A Study of Cobb-Douglas Production Function with Differential Geometry

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Abstract: In this paper we shall made an analysis of Cobb-Douglas production function from the differential point of view. We shall obtain some interesting results about the nature of the points of the surface, the total curvature, the conditions when a production function is minimal and finally we give the equations of the geodesics on the surface i.e. the curves of minimal length between two points.

Keywords: production functions; metric; curvature; geodesic; Cobb-Douglas

JEL Classification: E23

1. Introduction

In the theory of production functions, all computations and phenomenons are studied for a constant level of production. In order to detect many aspects of them, a complete analysis can be made only at the entire surface.

We therefore define on $\mathbf{R}^2$ – the **production space** for two resources: K – capital and L - labor as $SP=\{(K,L) \mid K,L\geq 0\}$ where $x\in SP$, $x=(K,L)$ is an **ordered set of resources**. Because in a production process, depending on the nature of applied technology, but also its specificity, not any amount of resources are possible, we restrict the production area to a subset $D_p\subset SP$ called **domain of production**.

It is called a **Cobb-Douglas production function** an application:

$$Q:D_p\to\mathbf{R}_+,\ (K,L)\to Q(K,L)=cK^{\alpha}L^{\beta}\in\mathbf{R}_+\ \forall(K,L)\in D_p,\ \alpha,\beta\in\mathbf{R}^*_+,\ c>0$$

The production function is C^{∞}-differentiable and homogenous of degree $\alpha+\beta$.

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2. The Differential Geometry of Cobb-Douglas Surface

The graph representation of a production function is a surface.

Let note in what follows:

$$(1)\ p=\frac{\partial Q}{\partial L},\ q=\frac{\partial Q}{\partial K},\ r=\frac{\partial^2 Q}{\partial L^2},\ s=\frac{\partial^2 Q}{\partial L\partial K},\ t=\frac{\partial^2 Q}{\partial K^2}$$

We have after simple calculations:

$$(2)\ p=\frac{\beta Q}{L},\ q=\frac{\alpha Q}{K},\ r=\frac{\beta(\beta-1)Q}{L^2},\ s=\frac{\alpha\beta Q}{KL},\ t=\frac{\alpha(\alpha-1)Q}{K^2}$$

The bordered Hessian:

$$(3)\ H_f=\begin{pmatrix} 0 & q & p \\ q & t & s \\ p & s & r \end{pmatrix}=\begin{pmatrix} 0 & \frac{\alpha Q}{K} & \frac{\beta Q}{L} \\ \frac{\alpha Q}{K} & \frac{\alpha(\alpha-1)Q}{K^2} & \frac{\alpha\beta Q}{KL} \\ \frac{\beta Q}{L} & \frac{\alpha\beta Q}{KL} & \frac{\beta(\beta-1)Q}{L^2} \end{pmatrix}$$

therefore, because:

$$(4)\ \Delta^B{}_1=\begin{vmatrix} 0 & \frac{\alpha Q}{K} \\ \frac{\alpha Q}{K} & \frac{\alpha(\alpha-1)Q}{K^2} \end{vmatrix}=-\frac{\alpha^2 Q^2}{K^2}<0,$$

$$\Delta^B{}_2=\begin{vmatrix} 0 & \frac{\alpha Q}{K} & \frac{\beta Q}{L} \\ \frac{\alpha Q}{K} & \frac{\alpha(\alpha-1)Q}{K^2} & \frac{\alpha\beta Q}{KL} \\ \frac{\beta Q}{L} & \frac{\alpha\beta Q}{KL} & \frac{\beta(\beta-1)Q}{L^2} \end{vmatrix}=\frac{\alpha\beta(\alpha+\beta)Q^3}{K^2L^2}>0$$

we obtain that Q is quasiconcave, that is for any $a\in\mathbf{R}$, $Q^{-1}([a,\infty))$ is convex in $\mathbf{R}^2$.

For a constant value of one parameter we obtain a curve on the surface, that is $Q=Q(K,L_0)$ or $Q=Q(K_0,L)$ are both curves on the production surface. They are obtained from the intersection of the plane $L=L_0$ or $K=K_0$ with the surface $Q=Q(K,L)$.

In the study of the surfaces, two quadratic forms are very useful.

The first fundamental quadratic form of the surface is:

(5) $g=g_{11}dL^2+2g_{12}dLdK+g_{22}dK^2$

where: $g_{11}=1+p^2$, $g_{12}=pq$, $g_{22}=1+q^2$.

In our case:

(6) $g_{11}=1+\beta^2c^2K^{2\alpha}L^{2\beta-2}$, $g_{12}=\alpha\beta c^2K^{2\alpha-1}L^{2\beta-1}$, $g_{22}=1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}$

The area element is:

(7) $d\sigma=\sqrt{g_{11}g_{22}-g_{12}^2}\,dKdL=\sqrt{\Delta}\,dKdL=\sqrt{1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}}\,dKdL$

and the surface area A when $(K,L)\in R$ (a region in the plane K-O-L) is $A=\iint_R d\sigma dKdL$.

The second fundamental form of the surface is:

(8) $h=h_{11}dL^2+2\,h_{12}dLdK+h_{22}dK^2$

where: $h_{11}=\frac{r}{\sqrt{1+p^2+q^2}}$, $h_{12}=\frac{s}{\sqrt{1+p^2+q^2}}$, $h_{22}=\frac{t}{\sqrt{1+p^2+q^2}}$.

In our case:

(9) $h_{11}=\frac{\beta(\beta-1)cK^{\alpha}L^{\beta-2}}{\sqrt{1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}}}$,

$h_{12}=\frac{\alpha\beta cK^{\alpha-1}L^{\beta-1}}{\sqrt{1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}}}$,

$h_{22}=\frac{\alpha(\alpha-1)cK^{\alpha-2}L^{\beta}}{\sqrt{1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}}}$

Considering the quantity $\delta=h_{11}h_{22}-h_{12}^2$ we have that:

(10) $$\delta=-\frac{\alpha\beta(\alpha+\beta-1)c^2K^{2\alpha-2}L^{2\beta-2}}{1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}}$$

- If $\delta>0$ in each point of the surface, we will say that it is elliptical. Such surfaces are the hyperboloid with two sheets, the elliptical paraboloid and the ellipsoid.
- If $\delta<0$ in each point of the surface, we will say that it is hyperbolic. Such surfaces are the hyperboloid with one sheet and the hyperbolic paraboloid.

- If δ=0 in each point of the surface, we will say that it is parabolic. Such surfaces are the cone surfaces and the cylinder surfaces.

From (10) we find that:

- $\alpha+\beta<1$: the production surface is elliptical;
- $\alpha+\beta=1$: the production surface is parabolic;
- $\alpha+\beta>1$: the production surface is hyperbolic

The curvature of a curve is, from an elementary point of view, the degree of deviation of the curve relative to a straight line. Considering a surface S and an arbitrary curve through a point P of the surface who has the tangent vector v in P, let the plane π determined by the vector v and the normal N in P at S. The intersection of π with S is a curve C_n named normal section of S. Its curvature is called normal curvature.

If we have a direction m=$\frac{dL}{dK}$ in the tangent plane of the surface in an arbitrary point P we have that the normal curvature is given by:

(11) $$k(m)=\frac{h_{11}m^2+2h_{12}m+h_{22}}{g_{11}m^2+2g_{12}m+g_{22}}$$

The extreme values k_1 and k_2 of the function k(m) are called the principal curvatures of the surface in that point. They satisfy also the equation:

(12) $$(g_{11}g_{22}-g_{12}^2)k^2-(g_{11}h_{22}-2g_{12}h_{12}+g_{22}h_{11})k+(h_{11}h_{22}-h_{12}^2)=0$$

The values of m, who give the extremes, call principal directions in that point.

They also satisfy the equation:

(13) $$(g_{11}s-g_{12}r)m^2+(g_{11}t-g_{22}r)m+(g_{12}t-g_{22}s)=0$$

The curve $\frac{dL}{dK}$=m (where m is one of the principal directions) is called line of curvature on the surface. On such a curve we have the maximum or minimum variation of the value of Q in a neighborhood of P.

The quantity $K=k_1k_2$ is named the total curvature in the considered point and $H=\frac{k_1+k_2}{2}$ is named the mean curvature of the surface in that point.

We have therefore:

(14) $$K=\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}=\frac{\delta}{\Delta}=-\frac{\alpha\beta(\alpha+\beta-1)c^2K^{2\alpha-2}L^{2\beta-2}}{\left(1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}\right)^2}$$

(15)
$$H=\frac{g_{11}h_{22}-2g_{12}h_{12}+g_{22}h_{11}}{g_{11}g_{22}-g_{12}^2}=$$
$$-\frac{cK^{\alpha-2}L^{\beta-2}\left(\alpha(1-\alpha)L^2+\beta(1-\beta)K^2+\alpha\beta(\alpha+\beta)c^2K^{2\alpha}L^{2\beta}\right)}{\left(1+\alpha^2c^2K^{2\alpha-2}L^{2\beta}+\beta^2c^2K^{2\alpha}L^{2\beta-2}\right)^{\frac{3}{2}}}$$

A surface with K=constant call surface with constant total curvature and if H=0 call minimal surface. In our case we can see that K=0 if and only if: $\alpha+\beta=1$.

If we consider now in the tangent plane π at the surface in a point P a direction m, if $h_{11}m^2+2\ h_{12}m+h_{22}=0$ we will say that m is an asymptotic direction, and the equation: $h_{11}\left(\frac{dL}{dK}\right)^2+2h_{12}\frac{dL}{dK}+h_{22}=0$ gives the asymptotic curves of the surface in the point P.

In our case, the asymptotic directions are:

(16) $$m_1=\frac{\alpha\beta+\sqrt{\alpha\beta(\alpha+\beta-1)}}{\beta(1-\beta)}\frac{L}{K},\ m_2=\frac{\alpha\beta-\sqrt{\alpha\beta(\alpha+\beta-1)}}{\beta(1-\beta)}\frac{L}{K}$$

If $\alpha+\beta=1$ then both asymptotic directions are equal.

With notations $x^1=L$, $x^2=K$, let define now the Christoffel symbols of first order:

(17) $$|ij,k|=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)$$

and of second order:

(18) $$\left|\begin{matrix} i \\ jk \end{matrix}\right|=g^{i1}|jk,1|+g^{i2}|jk,2|$$

where $g^{11}=\frac{1}{\Delta}G$, $g^{12}=-\frac{1}{\Delta}F$, $g^{22}=\frac{1}{\Delta}E$ are the components of the inverse matrix of $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$.

We have now:

(19) $|11,1| = \frac{1}{2}\frac{\partial g_{11}}{\partial L}$, $|11,2| = \frac{\partial g_{12}}{\partial L} - \frac{1}{2}\frac{\partial g_{11}}{\partial K}$, $|12,1| = \frac{1}{2}\frac{\partial g_{11}}{\partial K}$, $|12,2| = \frac{1}{2}\frac{\partial g_{22}}{\partial L}$,

$|22,1| = \frac{\partial g_{12}}{\partial K} - \frac{1}{2}\frac{\partial g_{22}}{\partial L}$, $|22,2| = \frac{1}{2}\frac{\partial g_{22}}{\partial K}$

(20) $\begin{vmatrix} 1 \\ 11 \end{vmatrix} = g^{11}|11,1| + g^{12}|11,2| = \frac{1}{\Delta}\left[\frac{1}{2}g_{22}\frac{\partial g_{11}}{\partial L} - g_{12}\left(\frac{\partial g_{12}}{\partial L} - \frac{1}{2}\frac{\partial g_{11}}{\partial K}\right)\right]$,

$\begin{vmatrix} 2 \\ 11 \end{vmatrix} = g^{21}|11,1| + g^{22}|11,2| = \frac{1}{\Delta}\left[-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial L} + g_{11}\left(\frac{\partial g_{12}}{\partial L} - \frac{1}{2}\frac{\partial g_{11}}{\partial K}\right)\right]$,

$\begin{vmatrix} 1 \\ 12 \end{vmatrix} = g^{11}|12,1| + g^{12}|12,2| = \frac{1}{\Delta}\left[\frac{1}{2}g_{22}\frac{\partial g_{11}}{\partial K} - \frac{1}{2}g_{12}\frac{\partial g_{22}}{\partial L}\right]$,

$\begin{vmatrix} 2 \\ 12 \end{vmatrix} = g^{21}|12,1| + g^{22}|12,2| = \frac{1}{\Delta}\left[-\frac{1}{2}g_{12}\frac{\partial g_{11}}{\partial K} + \frac{1}{2}g_{11}\frac{\partial g_{22}}{\partial L}\right]$,

$\begin{vmatrix} 1 \\ 22 \end{vmatrix} = g^{11}|22,1| + g^{12}|22,2| = \frac{1}{\Delta}\left[g_{22}\left(\frac{\partial g_{12}}{\partial K} - \frac{1}{2}\frac{\partial g_{22}}{\partial L}\right) - g_{12}\frac{1}{2}\frac{\partial g_{22}}{\partial K}\right]$,

$\begin{vmatrix} 2 \\ 22 \end{vmatrix} = g^{21}|22,1| + g^{22}|22,2| = \frac{1}{\Delta}\left[-g_{12}\left(\frac{\partial g_{12}}{\partial K} - \frac{1}{2}\frac{\partial g_{22}}{\partial L}\right) + \frac{1}{2}g_{11}\frac{\partial g_{22}}{\partial K}\right]$

From the upper we find that:

(21) $|11,1| = (\beta-1)\beta^2c^2K^{2\alpha}L^{-3+2\beta}$, $|11,2| = \alpha(\beta-1)\beta c^2K^{-1+2\alpha}L^{-2+2\beta}$,

$|12,1| = \alpha\beta^2c^2K^{-1+2\alpha}L^{-2+2\beta}$, $|12,2| = \alpha^2\beta c^2K^{-2+2\alpha}L^{-1+2\beta}$,

$|22,1| = \alpha(\alpha-1)\beta c^2K^{-2+2\alpha}L^{-1+2\beta}$, $|22,2| = \alpha^2(\alpha-1)c^2K^{-3+2\alpha}L^{2\beta}$

(22) $\begin{vmatrix} 1 \\ 11 \end{vmatrix} = \frac{(\beta-1)\beta^2c^2K^{2+2\alpha}}{\beta^2c^2K^{2+2\alpha}L + L^3\left(\alpha^2c^2K^{2\alpha} + K^2L^{-2\beta}\right)}$,

$\begin{vmatrix} 2 \\ 11 \end{vmatrix} = \frac{\alpha(\beta-1)\beta c^2K^{1+2\alpha}L^{2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta} + \alpha^2c^2K^{2\alpha}L^{2+2\beta} + K^2L^2}$,

$$\left|\begin{matrix}1\\12\end{matrix}\right| = \frac{\alpha\beta^2c^2K^{1+2\alpha}L^{2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2},$$

$$\left|\begin{matrix}2\\12\end{matrix}\right| = \frac{\alpha^2\beta c^2K^{2\alpha}L^{1+2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2},$$

$$\left|\begin{matrix}1\\22\end{matrix}\right| = \frac{\alpha(\alpha-1)\beta c^2K^{2\alpha}L^{1+2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2},$$

$$\left|\begin{matrix}2\\22\end{matrix}\right| = \frac{\alpha^2(\alpha-1)c^2L^{2+2\beta}}{K^{3-2\alpha}L^2+\beta^2c^2K^3L^{2\beta}+\alpha^2c^2KL^{2+2\beta}}$$

A geodesic is in common language the shortest curve between two points. It is useful when we try to determine the shortest way to go from a production at other in a minimum time. The equation of a geodesic is:

(23) $$\frac{d^2x^i}{ds^2}+\left|\begin{matrix}i\\jk\end{matrix}\right|\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

that is:

(24) $$\frac{d^2L}{ds^2}+\left|\begin{matrix}1\\11\end{matrix}\right|\left(\frac{dL}{ds}\right)^2+2\left|\begin{matrix}1\\12\end{matrix}\right|\frac{dL}{ds}\frac{dK}{ds}+\left|\begin{matrix}1\\22\end{matrix}\right|\left(\frac{dK}{ds}\right)^2=0$$

(25) $$\frac{d^2K}{ds^2}+\left|\begin{matrix}2\\11\end{matrix}\right|\left(\frac{dL}{ds}\right)^2+2\left|\begin{matrix}2\\12\end{matrix}\right|\frac{dL}{ds}\frac{dK}{ds}+\left|\begin{matrix}2\\22\end{matrix}\right|\left(\frac{dK}{ds}\right)^2=0$$

or, with the quantities determined:

(24)

$$\frac{d^2L}{ds^2}+\frac{(\beta-1)\beta^2c^2K^{2+2\alpha}}{\beta^2c^2K^{2+2\alpha}L+L^3\left(\alpha^2c^2K^{2\alpha}+K^2L^{-2\beta}\right)}\left(\frac{dL}{ds}\right)^2+$$

$$2\frac{\alpha\beta^2c^2K^{1+2\alpha}L^{2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2}\frac{dL}{ds}\frac{dK}{ds}+\frac{\alpha(\alpha-1)\beta c^2K^{2\alpha}L^{1+2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2}\left(\frac{dK}{ds}\right)^2=0$$

(25)

$$\frac{d^2K}{ds^2}+\frac{\alpha(\beta-1)\beta c^2K^{1+2\alpha}L^{2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2}\left(\frac{dL}{ds}\right)^2+$$

$$2\frac{\alpha^2\beta c^2K^{2\alpha}L^{1+2\beta}}{\beta^2c^2K^{2+2\alpha}L^{2\beta}+\alpha^2c^2K^{2\alpha}L^{2+2\beta}+K^2L^2}\frac{dL}{ds}\frac{dK}{ds}+\frac{\alpha^2(\alpha-1)c^2L^{2+2\beta}}{K^{3-2\alpha}L^2+\beta^2c^2K^3L^{2\beta}+\alpha^2c^2KL^{2+2\beta}}\left(\frac{dK}{ds}\right)^2=0$$

The equations of geodesics are: L=L(s), K=K(s) where s is the element of arc on the curves.

3. References

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