A study of Integers Using Software Tools – I

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Abstract. The paper reviews the Sophie German primes and then it introduces two new types related to them. Using software there will be determined the first sequences of such numbers.

Keywords: prime; Sophie Germain

1 Introduction

The prime number theory dates back to ancient times (see the Rhind papyrus or Euclid's Elements).

A number $p\in\mathbf{N}$, $p\geq 2$ is called prime if its only positive divisors are 1 and p. The remarkable property of primes is that any nonzero natural number other than 1 can be written as a unique product (up to a permutation of factors) of prime numbers to various powers.

All over in this paper, the software presented was written in Wolfram Mathematica 9.0.

2 Sophie Germain Primes

A prime number p is called a Sophie Germain prime if 2p+1 is also prime. The number 2p+1 is called safe prime.

A prime number $p\geq 5$ is of the form 6k-1 or 6k+1, $k\geq 1$. If p=6k-1 then 2p+1=12k-1 and if p=6k+1 then 2p+1=12k+3=3(4k+1) which is not prime. Therefore, Sophie Germain primes numbers $p\geq 5$ are necessarily of the form p=6k-1.

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Let note that p=2 and p=3 are Sophie Germain primes.

The software for determining the Sophie Germain primes and the related safe primes is:

```
Clear[“Global`*”];
(*Sophie Germain primes*)
limit=100;
For[k=1,k<limit,k++,If[PrimeQ[6*k-1]&& PrimeQ[12*k-1],Print[6*k-1,”
“,12*k-1]]]
```

and we find (with the limit 599 for Sophie Germain primes):

5,11	53,107	173,347	251,503	431,863
11,23	83,167	179,359	281,563	443,887
23,47	89,179	191,383	293,587	491,983
29,59	113,227	233,467	359,719	509,1019
41,83	131,263	239,479	419,839	593,1187

3 A New Type of Numbers: α,β-primes

Based on the idea of Sophie Germain prime numbers we define a number p to be α,β-prime if:
$p, \alpha p+1, \alpha^2 p+1, \ldots, \alpha^\beta p+1$ are all primes.

Obvious if $p\neq 2$, α must be an even number, otherwise $\alpha^s p+1=\text{even}\ \forall s=\overline{1,\beta}$.

If $p\geq 5$, $p=6k-1$ that is $p\equiv 5 \pmod 6$ then $\alpha p+1\equiv 5\alpha+1 \pmod 6$, $\alpha^2 p+1\equiv 5\alpha^2+1 \pmod 6$,..., $\alpha^\beta p+1\equiv 5\alpha^\beta+1 \pmod 6$.

If $\alpha=2\gamma$ we have therefore: $\alpha p+1\equiv 4\gamma+1 \pmod 6$, $\alpha^2 p+1\equiv 2\gamma^2+1 \pmod 6$,..., $\alpha^\beta p+1\equiv 5\cdot 2^\beta\cdot\gamma^\beta+1 \pmod 6$.

- If $\alpha\equiv 0 \pmod 6$ we have: $\alpha p+1\equiv 1 \pmod 6$, $\alpha^2 p+1\equiv 1 \pmod 6$, $\alpha^\beta p+1\equiv 1 \pmod 6$.
- If $\alpha\equiv 2 \pmod 6$ we have: $\alpha p+1\equiv 5 \pmod 6$, $\alpha^2 p+1\equiv 3 \pmod 6$ therefore we cannot have $12\gamma+2,\beta$-primes if $\beta>2$ and $p=6k-1$, $k\geq 1$.
- If $\alpha\equiv 4 \pmod 6$ we have: $\alpha p+1\equiv 3 \pmod 6$ therefore we cannot have $12\gamma+4,\beta$-primes if $\beta>1$ and $p=6k-1$, $k\geq 1$.

If now $p \geq 7$, $p=6k+1$ that is $p \equiv 1 \pmod 6$ then $\alpha p+1 \equiv \alpha+1 \pmod 6$, $\alpha^2 p+1 \equiv \alpha^2+1 \pmod 6$,..., $\alpha^\beta p+1 \equiv \alpha^\beta+1 \pmod 6$.

If $\alpha=2\gamma$ we have therefore: $\alpha p+1 \equiv 2\gamma+1 \pmod 6$, $\alpha^2 p+1 \equiv 4\gamma^2+1 \pmod 6$,..., $\alpha^\beta p+1 \equiv 2^\beta \cdot \gamma^\beta+1 \pmod 6$.

- If $\alpha \equiv 0 \pmod 6$ we have: $\alpha p+1 \equiv 1 \pmod 6$, $\alpha^2 p+1 \equiv 1 \pmod 6$,...,$\alpha^\beta p+1 \equiv 1 \pmod 6$.
- If $\alpha \equiv 2 \pmod 6$ we have: $\alpha p+1 \equiv 3 \pmod 6$ therefore we cannot have $12\gamma+2,\beta$-primes if $\beta>1$and $p=6k+1$, $k \geq 1$.
- If $\alpha \equiv 4 \pmod 6$ we have: $\alpha p+1 \equiv 5 \pmod 6$), $\alpha^2 p+1 \equiv 5 \pmod 6$,..., $\alpha^\beta p+1 \equiv 4^\beta+1 \equiv 5 \pmod 6$.

Finally, we have that α,β-primes for $p \geq 5$, $\beta \geq 3$ can exist only in the cases:

- $p \geq 5$, $p=6k-1$, $\alpha=6r$, $r \geq 1$;
- $p \geq 7$, $p=6k+1$, $\alpha=6r$, $r \geq 1$;
- $p \geq 7$, $p=6k+1$, $\alpha=6r+4$, $r \geq 0$.

The software for determining the α,β-primes with β=maximum (and limited to primes less than 100000 and $\alpha \leq 1000$) is:

```
Clear[“Global`*”];
(*α,β-primes*)
limit=100000;
basepower=1000;
number=2;
maximumnumber=0;
maximumtermsformaximumnumber=0;
maximumpower=0;
For[k=1,k≤limit,k++,maximumterms=0;
 For[α=2,α≤basepower,α++,
 valuetrue=PrimeQ[k];
 r=1;
 While[valuetrue,valuetrue=valuetrue&&PrimeQ[α^r*k+1];r++];
 If[maximumterms<r-2,maximumterms=r-2;number=k];
If[maximumtermsformaximumnumber<maximumterms,maximumtermsform
aximumnumber=maximumterms;maximumnumber=number;maximumpowe
r=α];
 ];
 ]
```

```
If[maximumtermsformaximumnumber≠0,Print[“Number=“,maximumnumber];Print[“Base power=“,maximumpower];Print[“Maximum terms=“, maximumtermsformaximumnumber];
For[r=1,r≤ maximumtermsformaximumnumber,r++,
Print[maximumpower^r*maximumnumber+1]]]
```

and we find:

Number=9319

Base power=100

Maximum terms=6

931901

93190001

9319000001

931900000001

93190000000001

9319000000000001

Therefore, the numbers: 9319, $100\times9319+1$, $100^2\times9319+1$, $100^3\times9319+1$, $100^4\times9319+1$, $100^5\times9319+1$, $100^6\times9319+1$ are all primes.

4 p,α-Sophie Germain Sequences

Another type of prime numbers, based also on the idea of Sophie Germain is the following: a sequence will be called a p,α-Sophie Germain sequence if $p_1=p$, $p_2=\alpha p_1+1=\alpha p+1$, $p_3=\alpha p_2+1=\alpha^2 p+\frac{\alpha^2-1}{\alpha-1}$,..., $p_n=\alpha p_{n-1}+1=\alpha^n p+\frac{\alpha^n-1}{\alpha-1}$ are all primes.

Obvious if $p\neq2$, α must be an even number, otherwise $p_2=\alpha p_1+1$=even.

If $p\geq5$, $p=6k-1$ that is $p_1=p\equiv5 \pmod 6$ then $p_2\equiv5\alpha+1\equiv1-\alpha \pmod 6$, $p_3\equiv5\alpha^2+\alpha+1\equiv1+\alpha-\alpha^2 \pmod 6$ etc.

If $\alpha=2\gamma$ then:

- If $\alpha\equiv0 \pmod 6$ we have: $p_k\equiv1 \pmod 6$ $\forall k=\overline{2,n}$.
- If $\alpha\equiv2 \pmod 6$ we have: $p_k\equiv5 \pmod 6$ $\forall k=\overline{2,n}$.

- If $\alpha\equiv4$ (mod 6) we have: $p_2\equiv3$ (mod 6) that is it cannot be prime.

If $p\geq7$, $p=6k+1$ that is $p_1=p\equiv1$ (mod 6) then $p_2\equiv\alpha+1$ (mod 6), $p_3\equiv\alpha^2+\alpha+1$ (mod 6) etc.

If $\alpha=2\gamma$ then:

- If $\alpha\equiv0$ (mod 6) we have: $p_k\equiv1$ (mod 6) $\forall k=\overline{2,n}$.
- If $\alpha\equiv2$ (mod 6) we have: $p_2\equiv3$ (mod 6) that is it cannot be prime.
- If $\alpha\equiv4$ (mod 6) we have: $p_2\equiv5$ (mod 6), $p_3\equiv3$ (mod 6) that is it cannot be prime.

Finally, we have that p,α-primes for $p\geq5$ can exist only in the cases:

- $p\geq5$, $p=6k-1$, $\alpha=6r$, $r\geq1$;
- $p\geq5$, $p=6k-1$, $\alpha=6r+2$, $r\geq1$;
- $p\geq7$, $p=6k+1$, $\alpha=6r$, $r\geq1$.

The software for determining the p,α-Sophie Germain sequence with maximum length (n=maximum) and limited to $p\leq10000$, $\alpha\leq100$ is:

```
Clear[“Global`*”];
(*p,α-Sophie Germain sequences*)
limit=10000;
basepower=100;
number=2;
maximumnumber=0;
maximumtermsformaximumnumber=0;
maximumpower=0;
For[k=1,k≤limit,k++,maximumterms=0;
 For[i=2,i≤basepower,i++,
 valuetrue=PrimeQ[k];
 p=1;
 While[valuetrue,valuetrue=If[p>1,valuetrue&&PrimeQ[i^p*k+(i^p-1)/(i-1)],
valuetrue&&PrimeQ[i*k+1]];p++];
 If[maximumterms<p-2,maximumterms=p-2;number=k];
If[maximumtermsformaximumnumber<maximumterms,maximumtermsform
aximumnumber=maximumterms;maximumnumber=number;maximumpowe
r=i];
 ]
]
If[maximumtermsformaximumnumber≠0,Print[“Number=“,maximumnumbe
r];Print[“Base power=“,maximumpower];Print[“Maximum
```

terms=",maximumtermsformaximumnumber];For[p=1,p≤maximumtermsfor maximumnumber, p++,Print[If[p>1,maximumpower^p*maximumnumber+(maximumpower^p-1)/(maximumpower-1),maximumpower*maximumnumber+1]]]]

We find:

Number=37

Base power=48

Maximum terms=5

1777

85297

4094257

196524337

9433168177

that is: $p_1=37$, $p_2=48\times p_1+1=1777$, $p_3=48\times p_2+1=85297$, $p_4=48\times p_3+1=4094257$,

$p_5=48\times p_4+1=196524337$, $p_6=48\times p_5+1=9433168177$ are all primes.

5 References

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