Microeconomics

A Type of a Rational Production Function

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Abstract: The article deals with a particular rational production function of two factors with constant scale return. It were determined from the compatibility conditions with the axioms of production function all the cases for a such function.

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1 Introduction

In what follows we shall presume there is a certain number of resources, supposedly indivisible needed for the proper functioning of the production process.

We define on $\mathbf{R}^2$ – the **production space** for two resources: K – capital and L - labor as $SP=\{(K,L) \mid K,L\geq 0\}$ where $x\in SP$, $x=(K,L)$ is an **ordered set of resources** and we restrict the production area to a subset $D_p\subset SP$ called **domain of production**.

It is called **production function** an application $Q:D_p\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$.

For an efficient and complex mathematical analysis of a production function, we impose a number of axioms both its definition and its scope.

1. The domain of production is convex;
2. $Q(0,0)=0$ (if it is defined on $(0,0)$);

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3. The production function is of class C^2 on D_p that is it admits partial derivatives of order 2 and they are continuous;
4. The production function is monotonically increasing in each variable;
5. The production function is quasiconcave that is: $Q(\lambda x+(1-\lambda)y)\geq\min(Q(x),Q(y))$ $\forall\lambda\in[0,1]$ $\forall x,y\in R_p$.

In a preceding paper ([5]), one of the authors define a rational production function with constant return to scale as:

$Q: D_p\subset\mathbf{R}^2\to\mathbf{R}_+$, $(K,L)\to Q(K,L)\in\mathbf{R}_+$ $\forall(K,L)\in D_p$

$$Q(K,L)=\frac{P(K,L)}{R(K,L)}\ \forall K,L>0$$

where P and R are homogenous polynomials in K and L, deg P=n, deg R=n-1, n≥2.

The compatibility conditions for that function to be of production were (from theorem 2):

$$\begin{cases} w_L'(\chi)>0 \\ w_K'(\chi)<0 \\ w_L''(\chi)<0 \end{cases}$$

where $\chi=\frac{K}{L}$ and w_L, w_K are the average productivity relative to L and K respectively.

2 A Type of a Rational Production Function

Let now:

$$Q(K,L)=\frac{aK^2+bKL+cL^2}{dK+eL},\ a,d\neq 0$$

We shall suppose that d=1 with loss of generality, after a simplification of the ratio with d.

Therefore, let: $Q(K,L)=\frac{aK^2+bKL+cL^2}{K+eL}=L\frac{a\chi^2+b\chi+c}{\chi+e}$.

The average productivity relative to K and L are:

$$w_K(\chi) = \frac{Q(K,L)}{K} = \frac{a\chi^2 + b\chi + c}{\chi(\chi + e)},\ w_L(\chi) = \frac{Q(K,L)}{L} = \frac{a\chi^2 + b\chi + c}{\chi + e}$$

The first and the second derivatives are:

$$w_K'(\chi) = \frac{(ae - b)\chi^2 - 2c\chi - ce}{\chi^2(\chi + e)^2}$$

$$w_L'(\chi) = \frac{a\chi^2 + 2ae\chi + be - c}{(\chi + e)^2}$$

$$w_L''(\chi) = 2\frac{ae^2 - be + c}{(\chi + e)^3}$$

The compatibility conditions become:

$$\begin{cases} \dfrac{a\chi^2 + 2ae\chi + be - c}{(\chi + e)^2} > 0 \\ \dfrac{(ae - b)\chi^2 - 2c\chi - ce}{\chi^2(\chi + e)^2} < 0 \\ 2\dfrac{ae^2 - be + c}{(\chi + e)^3} < 0 \end{cases}$$

or after simplifying:

$$\begin{cases} a\chi^2 + 2ae\chi + be - c > 0 \\ (ae - b)\chi^2 - 2c\chi - ce < 0 \\ (ae^2 - be + c)(\chi + e) < 0 \end{cases}$$

Let now the transformation: $\phi = \chi + e$, therefore $\phi > e$ and also: g= $ae^2 - be + c$. The conditions become:

$$\begin{cases} a\phi^2 - g > 0 \\ (ae - b)\phi^2 - 2g\phi + ge < 0 \\ g\phi < 0 \end{cases}$$

Case 1: g>0

From the third inequality, we have that $\phi < 0$. From the first: $a\phi^2 > g > 0$ therefore a>0. Also: $\phi \in \left(-\infty, -\sqrt{\frac{g}{a}}\right) \cup \left(\sqrt{\frac{g}{a}}, \infty\right)$ and because $\phi < 0$ we get: $\phi \in \left(-\infty, -\sqrt{\frac{g}{a}}\right)$ or $0 < \chi < -e - \sqrt{\frac{g}{a}}$.

But $0 < -e - \sqrt{\frac{g}{a}}$ is equivalent with: $\sqrt{\frac{g}{a}} < -e$ therefore e<0 and $be > c$.

Analysing the inequality: $f(\phi) = (ae - b)\phi^2 - 2g\phi + ge < 0$ we have first $\Delta = cg$.

If c<0 then Δ<0 therefore $\operatorname{sgn}(f(\phi)) = \operatorname{sgn}(ae - b) = \operatorname{sgn}\left(\frac{g - c}{e}\right)$=-1. In consequence:

$(ae - b)\phi^2 - 2g\phi + ge < 0 \ \forall \phi \in \mathbf{R}$.

If c>0 then Δ>0. The roots of the equation $(ae - b)\phi^2 - 2g\phi + ge = 0$ are: $\phi_{1,2} = \frac{g \pm \sqrt{cg}}{ae - b}$ therefore if $ae > b$ then: $\phi \in \left(\frac{g - \sqrt{cg}}{ae - b}, \frac{g + \sqrt{cg}}{ae - b}\right)$ which is possible because $\phi < 0$ if $\frac{g - \sqrt{cg}}{ae - b} < 0$ that is: $e(ae - b) < 0$ which is true. In this case: $\phi \in \left(\frac{g - \sqrt{cg}}{ae - b}, 0\right)$ or $\chi \in \left(\frac{c - \sqrt{cg}}{ae - b}, -e\right)$.

If now $ae < b$ then: $\phi \in \left(-\infty, \frac{g + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{g - \sqrt{cg}}{ae - b}, \infty\right)$. Because, in this case: $g > \sqrt{cg}$ we finally find that: $\phi \in \left(-\infty, \frac{g + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{g - \sqrt{cg}}{ae - b}, 0\right)$ or $\chi \in \left(\frac{c - \sqrt{cg}}{ae - b}, -e\right)$.

If ae=b then: $\phi > \frac{e}{2}$ which is true because e<0. In this case: $0 < \chi < -e - \sqrt{\frac{g}{a}}$.

Case 2: g<0

From the third inequality, we have that $\phi > 0$. From the first: $a\phi^2 > g$.

If now a>0 the inequality holds for all $\phi \in \mathbf{R}$. From the relation: $(ae - b)\phi^2 - 2g\phi + ge < 0$ we have $\Delta = cg$.

If c>0 we obtain Δ<0 therefore if $ae < b$ then the inequality holds for all $\phi \in \mathbf{R}$. If $ae > b$ then: $\phi \in \varnothing$.

If ae=b then: g=c>0 – contradiction.

If now c<0 we have: Δ>0 therefore, if $ae > b$: $\phi \in \left(\frac{g - \sqrt{cg}}{ae - b}, \frac{g + \sqrt{cg}}{ae - b}\right)$ or $\chi \in \left(\frac{c - \sqrt{cg}}{ae - b}, \frac{c + \sqrt{cg}}{ae - b}\right)$

If e≥0 then χ <0 therefore $\chi \in \varnothing$. If e<0 then $\chi \in \left(0, \frac{c + \sqrt{cg}}{ae - b}\right)$.

If now $ae < b$: $\phi \in \left(-\infty, \frac{g + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{g - \sqrt{cg}}{ae - b}, 0\right)$ that is: $\chi \in \left(-\infty, \frac{c + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{c - \sqrt{cg}}{ae - b}, -e\right)$.

If e≥0 then χ <0 therefore $\chi \in \varnothing$. If e<0 then: $\chi \in \left(0, \frac{c + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{c - \sqrt{cg}}{ae - b}, -e\right)$.

If c=0 then Δ=0 and we must have $ae < b$ and $\phi \neq \frac{g}{ae - b}$ or $\chi \neq 0$.

Let suppose now that a<0. In this case, from the first: $a\phi^2 > g$ therefore $\phi \in \left(-\infty, -\sqrt{\frac{g}{a}}\right) \cup \left(\sqrt{\frac{g}{a}}, \infty\right)$ and because $\phi > 0$ we get: $\phi \in \left(\sqrt{\frac{g}{a}}, \infty\right)$ or $\chi \in \left(-e + \sqrt{\frac{g}{a}}, \infty\right)$.

For the inequality $(ae - b)\phi^2 - 2g\phi + ge < 0$ we have $\Delta = cg$.

If c>0 we obtain Δ<0 therefore if $ae < b$ then the inequality holds for all $\phi \in \mathbf{R}$. If $ae > b$ then: $\phi \in \varnothing$.

If ae=b then: g=c>0 – contradiction.

If now c<0 we have: Δ>0 therefore, if $ae > b$: $\phi \in \left(\frac{g - \sqrt{cg}}{ae - b}, \frac{g + \sqrt{cg}}{ae - b}\right)$ or $\chi \in \left(\frac{c - \sqrt{cg}}{ae - b}, \frac{c + \sqrt{cg}}{ae - b}\right)$

If e≥0 then $\chi < 0$ therefore $\chi \in \varnothing$. If e<0 then $\chi \in \left(0, \frac{c + \sqrt{cg}}{ae - b}\right)$.

If now $ae < b$: $\phi \in \left(-\infty, \frac{g + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{g - \sqrt{cg}}{ae - b}, 0\right)$ that is: $\chi \in \left(-\infty, \frac{c + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{c - \sqrt{cg}}{ae - b}, -e\right)$.

If e≥0 then $\chi < 0$ therefore $\chi \in \varnothing$. If e<0 then: $\chi \in \left(0, \frac{c + \sqrt{cg}}{ae - b}\right) \cup \left(\frac{c - \sqrt{cg}}{ae - b}, -e\right)$.

If c=0 then Δ=0 and we must have $ae < b$ and $\phi \neq \frac{g}{ae - b}$ or $\chi \neq 0$.

Finally we have the following cases for all combinations of parameters:

1. $ae^2 - be + c > 0$, a>0, e<0, c<0, $be > c$, $0 < \chi < -e - \sqrt{\frac{g}{a}}$

2. $ae^2 - be + c > 0$, a>0, e<0, c>0, $be > c$, $ae > b$, $\frac{c-\sqrt{cg}}{ae-b} < \chi < -e$
3. $ae^2 - be + c > 0$, a>0, e<0, c>0, $be > c$, $ae < b$, $\frac{c-\sqrt{cg}}{ae-b} < \chi < -e$
4. $ae^2 - be + c > 0$, a>0, e<0, c>0, $be > c$, $ae = b$, $0 < \chi < -e - \sqrt{\frac{g}{a}}$
5. $ae^2 - be + c < 0$, a>0, c>0, $ae < b$, $\chi > 0$
6. $ae^2 - be + c < 0$, a>0, c<0, e<0, $ae > b$, $0 < \chi < \frac{c+\sqrt{cg}}{ae-b}$
7. $ae^2 - be + c < 0$, a>0, c<0, e<0, $ae < b$, $\chi \in \left(0, \frac{c+\sqrt{cg}}{ae-b}\right) \cup \left(\frac{c-\sqrt{cg}}{ae-b}, -e\right)$
8. $ae^2 - be < 0$, a>0, c=0, e>0, $ae < b$, $\chi > 0$
9. $ae^2 - be + c < 0$, a<0, c>0, $ae < b$, $be < c$ $\chi > -e + \sqrt{\frac{g}{a}}$
10. $ae^2 - be + c < 0$, a<0, c>0, $ae < b$, $be \geq c$ $\chi > 0$
11. $ae^2 - be + c < 0$, a<0, c<0, e<0, $ae > b$, $0 < \chi < \frac{c+\sqrt{cg}}{ae-b}$
12. $ae^2 - be + c < 0$, a<0, c<0, e<0, $ae < b$, $\chi \in \left(0, \frac{c+\sqrt{cg}}{ae-b}\right) \cup \left(\frac{c-\sqrt{cg}}{ae-b}, -e\right)$
13. $ae^2 - be < 0$, a<0, c=0, e<0, $ae < b$, $\chi > 0$

3 The Main Indicators of the Production Function

Considering now a production function: $Q(K,L) = \frac{aK^2 + bKL + cL^2}{K + eL}$ we have:

- the marginal productivity relative to K: $\eta_K = \frac{\partial Q}{\partial K} = \frac{aK^2 + 2aeKL + (be-c)L^2}{(K+eL)^2}$
- the marginal productivity relative to L: $\eta_L = \frac{\partial Q}{\partial L} = \frac{(b-ae)K^2 + 2cKL + ceL^2}{(K+eL)^2}$
- the average productivity relative to K: $w_K = \frac{Q}{K} = \frac{aK^2 + bKL + cL^2}{K(K+eL)}$

- the average productivity relative to L: $w_L = \frac{Q}{L} = \frac{aK^2 + bKL + cL^2}{L(K + eL)}$
- the partial marginal substitution rate of factors K and L: RMS(K,L)= $\frac{\eta_K}{\eta_L} =$

$$\frac{aK^2 + 2aeKL + (be - c)L^2}{(b - ae)K^2 + 2cKL + ceL^2}$$

- the elasticity of output with respect to K:

$$\varepsilon_K = \frac{\eta_K}{w_K} = \frac{K\left(aK^2 + 2aeKL + (be - c)L^2\right)}{(K + eL)\left(aK^2 + bKL + cL^2\right)}$$

- the elasticity of output with respect to L:

$$\varepsilon_L = \frac{\eta_L}{w_L} = \frac{K\left((b - ae)K^2 + 2cKL + ceL^2\right)}{(K + eL)\left(aK^2 + bKL + cL^2\right)}$$

- the elasticity of the marginal rate of technical substitution

$$\sigma = \frac{\frac{\partial RMS(K,L)}{\partial \chi}}{\frac{RMS(K,L)}{\chi}} = \frac{2\left(ae^2 - be + c\right)\chi\left(a\chi^2 + b\chi + c\right)}{\left(a\chi^2 + 2ae\chi + be - c\right)\left((b - ae)\chi^2 + 2c\chi + ce\right)}$$

4 Example for the Case 1

$ae^2 - be + c > 0$, a>0, e<0, c<0, $be > c$, $0 < \chi < -e - \sqrt{\frac{g}{a}}$

The graph for a=2, b=0, c=-1, e=-1 is:

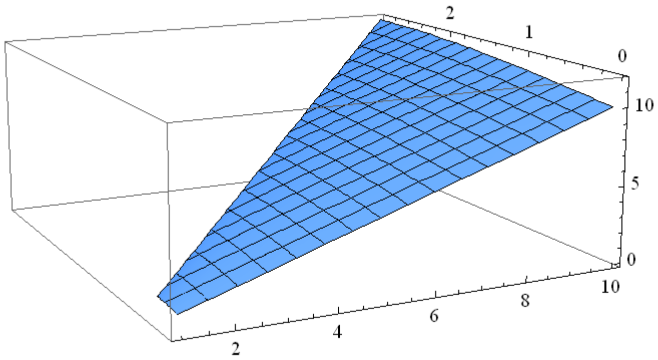

Figure 1

5 Conclusions

Rational production functions may occur in the process of determining specific method of least squares (leading to relatively simple systems solved) based on concrete data. Conditions compatibility axioms production function were analyzed and obtaining 13 cases for a ratio of polinomyals of degree 2 and 1 respectively.

6 References

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