. The property is equivalent to the convexity of the set $Q^{-1}[a,\infty)$ $\forall a\in\mathbf{R}$, where $Q^{-1}[a,\infty)=\{x\in R_p \mid Q(x)\geq a\}$.

2. The Main Indicators of Production Functions

Consider now a production function: $Q:D_p\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$.

We call **marginal productivity** relative to an input x_i: $\eta_{x_i}=\frac{\partial Q}{\partial x_i}$ and represents the trend of variation of production to the variation of x_i.

We call **average productivity** relative to an input x_i: $w_{x_i}=\frac{Q}{x_i}$ the value of production at a consumption of a unit of factor x_i.

We call **partial marginal substitution rate** of factors i and j the opposite change in the amount of factor j as a substitute for the amount of change in the factor i in the case of a constant level of production and we have: $RMS(i,j)=\frac{\eta_{x_i}}{\eta_{x_j}}$.

We call **elasticity of output** with respect to an input x_i: $\varepsilon_{x_i} = \dfrac{\frac{\partial Q}{\partial x_i}}{\frac{Q}{x_i}} = \dfrac{\eta_{x_i}}{w_{x_i}}$ and represents the relative variation of production to the relative variation of the factor x_i.

Considering now a production function Q:$D_p \to \mathbf{R}_+$ with constant return to scale that is Q(K,L)=$\frac{1}{\lambda}$Q(λK,λL), let note $\chi=\frac{K}{L}$. It is called the **elasticity of the marginal rate of technical substitution** $\sigma = \dfrac{\frac{\partial RMS(K,L)}{\partial \chi}}{\frac{RMS(K,L)}{\chi}}$.

3. The Generalized Cobb-Douglas Function for Two Inputs

Consider now a production function Q:$D_p \to \mathbf{R}_+$, (K,L)→Q(K,L)$\in \mathbf{R}_+$ $\forall$(K,L)$\in D_p$ with constant return to scale, where $\varepsilon_K = \rho(\chi) > 0$.

Considering the function q such that: Q(K,L)=L$q(\chi)$ we have: $\varepsilon_K = \dfrac{\eta_K}{w_K} = \dfrac{\frac{\partial q}{\partial \chi}}{\frac{q}{\chi}} = \dfrac{\chi q'(\chi)}{q} = \rho(\chi)$.

From here we find that: $\dfrac{q'(\chi)}{q} = \dfrac{\rho(\chi)}{\chi}$. Let F be a primitive function of $\dfrac{\rho(\chi)}{\chi}$. From $\dfrac{q'(\chi)}{q} = F'(\chi)$ we obtain: $q(\chi) = Ce^{F(\chi)}$ where C - constant strictly positive.

In particular, for $\rho(\chi)=\sum_{k=0}^{m}\beta_k\chi^k$ we have:

$$F(\chi) = \int \frac{\sum_{k=0}^{m}\beta_k\chi^k}{\chi}d\chi = \int\left(\frac{\beta_0}{\chi} + \sum_{k=1}^{m}\beta_k\chi^{k-1}\right)d\chi = \beta_0 \ln\chi + \sum_{k=1}^{m}\frac{\beta_k}{k}\chi^k + D$$

and the production function becomes (after an obvious renoting of C):

$$q(\chi) = Ce^{F(\chi)} = C\chi^{\beta_0} e^{\sum_{k=1}^{m} \frac{\beta_k}{k}\chi^k}$$

or other:

$$Q(K,L) = CK^{\beta_0} L^{1-\beta_0} e^{\sum_{k=1}^{m} \frac{\beta_k K^k}{kL^k}}$$

If $\rho(\chi) = \beta_0 + \beta_1\chi$ then:

$$Q(K,L) = CK^{\beta_0} L^{1-\beta_0} e^{\frac{\beta_1 K}{L}}$$

4. The Generalized Cobb-Douglas Function for Two Inputs and Linear Elasticity

Consider now the production function: $Q(K,L) = CK^{a} L^{1-a} e^{\frac{bK}{L}}$, K,L>0, a,b,C>0.

Because the function is elementary follows that it is of class C^∞ on the definition domain.

We now have:

$$\frac{\partial Q}{\partial K} = \frac{bK + aL}{KL} Q, \quad \frac{\partial Q}{\partial x_2} = -\frac{bK + (a-1)L}{L^2} Q$$

Considering bordered Hessian matrix:

$$H^B(Q) = \begin{pmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K \partial L} & \frac{\partial^2 Q}{\partial L^2} \end{pmatrix}$$

and the minors:

$$\Delta_1^B = \begin{vmatrix} 0 & \frac{\partial Q}{\partial K} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} \end{vmatrix} = -\left(\frac{\partial Q}{\partial K}\right)^2,$$

$$\Delta_2^B = \begin{vmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K \partial L} & \frac{\partial^2 Q}{\partial L^2} \end{vmatrix} = 2\frac{\partial Q}{\partial L}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K \partial L} - \left(\frac{\partial Q}{\partial L}\right)^2 \frac{\partial^2 Q}{\partial K^2} - \left(\frac{\partial Q}{\partial K}\right)^2 \frac{\partial^2 Q}{\partial L^2}$$

it is known that if $\Delta_1^B<0$, $\Delta_2^B>0$ the function is quasiconcave. Conversely, if the function is quasiconcave then: $\Delta_1^B\le 0$, $\Delta_2^B\ge 0$.

In the present case:

$$\Delta_1^B = -\frac{(bK+aL)^2}{K^2L^2}Q^2,\ \Delta_2^B = -\frac{b^2K^2+2abKL+a(a-1)L^2}{K^2L^4}Q^3$$

It is obvious that $\Delta_1^B<0$. For $\Delta_2^B>0$ it is necessary and sufficient that: $b^2K^2+2abKL+a(a-1)L^2<0 \quad \forall K,L>0$. With the substitution $\chi=\frac{K}{L}$ the statement is equivalent to $b^2\chi^2+2ab\chi+\left(a^2-a\right)<0 \ \forall\chi>0$.

Because the discriminant $\Delta=a^2b^2-b^2\left(a^2-a\right)=ab^2$ follows that if $\Delta\le 0$ then $\chi\in\varnothing$. Therefore: $\Delta>0$, $a>0$. We get: $\chi=\frac{K}{L}\in\left(\frac{-a-\sqrt{a}}{b},\frac{-a+\sqrt{a}}{b}\right)\cap(0,\infty)=\left(0,\frac{-a+\sqrt{a}}{b}\right)$. But $\frac{-a+\sqrt{a}}{b}>0$ is equivalent to $a\in(0,1)$.

From the above, for $a\in(0,1)$, $b>0$, the function is quasiconcave on $D_p=\left\{(K,L)\middle| 0<K<\frac{-a+\sqrt{a}}{b}L, L>0\right\}\subset\mathbf{R}^2$.

Also, relative to the monotonically increasing in each variable, we have: $\frac{\partial Q}{\partial K}=\frac{bK+aL}{KL}Q>0$ and because:

$bK+(a-1)L<b\frac{-a+\sqrt{a}}{b}L+(a-1)L=(\sqrt{a}-1)L<0$ we get $\frac{\partial Q}{\partial L}=-\frac{bK+(a-1)L}{L^2}Q>0.$

As an example, for C=1, a=0.2 and b=1 the graph is:

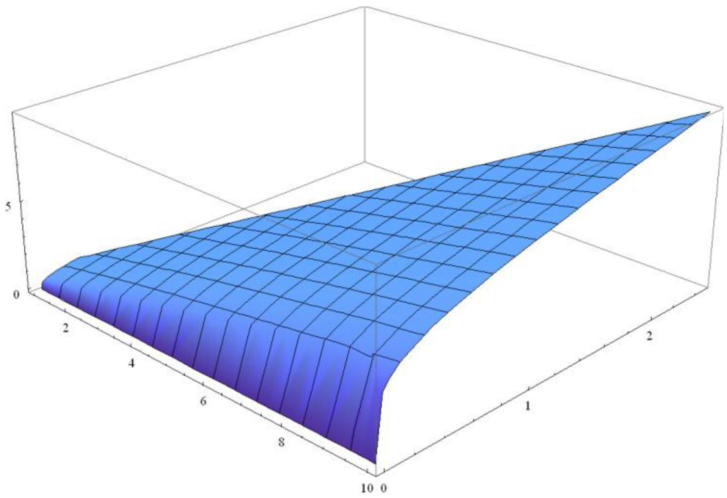

Figure 1

5 Main Indicators of the Generalized Cobb-Douglas Function for two Inputs and Linear Elasticity

We can compute, after section 2, the main indicators for the production function defined above. We have therefore:

- The marginal productivity:

$$\eta_K=\frac{\partial Q}{\partial K}=\frac{bK+aL}{KL}Q,\ \eta_L=\frac{\partial Q}{\partial L}=-\frac{bK+(a-1)L}{L^2}Q$$

- The average productivity:

$$w_K=\frac{Q}{K},\ w_L=\frac{Q}{L}$$

- The partial marginal substitution rate:

$$\text{RMS(K,L)}=-\frac{L(bK+aL)}{K(bK+(a-1)L)},\ \text{RMS(L,K)}=-\frac{K(bK+(a-1)L)}{L(bK+aL)}$$

- The elasticity of output:

$$\varepsilon_K = \frac{\frac{\partial Q}{\partial K}}{\frac{Q}{K}} = \frac{\eta_K}{w_K} = \frac{bK + aL}{L},\ \varepsilon_L = \frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}} = \frac{\eta_L}{w_L} = -\frac{bK + (a-1)L}{L}$$

- The elasticity of the marginal rate of technical substitution:

$$\sigma = \frac{\frac{\partial RMS(K,L)}{\partial \chi}}{\frac{RMS(K,L)}{\chi}} = -1 - \frac{b\chi}{(a + b\chi)(a + b\chi - 1)}$$

6. The Problem of Determining the Maximum of Production in terms of Given Total Cost

Let now the following problem:

$$\begin{cases} \max Q(K,L) \\ p_K K + p_L L = CT > 0 \\ K, L \geq 0 \end{cases}$$

where CT is the total cost of the production which is suppose to be a given constant.

From the Karush-Kuhn-Tucker conditions we have the necessary and sufficient conditions (taking into account that the restriction is affine):

$$\begin{cases} \frac{\frac{\partial Q}{\partial K}}{p_K} = \frac{\frac{\partial Q}{\partial L}}{p_L} \\ p_K K + p_L L = CT \end{cases}$$

From section 5 we get that the system becomes:

$$\begin{cases} \frac{\frac{bK + aL}{KL}}{p_K} = \frac{-\frac{bK + (a-1)L}{L^2}}{p_L} \\ p_K K + p_L L = CT \end{cases}$$

or:

$$\begin{cases} bp_K K^2 + ((a-1)p_K + bp_L)KL + ap_L L^2 = 0 \\ p_K K + p_L L = CT \end{cases}$$

In order to K,L exists, we must have from the first equation:

$$\Delta = ((a-1)p_K + bp_L)^2 - 4abp_K p_L = p_L^2\left[(a-1)^2\left(\frac{p_K}{p_L}\right)^2 - 2(a+1)b\left(\frac{p_K}{p_L}\right) + b^2\right] \geq 0$$

Because the discriminant of the paranthesis is: $\Delta' = (a+1)^2 b^2 - (a-1)^2 b^2 = 4ab^2 > 0$ we get:

$$\frac{p_K}{p_L} \in \left(0, \frac{1}{(1+\sqrt{a})^2} b\right] \cup \left[\frac{1}{(1-\sqrt{a})^2} b, \infty\right)$$

Also, from the existence condition of the production function, that is: $0 \leq \frac{K}{L} < \frac{-a+\sqrt{a}}{b}$, the first equation gives that it is always true.

Substituting now from second in the first equation, we get:

$$p_K^2 K^2 - CT((a+1)p_K - bp_L)K + aCT^2 = 0$$

and finally we find:

$$\begin{cases} K = \dfrac{(1+a)p_K - bp_L \pm \sqrt{((1+a)p_K - bp_L)^2 - 4ap_K^2}}{2p_K^2} CT \\ L = \dfrac{(1-a)p_K + bp_L \mp \sqrt{((1+a)p_K - bp_L)^2 - 4ap_K^2}}{2p_K p_L} CT \end{cases}$$

Because $\frac{p_K}{p_L} \in \left(0, \frac{1}{(1+\sqrt{a})^2} b\right] \cup \left[\frac{1}{(1-\sqrt{a})^2} b, \infty\right)$ we have that:

$$\Delta = ((1+a)p_K - bp_L)^2 - 4ap_K^2 = \left((1+\sqrt{a})^2 p_K - bp_L\right)\left((1-\sqrt{a})^2 p_K - bp_L\right) \geq 0$$

As a conclusion, we have that if $\frac{p_K}{p_L} \in \left(0, \frac{1}{\left(1+\sqrt{a}\right)^2} b\right] \cup \left[\frac{1}{\left(1-\sqrt{a}\right)^2} b, \infty\right)$ the combination of factors which maximize the production when the total cost remaining constant is:

$$\begin{cases} K = \dfrac{(1+a)p_K - bp_L \pm \sqrt{((1+a)p_K - bp_L)^2 - 4ap_K^2}}{2p_K^2} CT \\ L = \dfrac{(1-a)p_K + bp_L \mp \sqrt{((1+a)p_K - bp_L)^2 - 4ap_K^2}}{2p_K p_L} CT \end{cases}$$

7. Conclusions

The Generalized Cobb-Douglas function for two inputs and linear elasticity is determined from the condition that the linear elasticity of production with capital is linear expressed. The problem of determining the factors of production that maximizes output under a given total cost reveals that it has no choice but in terms of price limitations.

8. References

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