Mathematical and Quantative Methods

A Short History of the Theory of Numbers

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Abstract: The article treats some aspects of the history of the theory of algebraic numbers. The number theory is one of the oldest branches of mathematics which has its origins from second millennium BC, ancient documents dating from about 2000 BC being the Rhind papyrus and Golenischev papyrus, both of Egypt. Number theory is characterized by the simplicity of its fundamentals, its rigor and purity notions of its truths. One branch of number theory is the algebraic theory of numbers which, after Hilbert, is a wonderful monument of beauty and incomparable harmony.

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1. Introduction

The number theory is one of the oldest branches of mathematics which has its origins from second millennium BC, ancient documents dating from about 2000 BC being the Rhind papyrus and Golenischev papyrus, both of Egypt.

Euclid of Alexandria (ca. 300 BC) and Diofant of Alexandria (about 250 AD) are the best known theorists numbers in old age. However, this is considered as an independent and systematic science only in modern times.

One of the greatest merits in modern times has played one who considers the higher arithmetical as having "*those magical charm that makes her first favorite of mathematicians and scientists stemming from the inexhaustible riches that surpass all other parts of mathematics*" namely Gauss with his "Disquistiones Arithmeticae" where he created the modern theory of numbers in the true sense of the word. Amazement is becoming greater if we consider that he created a whole world of ideas without the help of any external stimulus, but due to the inspiration that he has proof.

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Number theory is characterized by the simplicity of its fundamentals, its rigor and purity notions of its truths. One branch of number theory is the algebraic theory of numbers which, after Hilbert, is a wonderful monument of beauty and incomparable harmony.

Deep insights that it provides the work of mathematicians who have dealt with this theory: Kummer, Kronecker, Dedekind, last two based on the ideas of the first, deepening and developing the manner of thinking of Kummer (who remained in history as the pioneer of the theory of ideals) shows that in this field of science a lot of the most beautiful treasures remain hidden.

Seductive as a nice reward, they should tempt the researcher which knows their unmatched value and practice with love the art to discover.

Some of the main objectives of number theory, mainly the Great Fermat's theorem, the laws of reciprocity or the binary quadratic forms theory had a strong instrumental character in the emergence of the theory of algebraic numbers.

Although the main problems were exposed in terms of rational integers, as time passed, it turned out that a better utility would the consideration of a richer sets of numbers, namely that of algebraic integers.

It was imperative to reconsider these objectives through the new numbers entered.

The study of the unique factorization in such rings became the basic problem of this branch of mathematics. Kummer has made it through the ideals of numbers, also Dedekind, as Kronecker help himself very much by the theory of divisors.

One of the greatest achievements of Kummer was that he proved that in the ring of cyclotomic integers any ideal is a unique product of prime ideal. His ideas were brilliant, but difficult and not clearly formulated.

The fundamental concept of ideal of numbers and prime ideal were not defined intrinsically because the Kummer's decomposition theory applies only to cyclotomic integers.

This was easily removed independently and in two different ways by Dedekind and Kronecker. Dedekind's work was the culmination of 17 years of investigation of the problems inherent in the unique factorization. He once created a new topic: the theory of algebraic numbers. He introduced, on his own way, some of the fundamental concepts of commutative algebra that is: ring ideal and prime ideal.

One of the significant innovations that followed Euclid period was the axiomatic method that has surfaced after 2000 years of stupor. Dedekind was instrumental in showing the power of mathematics and its pedagogical value. He inspired among others: David Hilbert and Emmy Noether.

2. Historical Highlights

There are a few main stages of development of algebraic number theory.

Between 1840 and 1846, Dirichlet published a series of notes relative to the ring $\mathbf{Z}[\theta]$ where θ is a complex number that verifies an equation of the form $x^n+a_1x^{n-1}+...+a_n=0$ where $a_i\in\mathbf{Z}$. In particular, he studied invertible elements (units) in the ring.

Starting in 1847, Kummer put the question to prove the Fermat's theorem and to extend the reciprocity theorems and was led to study in detail $\mathbf{Z}[\omega]$, $\omega^p=1$, $\omega\neq1$. To have a usable theory, Kummer introduced his concept of the ideal numbers and ideal class that will have an immense importance.

To refine the theory of $\mathbf{Z}[\omega]$, nothing important was made up to 1871, when Dedekind in an appendix to a number theory course of Dirichlet put the foundation of the modern theory of numbers. The 4th edition of the book, published in 1893, contains definitive exposure of Dedekind's theory.

In 1882, Kronecker published its statement about the fundamentals of the theory of algebraic numbers. Unfortunately his work was written in a very strange manner and had nor the impact of Dedekind. However the memoir contains ideas and suggestions that have greatly influenced the algebraic geometry and the germ of what was to happen with the theory of field of classes. Thanks to the efforts of Kronecker, Weber and Hilbert, the main directions of what would be the general theory of body classes were made between 1882 and 1905.

Activity in the study of algebraic numbers has pitched in 1896 when Hilbert asked the German Mathematical Society to publish a masterly treatise of the era. Treaty "Zahlbericht" was intended to be a tool and to stimulate further research. Certainly did.

Before primes development, Weber started in 1896 to apply analytical techniques in the study of distributions in ideal classes. The importance of Weber's work is the fact that to prove theorems, was led to postulate the existence of fields of algebraic numbers with some very special properties. These fields called fields of classes were the main object of study for the next 50 years.

After Weber's introduction of the concept of field of classes, Hilbert realized that it was necessary to study laws of reciprocity, quadratic forms and ideal factorization into prime ideals. Hilbert not demonstrated the existence of field of classes unless in some special circumstances which formed the basis of subsequent developments.

During research on the distribution of prime ideals, an important role in the theory of field of classes, plays Frobenius automorphism.

Near the end of the 19th century, Hensel began to introduce its p-adic numbers. First, they were used at a formal level until 1913 when the situation will change. Simple topological notions were introduced in p-adic numbers fields. These ideas stay at the base of a very convenient and suggestive language which would win the number theory.

In the decade 1920-1930, the main results of the theory of field of classes, conjectured by Weber and Hilbert, was proved by Takagi, Artin and Hasse.

The complex analysis was banished from the theory of field of classes by Chevalley in 1940, which gave complete algebraic demonstrations of all the main theorem, based on the theory of p-adic numbers.

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