About Andrica's Conjecture

Cătălin Angelo Ioan[1], Alin Cristian Ioan[2]

Abstract: The paper establishes an equivalence of the Andrica's conjecture in the direction of an increase of the difference of square root of primes by a power of a ratio of two consecutive primes.

Keywords: Andrica's conjecture; prime

JEL Classification: E23

1. Introduction

Even prime number theory dates back to ancient times (see the Rhind papyrus or Euclid's Elements) it retains its topicality and fascination to any mathematician due to numerous issues remain unresolved to this day.

A number $p \in \mathbf{N}$, $p \geq 2$ is called prime if its only positive divisors are 1 and p. The remarkable property of primes is that any nonzero natural number other than 1 can be written as a unique product (up to a permutation of factors) of prime numbers to various powers.

If there is not a formula, for the moment, generating prime numbers, there exist a lot of attempts (many successful, in fact) to determine some of their properties.

Unfortunately, many results are at the stage of conjectures (theorems that seem to be valid, but remained unproven yet).

A famous conjecture relative to prime numbers is that of Dorin Andrica. Denoting by p_n - the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$ etc.), Andrica's conjecture ([1]) states that:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \quad \forall n \geq 1$$

[1] Associate Professor, PhD, Danubius University of Galati, Faculty of Economic Sciences, Romania, Address: 3 Galati Blvd, Galati, Romania, Tel.: +40372 361 102, Fax: +40372 361 290, Corresponding author: catalin_angelo_ioan@univ-danubius.ro.

[2] Nicolae Oncescu College, Braila, Address: 1-3 ȘOS. Brăilei, City: Ianca, Brăila County, Tel.: +40239-668 494, E-mail: alincristianioan@yahoo.com.

Noting $g_n=p_{n+1}-p_n$ – the prime gap that is the difference between two successive prime numbers, the conjecture can be written in the form:

$$g_n < 2\sqrt{p_n} + 1 \ \forall n \geq 1$$

The equivalence is immediate because

$g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1 \Leftrightarrow p_{n+1} < \left(\sqrt{p_n} + 1\right)^2$ which is obviously equivalent to: $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$.

Even if Andrica's conjecture is weaker than that resulting from Oppermann's conjecture which states that $g_n < \sqrt{p_n}$ the attempts to prove have not been successful to this moment.

In the following, we shall prove a theorem of equivalence of Andrica's conjecture with another conjecture of increasing the difference of square root of consecutive primes with a power of their ratio.

2. Main Theorem

Theorem

Let p_n the n-th prime number. The following statements are equivalent for $n\geq 5$:

1. $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$;
2. $\exists \alpha \geq 0$ such that: $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{p_n}{p_{n+1}}\right)^{\alpha}$.

Proof

<u>2⇒1</u> Because $\frac{p_n}{p_{n+1}}$ <1 follows that: $\left(\frac{p_n}{p_{n+1}}\right)^{\alpha} < \left(\frac{p_n}{p_{n+1}}\right)^{0}$ =1.

<u>1⇒2</u> If we take the logarithm in the relationship, it becomes:

$$\ln(\sqrt{p_{n+1}} - \sqrt{p_n}) < \alpha \ln\left(\frac{p_n}{p_{n+1}}\right) \Leftrightarrow \ln(\sqrt{p_{n+1}} - \sqrt{p_n}) < \alpha\left(\ln p_n - \ln p_{n+1}\right) \Leftrightarrow$$

$$\frac{\ln(\sqrt{p_{n+1}} - \sqrt{p_n})}{\ln p_{n+1} - \ln p_n} < -\alpha \Leftrightarrow$$

$\frac{\ln(\sqrt{p_{n+1}} - \sqrt{p_n})}{\ln\sqrt{p_{n+1}} - \ln\sqrt{p_n}} < -2\alpha$.

Let now the function:

$$f:(a,\infty)\to\mathbf{R},\ f(x)=\frac{\ln(x-a)}{\ln x - \ln a} \text{ with } a>2$$

We have now: $f'(x)=\frac{x(\ln x - \ln a) - (x-a)\ln(x-a)}{x(x-a)(\ln x - \ln a)^2}$.

Noting $g(x)=x(\ln x - \ln a) - (x-a)\ln(x-a)$ we get: $g'(x)=\ln x - \ln a - \ln(x-a)=h(x)$.

Because $h'(x)=-\frac{a}{x(x-a)}<0$ we have that h is strictly decreasing. How $h(a+1)=\ln(a+1)-\ln a>0$, $h(a+2)=\ln\frac{a+2}{2a}<0$, follows that $\exists\xi\in(a+1,a+2)$ such that: $h(\xi)=0$ that is:

(1) $\ln\xi - \ln a = \ln(\xi - a)$

and after the monotony: $h(x)>0\ \forall x<\xi$ and $h(x)<0\ \forall x>\xi$.

Because $g'=h$ we obtain that g is strictly increasing on (a,ξ) and strictly decreasing on (ξ,∞).

But $\lim_{x\to a} g(x)=0$, $\lim_{x\to\infty} g(x)=-\infty$, $g(\xi)=\xi(\ln\xi - \ln a) - (\xi - a)\ln(\xi - a)$ and from (1):

$$g(\xi)=a\ln(\xi - a)>0$$

It results that $\exists\eta>\xi>a+1$ such that: $g(\eta)=0$ that is:

(2) $\eta(\ln\eta - \ln a) = (\eta - a)\ln(\eta - a)$

From monotonicity, we obtain that: $g(x)>0\ \forall x\in(a,\eta)$ and $g(x)<0\ \forall x\in(\eta,\infty)$.

Therefore, f is strictly increasing on (a,η) and strictly decreasing on (η,∞).

As $f(a+1)=0$, $f(\eta)=\frac{\ln(\eta - a)}{\ln\eta - \ln a}$, and from (2): $f(\eta)=\frac{\eta}{\eta - a}>0$, $\lim_{x\to\infty} f(x)=1$ it follows that: $f(x)<0\ \forall x\in(a,a+1)$.

From hypothesis 1 (Andrica's conjecture), we have: $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ and noting: $x = \sqrt{p_{n+1}}$, $a = \sqrt{p_n}$ we have: $x \in (a, a+1)$.

Be so: $\alpha = -\frac{1}{2} \sup_n \frac{\ln(\sqrt{p_{n+1}} - \sqrt{p_n})}{\ln\sqrt{p_{n+1}} - \ln\sqrt{p_n}} \geq 0$. The statement 2 is now obvious. Q.E.D.

3. Determination of the Constant α

Using the Wolfram Mathematica software, in order to determine the constant α (for the first 100000 prime numbers):

```
alfa:=1000.
For[n=5,n<100000,n++,alfa1=Log[Prime[n]/Prime[n+1],Sqrt[Prime[n+1]]-
Sqrt[Prime[n]]];alfa=Min[alfa1,alfa]];
Print[N[alfa,1000]]
```

we found that the first 1000 decimals are:

α=2.281103221027218229423232822443286599017584100009984588220198770
7555016956079556125523902095828355521972189294678153434257651743655
1726362923880678160746758935639657414508211919857643587356785905060
6949765734239636479131860224037385179731071495942268916184068058269
6177512720818817490245596293254286947503092584044428951753449292269
7389861883902366389124093450412998636957454073445598539595694269058
9958313834222509660041292032936969131875482708936809950924268932852
6997409051212237021151286038659114545358509297427153361178689719192
7351545951228711831887776623070206318211691345478696428460823105785
4097720248263859745150386851610652371395957541613534650131084885714
3384952309151433452360237879606095184289487480462134479859039214689
8208626417098165357501169756266653666858624450374358872935039206880
8501947089500551068277858220736778549760090107511452195156335525727
1942429070573882246361510791556298086569047772964502980196653160300
3464271617005795731370452461062187687938362733212031545405755221 9

4. References

Andrica, D. (1986). Note on a conjecture in prime number theory. *Studia Univ. Babes-Bolyai Math*. 31 (4), pp. 44–48.

Guy, R.K. (1994). *Unsolved Problems in Number Theory*. Second Edition. New York: Springer Verlag.

Wolf, M. (2010). *A Note on the Andrica Conjecture*. arXiv:1010.3945v1 [math.NT] 19 Oct.