Microeconomics

A Rational Production Function

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Abstract: The article deals with a rational production function of two factors with constant scale return. It was determined the compatibility conditions with the axioms of production function resulting inequality of a single variable.

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1. Introduction

In what follows we shall presume there is a certain number of resources, supposedly indivisible needed for the proper functioning of the production process.

We define on $\mathbf{R}^2$ – the **production space** for two resources: K – capital and L - labor as $SP=\{(K,L) \mid K,L\geq 0\}$ where $x\in SP$, $x=(K,L)$ is an **ordered set of resources** and we restrict the production area to a subset $D_p\subset SP$ called **domain of production**.

It is called **production function** an application $Q:D_p\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$.

For an efficient and complex mathematical analysis of a production function, we impose a number of axioms both its definition and its scope.

1. The domain of production is convex;
2. $Q(0,0)=0$ (if it is defined on $(0,0)$);
3. The production function is of class C^2 on D_p that is it admits partial derivatives of order 2 and they are continuous;

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4. The production function is monotonically increasing in each variable;
5. The production function is quasiconcave that is: $Q(\lambda x+(1-\lambda)y)\geq\min(Q(x),Q(y))$ $\forall\lambda\in[0,1]$ $\forall x,y\in R_p$.

From a geometric point of view, a quasiconcave function having the property of being above the lowest value recorded at the end of a certain segment. The property is equivalent to the convexity of the set $Q^{-1}[a,\infty)$ $\forall a\in\mathbf{R}$, where $Q^{-1}[a,\infty)=\{x\in R_p \mid Q(x)\geq a\}$.

2. The Main Indicators of Production Functions

Consider now a production function: $Q:D_p\rightarrow\mathbf{R}_+$, $(K,L)\rightarrow Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$.

We call **marginal productivity** relative to an input x_i: $\eta_{x_i}=\frac{\partial Q}{\partial x_i}$ and represents the trend of variation of production to the variation of x_i.

We call **average productivity** relative to an input x_i: $w_{x_i}=\frac{Q}{x_i}$ the value of production at a consumption of a unit of factor x_i.

We call **partial marginal substitution rate** of factors i and j the opposite change in the amount of factor j as a substitute for the amount of change in the factor i in the case of a constant level of production and we have: $RMS(i,j)=\frac{\eta_{x_i}}{\eta_{x_j}}$.

We call **elasticity of output** with respect to an input x_i: $\varepsilon_{x_i}=\frac{\frac{\partial Q}{\partial x_i}}{\frac{Q}{x_i}}=\frac{\eta_{x_i}}{w_{x_i}}$ and represents the relative variation of production to the relative variation of the factor x_i.

Considering now a production function $Q:D_p\to\mathbf{R}_+$ with constant return to scale that is $Q(K,L)=\frac{1}{\lambda}Q(\lambda K,\lambda L)$, let note $\chi=\frac{K}{L}$. It is called the **elasticity of the marginal rate of technical substitution** $\sigma=\dfrac{\dfrac{\partial \mathrm{RMS}(K,L)}{\partial\chi}}{\dfrac{\mathrm{RMS}(K,L)}{\chi}}$.

3. A Rational Production Function

Consider now a production function $Q:D_p\subset\mathbf{R}^2\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$ with constant return to scale:

$$Q(K,L)=\frac{P(K,L)}{R(K,L)}\quad \forall K,L>0$$

where P and R are homogenous polynomials in K and L, deg P=n, deg R=n-1, n≥2.

Because the function is elementary follows that it is of class C^∞ on the definition domain.

Let note also: $\chi=\frac{K}{L}$.

In what follows we put the question of determining the conditions so that the axioms 4 and 5 to be verified.

We now have:

$$\frac{\partial Q}{\partial K}=\frac{\frac{\partial P}{\partial K}R-P\frac{\partial R}{\partial K}}{R^2},\quad \frac{\partial Q}{\partial L}=\frac{\frac{\partial P}{\partial L}R-P\frac{\partial R}{\partial L}}{R^2}$$

Because of homogeneity, we have:

$$\begin{cases} K\frac{\partial P}{\partial K}+L\frac{\partial P}{\partial L}=nP\\ K\frac{\partial R}{\partial K}+L\frac{\partial R}{\partial L}=(n-1)R\end{cases}$$

that is:

$$\begin{cases} \dfrac{\partial P}{\partial L} = \dfrac{nP - K\dfrac{\partial P}{\partial K}}{L} \\ \dfrac{\partial R}{\partial L} = \dfrac{(n-1)R - K\dfrac{\partial R}{\partial K}}{L} \end{cases}$$

Note now: $\alpha=\dfrac{\partial P}{\partial K}$, $\beta=\dfrac{\partial R}{\partial K}$, $\gamma=\dfrac{\partial^2 P}{\partial K^2}$, $\delta=\dfrac{\partial^2 R}{\partial K^2}$. We have:

$$\begin{cases} \dfrac{\partial P}{\partial L} = \dfrac{nP - \alpha K}{L} \\ \dfrac{\partial R}{\partial L} = \dfrac{(n-1)R - \beta K}{L} \end{cases}$$

$$\frac{\partial^2 P}{\partial L \partial K} = \frac{(n-1)\alpha - \gamma K}{L}$$

$$\frac{\partial^2 P}{\partial L^2} = \frac{n(n-1)P - 2(n-1)\alpha K + \gamma K^2}{L^2}$$

$$\frac{\partial^2 R}{\partial L \partial K} = \frac{(n-2)\beta - \delta K}{L}$$

$$\frac{\partial^2 R}{\partial L^2} = \frac{(n-1)(n-2)R - 2(n-2)\beta K + \delta K^2}{L^2}$$

After many computations, we obtain:

$$\frac{\partial Q}{\partial K} = \frac{\alpha R - \beta P}{R^2}$$

$$\frac{\partial Q}{\partial L} = \frac{K}{L}\left(\frac{Q}{K} - \frac{\partial Q}{\partial K}\right) = \frac{PR - \alpha KR + \beta PK}{LR^2}$$

$$\frac{\partial^2 Q}{\partial K^2} = \frac{(\gamma R - \delta P)R - 2\beta(\alpha R - P\beta)}{R^3}$$

$$\frac{\partial^2 Q}{\partial K \partial L} = K\frac{-\gamma R^2 + 2\alpha\beta R + \delta PR - 2\beta^2 P}{LR^3}$$

$$\frac{\partial^2 Q}{\partial L^2} = K^2 \frac{\gamma R^2 - 2\alpha\beta R - \delta P R + 2\beta^2 P}{L^2 R^3}$$

The conditions that: $\frac{\partial Q}{\partial K}>0$, $\frac{\partial Q}{\partial L}>0$ become:

$$\begin{cases} \frac{\alpha R - \beta P}{R^2} > 0 \\ \frac{PR - \alpha KR + \beta PK}{LR^2} > 0 \end{cases}$$

Considering now the bordered Hessian matrix:

$$H^B(Q) = \begin{pmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K \partial L} & \frac{\partial^2 Q}{\partial L^2} \end{pmatrix}$$

and the minors:

$$\Delta_1^B = \begin{vmatrix} 0 & \frac{\partial Q}{\partial K} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} \end{vmatrix} = -\left(\frac{\partial Q}{\partial K}\right)^2 = -\frac{(\alpha R - \beta P)^2}{R^4}$$

$$\Delta_2^B = \begin{vmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K \partial L} & \frac{\partial^2 Q}{\partial L^2} \end{vmatrix} = 2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K \partial L} - \left(\frac{\partial Q}{\partial L}\right)^2 \frac{\partial^2 Q}{\partial K^2} - \left(\frac{\partial Q}{\partial K}\right)^2 \frac{\partial^2 Q}{\partial L^2} =$$

$$\frac{P^2}{L^2 R^5}\left(-\gamma R^2 + 2\alpha\beta R + \delta P R - 2\beta^2 P\right)$$

it is known that if $\Delta_1^B<0$, $\Delta_2^B>0$ the function is quasiconcave. Conversely, if the function is quasiconcave then: $\Delta_1^B \le 0$, $\Delta_2^B \ge 0$. Therefore, a sufficient condition for the validity of axiom 5 is:

$$\begin{cases} -\dfrac{(\alpha R - \beta P)^2}{R^4} < 0 \\ \dfrac{P^2}{L^2 R^5}\left(-\gamma R^2 + 2\alpha\beta R + \delta P R - 2\beta^2 P\right) > 0 \end{cases}$$

From these two sets of conditions we obtain finally:

$$\begin{cases} \dfrac{\alpha R - \beta P}{R^2} > 0 \\ \dfrac{PR - \alpha K R + \beta P K}{LR^2} > 0 \\ \dfrac{P^2}{L^2 R^5}\left(-\gamma R^2 + 2\alpha\beta R + \delta P R - 2\beta^2 P\right) > 0 \end{cases}$$

or, more simple (taking inot account that Q,K,L>0):

$$\begin{cases} \alpha R - \beta P > 0 \\ PR - K(\alpha R - \beta P) > 0 \\ \dfrac{2\beta(\alpha R - \beta P) + R(\delta P - \gamma R)}{R} > 0 \end{cases}$$

Theorem 1

A function $Q: D_p \subset \mathbf{R}^2 \to \mathbf{R}_+$, $(K,L) \to Q(K,L) \in \mathbf{R}_+$ $\forall (K,L) \in D_p$ with constant return to scale:

$$Q(K,L) = \frac{P(K,L)}{R(K,L)} \quad \forall K,L>0$$

is a production function if:

$$\begin{cases} \alpha R - \beta P > 0 \\ PR - K(\alpha R - \beta P) > 0 \\ \dfrac{2\beta(\alpha R - \beta P) + R(\delta P - \gamma R)}{R} > 0 \end{cases}$$

where: $\alpha = \dfrac{\partial P}{\partial K}$, $\beta = \dfrac{\partial R}{\partial K}$, $\gamma = \dfrac{\partial^2 P}{\partial K^2}$, $\delta = \dfrac{\partial^2 R}{\partial K^2}$.

Because P and R are homogenous, we have:

$$P(K,L)=L^n P\left(\frac{K}{L},1\right)=L^n S(\chi),\ R(K,L)=L^{n-1}R\left(\frac{K}{L},1\right)=L^{n-1}T(\chi)$$

with the obviously notations: $S(\chi)=P(\chi,1)$, $T(\chi)=R(\chi,1)$.

If $P(K,L)=\sum_{i=0}^{n} a_i K^i L^{n-i}$ we have:

$$\alpha=\frac{\partial P}{\partial K}=\sum_{i=1}^{n} i a_i K^{i-1}L^{n-i}=L^{n-1}\sum_{i=1}^{n} i a_i \chi^{i-1}=L^{n-1}S'(\chi)$$

$$\gamma=\frac{\partial^2 P}{\partial K^2}=\sum_{i=2}^{n} i(i-1)a_i K^{i-2}L^{n-i}=L^{n-2}\sum_{i=1}^{n} i(i-1)a_i \chi^{i-1}=L^{n-2}S''(\chi)$$

Analogously: $\beta=\frac{\partial R}{\partial K}=L^{n-2}T'(\chi)$, $\delta=\frac{\partial^2 R}{\partial K^2}=L^{n-3}T''(\chi)$

We obtain therefore that the conditions of the above theorem become:

$$\begin{cases} S'(\chi)T(\chi)-T'(\chi)S(\chi)>0 \\ S(\chi)T(\chi)-\chi(S'(\chi)T(\chi)-T'(\chi)S(\chi))>0 \\ \dfrac{2T'(\chi)(S'(\chi)T(\chi)-T'(\chi)S(\chi))+T(\chi)(T''(\chi)S(\chi)-S''(\chi)T(\chi))}{T(\chi)}>0 \end{cases}$$

If we note for simplify: $V(\chi)=\frac{S(\chi)}{T(\chi)}$ we finally have:

$$\begin{cases} V'(\chi)>0 \\ V(\chi)-\chi V'(\chi)>0 \\ V''(\chi)<0 \end{cases}$$

Because $Q(K,L)=\frac{P(K,L)}{R(K,L)}=\frac{L^n S(\chi)}{L^{n-1}T(\chi)}=LV(\chi)$ we easily see that:

$V(\chi)=\frac{Q(K,L)}{L}=w_L$, $\frac{V(\chi)}{\chi}=\frac{Q(K,L)}{K}=w_K$ therefore:

Theorem 2

A function Q:$D_p\subset\mathbf{R}^2\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$ with constant return to scale:

$$Q(K,L)=\frac{P(K,L)}{R(K,L)}\ \forall K,L>0$$

is a production function if:

$$\begin{cases} w_L'(\chi)>0 \\ w_K'(\chi)<0 \\ w_L''(\chi)<0 \end{cases}$$

where $\chi=\frac{K}{L}$ and w_L, w_K are the average productivity relative to L and K respectively.

Because $P(K,L)=L^nS(\chi)$, $R(K,L)=L^{n-1}T(\chi)$ we find that:

$$\alpha=\frac{\partial P}{\partial K}=L^{n-1}S'(\chi),\ \beta=\frac{\partial R}{\partial K}=L^{n-2}T'(\chi),\ \gamma=\frac{\partial^2 P}{\partial K^2}=L^{n-2}S''(\chi),\ \delta=\frac{\partial^2 R}{\partial K^2}=L^{n-3}T''(\chi)$$

$$\frac{\partial Q}{\partial K}=\frac{\alpha R-\beta P}{R^2}=\frac{S'(\chi)T(\chi)-T'(\chi)S(\chi)}{T(\chi)^2}=V'(\chi)$$

$$\frac{\partial Q}{\partial L}=\frac{PR-\alpha KR+\beta PK}{LR^2}=\frac{S(\chi)T(\chi)-\chi S'(\chi)T(\chi)+\chi T'(\chi)S(\chi)}{T(\chi)^2}=V(\chi)-\chi V'(\chi)$$

The main indicators for this function are:

- the marginal productivity relative to L: $\eta_L=\frac{\partial Q}{\partial L}=V(\chi)-\chi V'(\chi)$
- the marginal productivity relative to K: $\eta_K=\frac{\partial Q}{\partial K}=V'(\chi)$
- the average productivity relative to L: $w_L=\frac{Q}{L}=V(\chi)$
- the average productivity relative to K: $w_K=\frac{Q}{K}=\frac{V(\chi)}{\chi}$

- the partial marginal substitution rate of factors K and L: RMS(K,L)= $\frac{\eta_K}{\eta_L} = \frac{V'(\chi)}{V(\chi)-\chi V'(\chi)}$
- the elasticity of output with respect to K: $\varepsilon_K = \frac{\eta_K}{w_K} = \frac{\chi V'(\chi)}{V(\chi)}$
- the elasticity of output with respect to L: $\varepsilon_L = \frac{\eta_L}{w_L} = \frac{V(\chi)-\chi V'(\chi)}{V(\chi)}$
- the elasticity of the marginal rate of technical substitution $\sigma = \frac{\frac{\partial \mathrm{RMS}(K,L)}{\partial \chi}}{\frac{\mathrm{RMS}(K,L)}{\chi}} = \frac{\chi V(\chi) V''(\chi)}{V'(\chi)(V(\chi)-\chi V'(\chi))}$

4. Example

Let now, the function of production: Q:$D_p \subset \mathbf{R}^2 \to \mathbf{R}_+$, $(K,L) \to Q(K,L) \in \mathbf{R}_+$ $\forall (K,L) \in D_p$ with constant return to scale:

$$Q(K,L) = \frac{4K^2 - 2KL - L^2}{K-L} \quad \forall K,L,\ K < \frac{L}{2}$$

We have: $V(\chi) = \frac{4\chi^2 - 2\chi - 1}{\chi - 1}$, $w_K(\chi) = \frac{V(\chi)}{\chi} = \frac{4\chi^2 - 2\chi - 1}{\chi^2 - \chi}$, $w_L(\chi) = V(\chi) = \frac{4\chi^2 - 2\chi - 1}{\chi - 1}$, $w_K'(\chi) = \frac{-2\chi^2 + 2\chi - 1}{\chi^2(\chi-1)^2}$, $w_L'(\chi) = \frac{4\chi^2 - 8\chi + 3}{(\chi-1)^2}$, $w_L''(\chi) = \frac{2}{(\chi-1)^3}$.

The conditions from the theorem 2 become:

$$\begin{cases} \frac{4\chi^2 - 8\chi + 3}{(\chi-1)^2} > 0 \\ \frac{-2\chi^2 + 2\chi - 1}{\chi^2(\chi-1)^2} < 0 \\ \frac{2}{(\chi-1)^3} < 0 \end{cases}$$

which are equivalent with:

$$\begin{cases} \chi \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{3}{2}, \infty\right) \\ \chi \in \mathbf{R} \\ \chi < 1 \end{cases}$$

that is, with $\chi > 0$: $\chi \in \left(0, \frac{1}{2}\right)$ or $K < \frac{L}{2}$.

The graph of the production function is:

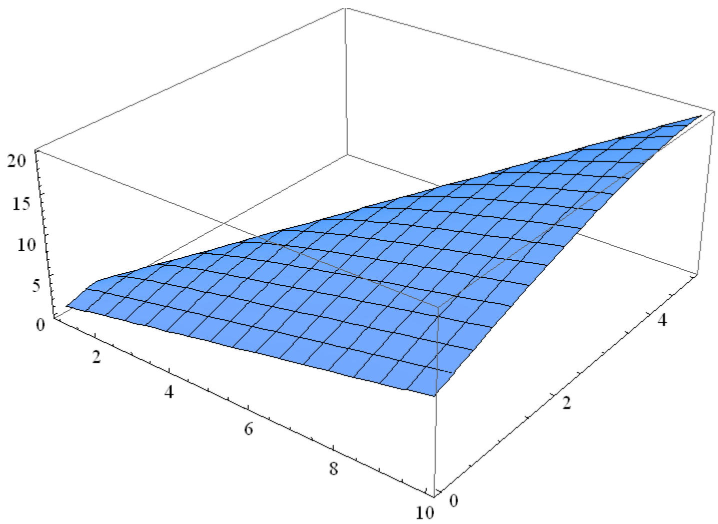

Figure 1

5. Conclusions

Rational production functions may occur in the process of determining specific method of least squares (leading to relatively simple systems solved) based on concrete data. Conditions compatibility axioms production function were simplified by using the factor $\chi = \frac{K}{L}$, generating inequalities of a single variable.

6. References

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