A New Approach to Utility Function

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Abstract: This paper treats from an axiomatic point of view the notions of indifference and preference, relative to the consumption of goods. After the introduction and analysis of indifference classes, the notion of the utility function is introduced naturally, a number of axioms giving consistency and rigor. The concept of marginal utility is presented in both differentiable and, especially, in the discretized case. There are introduced new types of discretized marginal utility that adapts better when analyzing discrete the differential situation. The marginal rate of substitution is addressed globally, for n goods, obtaining the notions of hyperplane or minimal vector of substitution. Also, in the discretized case, there are introduced the marginal rates of substitution to the left, right or bilateral, as well as the adjusted rates, which give more precisely the possibility of consumption of a good when replaced another.

Keywords indifference; preference utility; marginal rate of substitution

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1. Introduction

From moments of impasse that has passed mathematics at the end of the nineteenth century, when it was forced relocation and reconstruction of the foundations of rigorous, any scientific theory that any aims to be sustainable and, above all, rigorous, has a urge to be built on solid bases, axiomatized. This field theory make a distance from the field of speculations or circumstantial situations, giving durability and at the same time, rigorous scientific reasoning. Any economic activity involves the existence of two distinct entities, but complementary, namely at least one manufacturer and a single consumer.

A manufacturer can not operate without a specific guarantee the possibility of purchase of his goods by at least one buyer as such it can not exist an applicant without the real creator of the product to be asked. It is natural to assume that each of the two parties follows a well-defined purpose. A manufacturer which would not pursue its profit maximization (even if this approach is somewhat simplistic) were closer to a charitable institution, rather than an economic entity. On the other hand, a beneficiary (which departs net from the notion of consumer) that would purchase

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products without having to follow a specific purpose (food needs, comfort, travel, etc.) could easily turn into a collector therefore, it would affect another individual needs.

It is very difficult to be measured or quantified the consumer's "need". A concept, largely controversial, satisfying to some extent, provided the "primary concept" in any axiomatic theory, is the **utility**.

There are a number of theories that define, more or less axiomatic notion of utility directly related to consumer preference for certain combinations of assets. What is a consumer preference but a good over another? The answer can only return to the same concept just try to explain it. We will not deviate too much from this line (even it is questionable from many points of view), but we will try a systematic and consistent increase in scientific endeavor.

2. Consumer Preferences

Before defining the consumer space, we consider first, that all goods for consumption are indefinitely divisible. We will see, a little later, that this convention is to a point, benefit, meaning that differential techniques can be applied in the analysis of consumer behavior. On the other hand, the findings obtained will be applied with great caution, especially when we want to establish a consumer verdict.

We thus define the **consumer space** on $\mathbf{R}^n$ for n fixed assets as $SC=\{(x_1,...,x_n) \mid x_i\geq 0,\ i=\overline{1,n}\}$ where $x\in SC$, $x=(x_1,...,x_n)$ is a **consumption basket** or **basket of goods**.

In relation to the issues raised above, is a natural question: considering two elements $x,y\in SC$, how do we characterize that a consumer will choose the basket x or y? It seems then that will have to establish a certain choice between a basket or another. In order not to enter the above vicious circle, we define the so-called preference relations, external in the generation of the rigorous theory, but effective in implementation.

We will define the **relationship of indifference** on SC noted, in what follows, with: $\sim$. If two baskets x and y are in relation $x\sim y$, this means that any combination of goods x and y is indifferent for the consumer. Also, we note that $x \nsim y$ the fact that x is not indifferent to y.

We will impose the condition of indifference to be a relationship of equivalence that is:

I.1. $\forall x \in SC \Rightarrow x \sim x$ (reflexivity);

I.2. $\forall x,y \in SC$, $x \sim y \Rightarrow y \sim x$ (symmetry);

I.3. $\forall x,y,z \in SC$, $x \sim y$, $y \sim z \Rightarrow x \sim z$ (transitivity).

The interpretation of these axioms is natural. Thus, reflexivity is not merely say that a basket of goods is indifferent in his choice himself, and symmetry that indifference between x and y implies choice, inevitably, the indifference of y and x. Transitivity is not always obvious, in that there may be situations (more or less forced) the indifference between x and y, then y and z between not involve the binding of x and z. Usually, the deviation from transitivity can occur when the relation of indifference is not "perfect", small differences between the three baskets leading to a significant distance between the extremes.

Let therefore the consumer space endowed with the relationship of indifference defined upper (SC,~) and $x \in SC$. The equivalence class of x: $[x]=\{y \in SC \mid y \sim x\}$ will consist in all consumer's baskets indifferent respected to x. We will call [x] – **the indifference class** of x.

From the properties of equivalence classes follow some remarkable conclusions, namely:

- x and y are indifferent if and only if they have identical indifference classes;
- for any two baskets of goods x and y, their indifference classes are either identical or disjoint (i.e. if exists z such that $x \sim z$, but $y \nsim z$ then for any $u \sim x$ will result that $u \nsim y$;
- the set of all baskets of goods or, in other words, the **consumer space** is the union of all classes of consumer indifference.

A system of representatives for the relationship of indifference $\sim$ will consist of all consumer baskets such that any two such entities are not indifferent and any consumer basket are whatever exactly one of the elected representatives.

Before continuing, let recall that a norm on the linear space $\mathbf{R}^n$ is an application: $\|\cdot\|$ $:\mathbf{R}^n \to \mathbf{R}$, $x \to \|x\|$ $\forall x \in \mathbf{R}^n$ such that the following axioms are satisfied:

N.1. $\|x\|=0 \Rightarrow x=0$;

N.2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ $\forall x \in \mathbf{R}^n$ $\forall \alpha \in \mathbf{R}$;

N.3. $\|x+y\| \le \|x\| + \|y\|$ $\forall x,y \in \mathbf{R}^n$.

The pair $(\mathbf{R}^n, \|\cdot\|)$ is called the normed n-dimensional linear space.

Considering therefore an arbitrary norm $\|\cdot\|$ on $\mathbf{R}^n$ we will add a first additional axiom to the relationship of indifference named **the axiom of continuity**:

I.4. $\forall x,y \in SC$, $x \sim y$, $\|x\| < \|y\| \Rightarrow \exists z \in SC$ such that: $x \sim z$ and $\|x\| < \|z\| < \|y\|$.

The axiom of continuity, not simply say that switching to a basket of goods to another indifferent with it, is done continuously, without "jumps".

For $x \in SC$, we will call, in the assumption of continuity axiom, the indifference class of x as **the indifference hypersurface** or for n=2 – **the indifference curve**. Because the indifference classes are either identical or disjoint, we have that the intersection of indifference hypersurfaces or curves are impossible.

A second additional axiom of indifference with respect to the above relationship refers to the condition of the lower bound of indifference classes namely:

I.5. $\forall x \in SC \Rightarrow \exists u \sim x$ such that $\|u\| \le \|v\|$ $\forall v \sim x$

The axiom I.5 describes the condition that in a class of indifference to be a basket at a least "distance" of origin or, in other words, with the lowest total (with respect to the norm) number of goods in his structure.

We will call a basket of goods like in the axiom I.5 - **minimal basket in the sense of norm with respect to the indifference class** of $x \in SC$ and we will note m(x). It should be noted that we do not necessarily guarantee the uniqueness of the existence of such a basket, but his norm is really unique.

Moreover, if $x \sim y$ then $\|m(x)\| = \|m(y)\|$. Indeed, if $x \sim y$ then $\|m(x)\| \le \|v\|$ $\forall v \sim x$ so , in particular: $\|m(x)\| \le \|m(y)\|$ because $m(y) \in [x]$ and hence $m(y) \sim x$. Analogously, $\|m(y)\| \le \|m(x)\|$ hence the above statement.

To define the relationship of preference, we will formulate differently the problem. If a basket of goods will be some x preferred to another y, it is logical to assume that any other basket z indifferent to x will also preferred to y. Therefore, we will consider instead of SC, the factor set SC relative to ~ which consists in the indifference classes of SC, denoted with SC/~.

Thus, to define **the relationship of the classes** marked in the following with $\succeq$ through the following axioms:

P.1. $\forall [x] \in SC/\sim \Rightarrow [x] \succeq [x]$ (reflexivity);
P.2. $\forall [x],[y] \in SC/\sim$, $[x] \succeq [y]$, $[y] \succeq [x] \Rightarrow [x]=[y]$ (antisymmetry);
P.3. $\forall [x],[y],[z] \in SC/\sim$, $[x] \succeq [y]$, $[y] \succeq [z] \Rightarrow [x] \succeq [z]$ (transitivity)

We will impose to this relationship four additional axioms:

P.4. $\forall x,y \in SC \Rightarrow [x] \succeq [y]$ or $[y] \succeq [x]$ (the condition of total ordering);

P.5. $\forall x \in SC \Rightarrow \exists y \in SC$ such that $y \nsim x$ and $[y] \succeq [x]$;

P.6. $[x] \succeq [y]$ if and only if $\|m(x)\| \geq \|m(y)\|$ (the condition of the compatibility with the existence of minimal baskets);

P.7. $\forall x,y \in SC$, $x>y \Rightarrow [x] \succeq [y]$ and $x \nsim y$ (the condition of the compatibility with the strict inequality relationship).

At first glance, the relationship of preference seems to depart from the nature of the goods, operating on indifference classes which represents set of goods indifferent between them.

On the other hand, however, the advantage of considering the indifference classes lies from the axiom P.2 which with P.1 and P.3 give the order relation character. Otherwise, if the relationship would be defined strictly preferably baskets of goods, from the fact that the other one was preferred and the second to the first, result that they are not identical, but that they are indifferent, so just classes equal indifference.

The total ordering condition states that any two baskets of goods are comparable in meaning preference for one of them.

The P.5 axiom guarantees the existence for any basket of goods of one not indifferent with it with and be at least as much preferred the former.

The axiom P.6 states that a class is preferred to another if and only if the norm of the first basket is greater than or equal to those of the minimal basket of the second.

The P.7 axiom states that a small additional quantity of a good from a basket leads to a preference superior to the original. The axiom also shows the existence for any basket of goods of one superior relative to the preference and, analogously, lower like preference.

Let now $x \in SC$ and $m(x)=(\overline{x}_1,...,\overline{x}_n) \in [x]$.

We will define **the relationship of preference** noted, in the following, without the danger of ambiguity, also with $\succeq$, by: $\forall x,y \in SC$ $x \succeq y$ if and only if $[x] \succeq [y]$.

The relationship will keep the properties of reflexivity, transitivity and total ordering, but the antisymmetry becomes:

P.2'. $\forall x,y \in SC/\sim$ $x \succeq y$, $y \succeq x \Rightarrow x \sim y$

The interpretation of the axioms is obvious. If any indifference class of a basket of consumer is preferred at least as much itself (reflexivity), follows that for an arbitrary basket x, any basket y indifferent to x, will be at least as preferred much as x.

The symmetry states that if a basket z indifferent to x and a basket t indifferent to y are each preferred by at least as much the other, then the two consumer baskets belonging to the same classes, so they are indifferent between them.

The transitivity can have violations in practice, but will usually be excluded from the analysis. It is possible that if relations preferably slightly offset time or applied to different situations, not to achieve transitivity. To have ensured transitivity, it must first be satisfied the simultaneity of the moments of choice and, on the other hand, it must apply to the same circumstantial situation.

We will define now on SC **the strict preference relationship** as a relationship class, denoted $\succ$ and defined by $[x]\succ[y]$ if and only if $[x]\succeq[y]$ and $[x]\neq[y]$ (which is equivalent to $[x]\cap[y]=\varnothing$).

Similarly, we now define **the strict preference relationship** denoted in the following, without the danger of confusion, by $\succ$: $\forall$ $x,y\in SC$ $x\succ y$ if and only if $[x]\succ[y]$.

The strict preference relationship on classes is not, obviously, reflexive because if for $[x]\in SC/\sim$ implies $[x]\succ[x]$ then $[x]\neq[x]$ which is a contradiction. Also, the antisymmetry states that: $\forall[x],[y]\in SC/\sim$ $[x]\succ[y]$, $[y]\succ[x]\Rightarrow[x]=[y]$. But $[x]\succ[y]$ implies $[x]\neq[y]$ so a contradiction with the statement of conclusion. Relative to transitivity, if $\forall[x]$, $[y]$, $[z]\in SC/\sim$, $[x]\succ[y]$, $[y]\succ[z]$ implies $[x]\succeq[y]$, $[x]\neq[y]$, $[y]\succeq[z]$, $[y]\neq[z]$ therefore: $[x]\succeq[z]$. The fact that $[x]\neq[z]$ does not result from any assertion, therefore it can not be proven the transitivity in this axiomatic framework.

For this reason, we will use below only indifference or preference relations (not strictly), in order to make full use of “power” property of the relations of equivalence or order.

Consider now an arbitrary consumer basket $x\in SC$. We will call **the preferred area of consumer** of x, the set: $ZC(x)=\{y\in SC \mid [y]\succeq[x]\}$ that is the set of those baskets that consumer prefers at least as much of x.

It notes that under the axiom P.5, $ZC(x)-[x]\neq\varnothing$ that is in the preferred are of consumption of x is at least one basket y strictly preferred to x.

Let us note that if $y\in ZC(x)$ then for any $z\in ZC(y)$ we have: $[z]\succeq[y]\succeq[x]$ from where, by virtue of transitivity: $[z]\succeq[x]$ so $z\in ZC(x)$. Therefore:

$$\forall y\in ZC(x)\Rightarrow ZC(y)\subset ZC(x)$$

From the axiom P.5 $\exists z\in SC$ such that $z\not\sim x$ and $[z]\succeq[x]$. From the above results we have that $ZC(z)\subset ZC(x)$. It is clear that $x\notin ZC(z)$, otherwise having $[x]\succeq[z]$ and from antisymmetry, results $[x]=[z]$ so $x\sim z$ – contradiction. After this observation we have that for any $x\in SC$ $\exists y_1\in ZC(x)$ such that $ZC(y_1)\subset ZC(x)$, $ZC(x)\text{-}ZC(y_1)\neq\varnothing$ (that is the inclusion is strictly). Analogously $\exists y_2\in ZC(y_1)$ such that $ZC(y_2)\subset ZC(y_1)$, $ZC(y_1)\text{-}ZC(y_2)\neq\varnothing$.

Therefore, for any $x\in SC$ $\exists(y_n)_{n\geq1}\subset SC$ such that:

$$ZC(x)\supset ZC(y_1)\supset ZC(y_2)\supset\ldots\supset ZC(y_n)\supset\ldots$$

the inclusions being strictly, so the underlying consumption of some basket, contains an infinity of different baskets.

Examples

1. Considering any two goods, the relationship $x\sim y$ defined by: $ax_1+bx_2=ay_1+by_2$ $\forall x=(x_1,x_2)$, $y=(y_1,y_2)\in SC$ where $a,b>0$ is one of indifference. The indifference classes relative to $\sim$ are for any $x=(x_1,x_2)$: $[x]=\{y\in SC \mid ay_1+by_2=ax_1+bx_2\}$

2. Considering any two goods and the indifference relation defined in the first example, the relationship $[x]\succeq[y]$ defined by: $ax_1+bx_2\geq ay_1+by_2$ $\forall x=(x_1,x_2)\in[x]$, $y=(y_1,y_2)\in[y]\in SC$ where $a,b>0$ is a preference relationship. Considering now $x\in SC$, $x=(x_1,x_2)$, $ax_1+bx_2=U$, we have $ZC(x)=\{(y_1,y_2)\mid ay_1+by_2\geq U\}$.

At the end of this section, we ask the normal question: how can we define concretely in practice, the relations of indifference or preference?

A first approach would be the income of the consumer is willing to spend on a basket of some goods. Considering two baskets of goods $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ we can believe that $x\sim y$ if a consumer is willing to devote the same amount of money for the purchase of x and y, respectively. The problem of preference is much more complicated. Considering the amount of the money S that the consumer is willing to spend to purchase a basket of goods (with some fixed structure) we could say that $x\succeq y$ if the sum S_x necessary to obtain x is greater than or equal to the corresponding S_y for the purchasing y, both amounts being less than or equal to S. This type of choice is quite limited but its concrete applicability. On the one hand, even if the price of a particular good would be identical to the market (otherwise, the consumer could purchase basket of goods from various sources and the relationship of preference could be, in some cases, reverse to the income allocated) the internal structure basket could lead to situations of exclusion in certain parts of it.

Consider, for example, a customer that has disposable an income of 12 monetary units wishing to purchase two products, namely bread whose price is 3 u.m./pcs. and toothpaste with the price 5 u.m./pcs. Considering pairs of goods of the form (p,d) where p - number of breads and d - the number of tubes of toothpaste, all baskets will be made admissible in pairs: (0,1) - 5 u.m., (0,2) - 10 u.m., (1,0) - 3 u.m., (1,1) - 8 u.m., (2,0) - 6 u.m., (2,1) - 11 u.m., (3,0) - 9 u.m., (4,0) - 12 u.m. The consumption basket that will surpass all others will contain, from this point of view, 4 breads and no toothpaste. The consumer then allocated the entire amount available, but satisfaction does not seem in any case, being the greatest because, on the one hand, did not buy any toothpaste (which had actually needed), and on the other, bought four breads that, if he lives alone, could be much more than its food needs. In the idea that he can not eat more one bread per day, most rational choice would be (1,1), but not willing to be spent maximizing income! Another choice that would ensure the two products could be (2,1) but, again, would bring an extra supply of breads which may not need them.

We see therefore that, in principle, the space of consumption SC should be limited according to consumer needs. On the other hand, a strict monetary approach to consumer preferences may lead to extreme situations that cause, in fact, dissatisfaction.

3. The Convexity of the Areas of Consumption

Considering a set A in $\mathbf{R}^n$ this is called convex if $\forall x,y\in A\ \forall\lambda\in[0,1]\Rightarrow\lambda x+(1-\lambda)y\in A$. Considering the line segment passing through points M(x) and N(y) we have that a set is convex if the segment MN (noted also [x,y]) is entirely in it.

In particular, we assume in what follows, that for any consumption basket $x\in SC$, ZC (x) is a convex set.

What significance has this fact and where it is the origin for this restriction?

The problem is quite complicated and, at first glance, seems somewhat common sense to take this restriction. If $y,z\in SC(x)$ then y and z are preferred to x. It seems natural to believe that any combination of intermediate goods between y and z will be preferred to x.

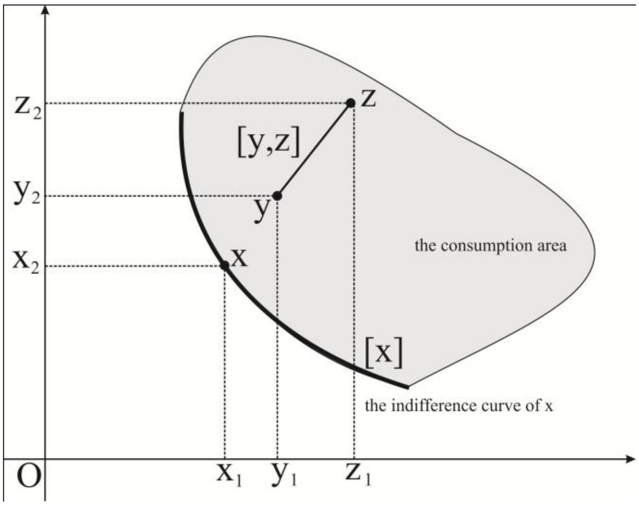

Figure 1. The convexity of the consumption area

Unfortunately, not always so! Consider, as an example a person who wants to travel to work effectively. If x="walking", then y="bus travel" and z="travel by cab" will be the preferred choices of x (ignoring here the actual distances or transport costs). A combination of y and z will be, for a relatively short distance, always disadvantageous to the first, because waiting times flowed into the travel mode change.

We believe however that most of the areas of consumption is convex, for several reasons. On the one hand, a basket of goods x, generating non-convex consumer area, is too unstable to be taken into account by a "rational" consumer. Any combination of consumer goods within the area can lead potentially to a reduction of its satisfaction with respect to x. On the other hand, even if our analysis is static, in reality, the migration is dynamic (it takes place in some time) and it is hard to believe that the consumer will go through a period of consumer dissatisfaction reach, for example, from y to z.

Formalizing, we will say that $\forall x\in SC$, ZC(x) is a convex set that is $\forall y,z\in SC$ such that $y\succeq x$ and $z\succeq x$ follows $\lambda y+(1-\lambda)z\succeq x$ $\forall\lambda\in[0,1]$.

From the condition of convexity of ZC, we get that for any $x,y,z\in SC$ such that $[y]\succeq[x]$ and $[z]\succeq[x]$ then $[\lambda y+(1-\lambda)z]\succeq[x]$ $\forall\lambda\in[0,1]$.

4. The Utility Functions

In the previous section, we have noted the difficulty of the mathematical approach of indifference and preference concepts. We will try in this part an axiomatic introducing of a concept, even though disputed by many economists, will bring some light in the treatment of previous notions. For a mathematical analysis of efficient consumer preferences could be useful to introduce a function with numerical values to enable their hierarchy.

We thus define **the utility function** as:

$$U:SC\to\mathbf{R}_+,\ (x_1,...,x_n)\to U(x_1,...,x_n)\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in SC$$

satisfying the following axioms:

U.1. $\forall x,y\in SC$: $x\sim y \Leftrightarrow U(x)=U(y)$;
U.2. $\forall x,y\in SC$: $x\succeq y \Leftrightarrow U(x)\geq U(y)$;
U.3. $U(0)=0$

We can reformulate the definition of utility function in terms of indifference classes as follows:

$$U:SC/\sim\cup\{0\}\to\mathbf{R}_+,\ [x]\to U([x])\in\mathbf{R}_+\ \forall[x]\in SC/\sim$$

satisfying the following axioms:

U.1'. $\forall x,y\in SC/\sim$: $[x]=[y] \Leftrightarrow U([x])=U([y])$;
U.2'. $\forall x,y\in SC/\sim$: $[x]\succeq[y] \Leftrightarrow U([x])\geq U([y])$;
U.3'. $U(0)=0$

We note that axiom U.1' does not mean anything other than injectivity of the utility function on the set of indifference classes.

Analyzing carefully the definition of utility, we see that, in fact, it brings nothing new concept in relation to the relations of indifference or preference.

Indeed, considering an arbitrary function, strictly increasing (with respect to the non-total order relation $\geq$), $U:SC\to\mathbf{R}_+$, we can define on SC the relationship of indifference as: $\forall x,y\in SC$: $x\sim y\Leftrightarrow U(x)=U(y)$. The relationship satisfies the axioms I.1, I.2 and I.3 of the previous definition of indifference. We can also define the relationship of preference by: $\forall x,y\in SC/\sim$: $[x]\succeq[y]\Leftrightarrow U([x])\geq U([y])$. The axioms P.1, P.2, P.3, P.4 and P.5 are also satisfied.

From the axiom P.7 we have that if $x,y\in SC$ such that $x>y$ then $[x]\succeq[y]$ therefore $U(x)>U(y)$. The utility function is therefore strictly increasing relatively to the relationship of strictly inequality. Let us note however that due to the impossibility of defining a relationship of total order on $\mathbf{R}^n$ we can not speak of a strict monotony of the overall definition scope.

Under the two definitions, we can characterize the class of indifference relative to a basket $x \in SC$ like $[x]=\{y \in SC \mid U(y)=U(x)\}$ and the consumer's area of x: $ZC(x)=\{y \in SC \mid U(y) \geq U(x)\}$.

Consider now $x \in SC$ and $U(x)=a \in \mathbf{R}_+$. We have therefore:

$$[x]=\{y \in SC \mid U(y)=a\}$$

If $y,z \in [x]$ then: $U(y)=U(z)=a$. We have seen, above, from the convexity of ZC(x) that: $[\lambda y+(1-\lambda)z] \succeq [x]$ or, in terms of utility: $U(\lambda y+(1-\lambda)z) \geq U(x)=a=\lambda a+(1-\lambda)a=\lambda U(y)+(1-\lambda)U(z)$.

We obtained thus:

$$U(\lambda y+(1-\lambda)z) \geq \lambda U(y)+(1-\lambda)U(z) \ \forall \lambda \in [0,1] \ \forall z,y \in [x] \ \forall x \in SC$$

The above condition is nothing but than the concavity of a function. In the case of a continuous function, the concavity is expressed geometrically by the fact that any chord determined by two points on the graph function is located below it. Therefore, the restriction of the utility function to a class of indifference of an arbitrary basket is concave.

We will extend this requirement to the whole space SC, thus requiring the utility function the following condition:

UC.1. The utility function is concave.

While not necessarily essential to the fundamental properties, we must sometimes still an additional condition:

UC.2. The utility function is of class C^2 on the inside of SC.

The differentiability of the utility function automatically implies its continuity on the interior domain of definition. In the case UC.2 the concavity axiom that function is equivalent to the fact that the second differential of U is defined negatively.

Considering $d^2U=\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i \partial x_j} dx_i dx_j$ and the attached quadratic form: H= $\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i \partial x_j} h_i h_j$, the fact that H is negatively defined is shown by Gauss method or by that of Jacobi.

Also, in the case of the differentiability, let note that $\sum_{i=1}^{n}\left(\frac{\partial U}{\partial x_i}\right)^2 \neq 0$ that is at least one of the first order partial derivatives is nonzero at any point. Indeed, if there is a point such that: $\frac{\partial U}{\partial x_i}=0\ \forall i=\overline{1,n}$ then, from the concavity of U it follows that it is a local maximum. On the other hand, the axiom P.7 imposed the hypothesis of non-existence of local maximum or minimum points.

We can not conclude this section without a perfectly legitimate question: how we will effectively build the utility function?

In principle, we can assign arbitrary values to the indifference classes, which will be satisfied only condition being that if x is preferred to y then the value assigned to the class of x to be equal to or greater than that attributed to class y. In this case, the detailed rules for the award is very relative.

If we are not interested than order of preference for a basket of goods and another, serial numbers can be assigned arbitrarily (e.g. order of preference indexed by non-zero natural numbers), that will do a hierarchy of the baskets of goods. In this case, we say that we are dealing with an **ordinal utility**.

Its disadvantage is, on the one hand, that we have not an uniqueness in assignment and, on the other hand, the utility thus defined can not be used in complex mathematical calculations (because of dependence by the arbitrary allocation).

Another way the award is related to external factors which contribute to the expression of preference for a basket of goods or another. We can define the utility for the purposes of income the consumer is willing to allocate for purchase a basket of goods. Thus, a consumer who has 7 u.m. put in a position to choose between buying a basket of soft drink with a price 2 u.m. and a sandwich of 3 u.m. and one of two drinks a 3 u.m. (together) and a sandwich for the same price he chooses, most often, the latter combination.

Also, we can define the utility as the consumer's economy that makes reference to a standard basket of goods in the choice amounts to the same invoice. We could give an example where a person is indifferent where to go in it's free time: to the theater, the cinema or a concert. If a theater ticket will cost 20 u.m., at the cinema - 10 u.m. and 30 u.m. at the concert, he will take the concert like standard and if he go to the theater will have a utility of 10 u.m. (30-20) and analogously, to the cinema - 20 u.m. (30-10).

Another approach may be of utility in terms of satisfaction in the future purchase act. Thus, an individual who is in a position to choose between a TV and a computer having identical prices, choose the TV if it has no notions about computers and choose the computer that is definitely going to write a book about the theory of utility!

However we put the problem, it is agreed that an allocation of utility which abides the axioms and will meet the above conditions can be addressed mathematically more correct once it has been precisely defined. We call such an allocation: **cardinal utility**.

Let now a concrete way to approach the construction of utility functions.

Considering $x\in SC$, we will define $U(x)=\|m(x)\|$. The definition is correct under axiom I.5 that for any class of indifference to a basket of certain guarantees the existence of a basket of minimum norm.

From the axiom I.5, we saw that if $x\sim y$ then $\|m(x)\|=\|m(y)\|$ so $U(x)=U(y)$. Therefore, U.1 axiom is satisfied.

If $x\succeq y$ then, from the axiom P.6, we have that $m(x)\geq m(y)$ therefore: $U(x)\geq U(y)$ so just axiom U.2.

Considering now a utility function $U:SC\to\mathbf{R}_+$ and an application monotonically increasing $f:\mathbf{R}_+\to\mathbf{R}_+$, the function $f\circ U$ defined by $f\circ U(x)=f(U(x))$ is also an utility function. Indeed, if $x\sim y$ then $U(x)=U(y)$ from where $f(U(x))=f(U(y))$ and if $x\succeq y$ then $U(x)\geq U(y)$ and $f(U(x))\geq f(U(y))$. We therefore conclude that the utility function is determined up to a monotone increasing application.

Finally, let mention that for $a\in\mathbf{R}_+$, the graph corresponding to the equation solutions $U(x)=a$ is called isoutility curve (in $\mathbf{R}^2$) or isoutility hypersurface (in $\mathbf{R}^n$).

From [3] and the fact that U is a concave function and partial derivatives of first order are positive (as we shall see later), it follows that the isoutility hypersurfaces are convex.

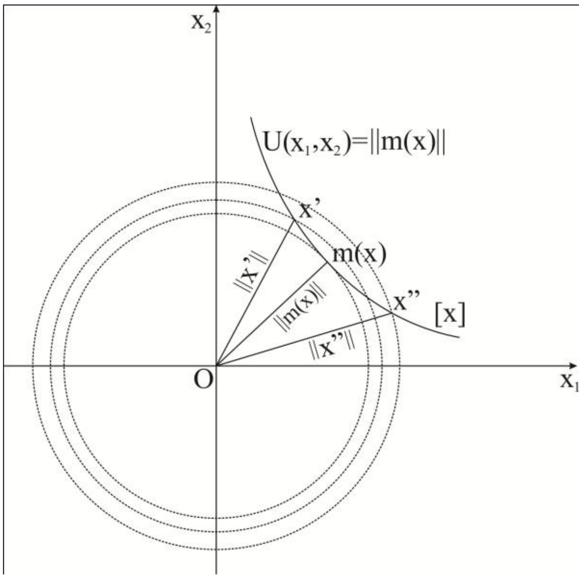

Figure 2. The definition of the utility function

Let consider now n classes of basket of goods whose consumption spaces are $SC_1 \subset \mathbf{R}_+^{k_1}, \dots, SC_n \subset \mathbf{R}_+^{k_n}$ and $U_1, \dots, U_n$ – corresponding utility functions. We will call the n classes **independent in the sense of utility** if the function $U: SC_1 \times \dots \times SC_n \to \mathbf{R}_+$, $U(X_1, \dots, X_n) = U_1(X_1) + \dots + U_n(X_n)$ $\forall X_i = \left(x_{i1}, \dots, x_{ik_i}\right) \in SC_i$, $i = \overline{1, n}$ is a utility for all goods.

In particular, n goods will call **independent in the sense of utility** if $U(x_1, \dots, x_n) = U_1(x_1) + \dots + U_n(x_n)$ $\forall (x_1, \dots, x_n) \in SC$.

One can easily see that if the functions $U_1, \dots, U_n$ are concave and of class C^2 then: $d^2U = \sum_{i=1}^{n} U_i''(x_i) dx_i^2 \le 0$ therefore U is concave.

Example

Considering for any $n \ge 2$ goods the relationship of indifference $x \sim y$ defined by: $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} = y_1^{k_1} y_2^{k_2} \dots y_n^{k_n}$ $\forall x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in SC$, $k_1, \dots, k_n > 0$, we will define after foregoing the utility function:

$$U(x) = \|m(x)\| = \sqrt{\sum_{i=1}^{n} k_i}\, k_1^{-\frac{k_1}{2\sum_{i=1}^{n} k_i}} \dots k_n^{-\frac{k_n}{2\sum_{i=1}^{n} k_i}} x_1^{\frac{k_1}{\sum_{i=1}^{n} k_i}} x_2^{\frac{k_2}{\sum_{i=1}^{n} k_i}} \dots x_n^{\frac{k_n}{\sum_{i=1}^{n} k_i}} \quad \forall x = (x_1, x_2, \dots, x_n) \in SC$$

5. The Marginal Utility

Let $U:SC\to\mathbf{R}_+$ an utility function. We saw above that the utility is an increasing function with respect to the preference relation of goods basket and strictly increasing with respect to the relationship of strictly inequality on $\mathbf{R}^n$.

Considering $1\le i\le n$ – fixed and $a_k\in\mathbf{R}_+$, $k=\overline{1,n}$, $k\ne i$, we will note synthetic $x=(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)\in SC$.

We define **the discretized marginal utility** in relation to the i-th good, while the consumption of other goods is constant as:

$$U_{m,i}(x)=\frac{\Delta U}{\Delta x_i}=\frac{U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i-\Delta x_i,a_{i+1},...,a_n)}{\Delta x_i}$$

therefore the variation of the utility U at the variation of the consumption of good i.

In relation to the above definition, we deduce easily:

$$\Delta U=U_{m,i}(x)\Delta x_i$$

It is necessary here to make an interesting observation! The classic definition of marginal utility essentially uses the variation of the utility function from one direction. Considering thus $\Delta x_i=h$, we get from above:

$$U_{m,i}(x)=\frac{U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i-h,a_{i+1},...,a_n)}{h}$$

therefore the variation at left in the point x.

If $h>0$ then the marginal utility at the point x is the change in utility of the "past" in "now" and can not be used to estimate the utility in the "future". Analogously, if $h=-s<0$ then the marginal utility at the point x becomes:

$$U_{m,i}(x)=\frac{U(a_1,...,a_{i-1},x_i+s,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)}{s}$$

and represents the variation of the utility from "present" in the "future" and can not be used to calculate the utility in the "past".

A more accurate way of calculating the marginal utility is the arithmetic mean of the marginal utility to the left and right:

$$U_{m,i}(x)=\frac{U(a_1,...,a_{i-1},x_i+h,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x_i-h,a_{i+1},...,a_n)}{2h}$$

for all points inside the space consumption, and to the left and right of it, calculating the marginal utility to right, respectively left.

In what follows, we will note the discretized marginal utility at left with $U_{ml,i}$, the discretized marginal utility at right with $U_{mr,i}$ and the discretized marginal utility two-sided with $U_{mb,i}$.

We obtain that:

- $\Delta_l U=U_{ml,i}(x)\Delta x_i$ for $\Delta_l U= U(a_1,...,x_i,...,a_n)-U(a_1,...,x_i-\Delta x_i,...,a_n)$;
- $\Delta_r U=U_{mr,i}(x)\Delta x_i$ for $\Delta_r U= U(a_1,...,x_i+\Delta x_i,...,a_n)-U(a_1,...,x_i,...,a_n)$;
- $\Delta_b U=U_{mb,i}(x)\Delta x_i$ for $\Delta_b U= \dfrac{U(a_1,...,x_i+\Delta x_i,...,a_n)-U(a_1,...,x_i-\Delta x_i,...,a_n)}{2}$

where $\Delta x_i>0$.

Before concluding this discussion let note that $U_{ml,i}$ in $(a_1,...,x_i,...,a_n)$ coincides with $U_{mr,i}$ in $(a_1,...,x_i-\Delta x_i,...,a_n)$ and also $U_{mr,i}$ in $(a_1,...,x_i,...,a_n)$ coincides with $U_{ml,i}$ in $(a_1,...,x_i+\Delta x_i,...,a_n)$. Also, from the above definition: $U_{mb.i}=\dfrac{U_{ml,i}+U_{mr,i}}{2}$ therefore: $\min\{U_{ml,i},U_{mr,i}\}\le U_{mb,i}\le\max\{U_{ml,i},U_{mr,i}\}$.
If the case of a differentiable utility of class C^1, we define **the marginal utility** in relation to the i-th good, while the consumption of other goods is constant, as:

$$U_{m,i}(x)=\frac{\partial U}{\partial x_i}(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)=$$

$$\lim_{\Delta x_i\to 0}\frac{U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_n)-U(a_1,...,a_{i-1},x-\Delta x_{ii},a_{i+1},...,a_n)}{\Delta x_i}$$

therefore the differentiable marginal utility is the limit when of the discretized marginal utility when the variation of the good's consumption tends to 0.

The general approach of the utility function, requires it to be concave (the UC.1 axiom). But we have $d^2U=\sum_{i,j=1}^{n}\frac{\partial^2 U}{\partial x_i\partial x_j}dx_i dx_j=\frac{\partial^2 U}{\partial x_i^2}dx_i^2=\frac{\partial}{\partial x_i}\left(\frac{\partial U}{\partial x_i}\right)dx_i^2=\frac{\partial U_{m,i}}{\partial x_i}dx_i^2$ (caeteris paribus). The negatively defined character of d^2U implies $\frac{\partial U_{m,i}}{\partial x_i}<0$ therefore $U_{m,i}$ is decreasing caeteris paribus (Gossen's First Law).

Let now reconsider the situation of the discretized marginal utility. We saw that: $\Delta U=U_{m,i}(x)\Delta x_i$ caeteris paribus for each type of the marginal utility (but with different meanings of ΔU). Considering a number of k units of good i consumed, we get (with abbreviated notation $U_i(j)=U_i(a_1,...,a_{i-1},j,a_{i+1},...,a_n)$ and analogously for $U_{m,i}$), successively, for the left marginal utility:

$$U_i(j+1)-U_i(j)=U_{ml,i}(j+1)\cdot 1\ \forall j=\overline{0,k-1}$$

and after summing and reductions:

$$U_i(k)-U_i(0)=\sum_{j=1}^{k} U_{ml,i}(j)$$

We got that the total utility is the sum of discretized marginal utilities to the left. If it is one single good, we have $U_i(0)=0$ (the axiom U.3) thus:

$$U_i(k)=\sum_{j=1}^{k} U_{ml,i}(j)$$

We have obtained that the total utility corresponding to the consumption of k units of a good equals the sum of discretized marginal utilities to the left (for goods 1,...,k).

For the right marginal utility, we have:

$$U_i(j+1)-U_i(j)=U_{mr,i}(j)\cdot 1\ \forall j=\overline{0,k-1}$$

and after summing and reductions:

$$U_i(k)-U_i(0)=\sum_{j=0}^{k-1} U_{mr,i}(j)$$

We got that the total utility is the sum of discretized marginal utilities to the right. If it is one single good, we have $U_i(0)=0$ (the axiom U.3) thus:

$$U_i(k)=\sum_{j=0}^{k-1} U_{mr,i}(j)$$

We have obtained that the total utility corresponding to the consumption of k units of a good equals the sum of discretized marginal utilities to the right (for goods 0,...,k-1 where the good 0 is formal in order to use the right utility).

For bilateral marginal utility, we have for a total number N of copies of good i:

$$\begin{cases} U_i(1) - U_i(0) = U_{mb,i}(0) \cdot 1 \\ \dfrac{U_i(j) - U_i(j-2)}{2} = U_{mb,i}(j-1) \cdot 1,\ j = \overline{2,N} \\ U_i(N) - U_i(N-1) = U_{mb,i}(N) \cdot 1 \end{cases}$$

After recurrence, it follows:

$$U_i(k) = U_i(k-2s) + 2\left(U_{mb,i}(k-2s+1) + \ldots + U_{mb,i}(k-1)\right) \ \forall s>0 \text{ such that } k-2s \geq 0$$

In particular, as $U_i(0)=0$ and $U_i(1)=U_{mb,i}(0)$ we have:

$$U_i(k) = 2\left(U_{mb,i}(1) + \ldots + U_{mb,i}(k-1)\right) \text{ for } k=\text{even}$$

$$U_i(k) = 2\left(U_{mb,i}(0) + U_{mb,i}(2) + \ldots + U_{mb,i}(k-1)\right) - U_{mb,i}(0) \text{ for } k=\text{odd}$$

Like a conclusion we have that the total utility corresponding to the consumption of k units of a good is equal to twice the sum of discretized bilateral marginal utilities of odd order less than k, and for k = odd with twice the sum of discretized bilateral marginal utilities of even order, less than k, minus the bilateral utility of the null good.

If the utility is differentiable, then:

$$U_i(k) = \int_0^k U_{m,i}(x)dx_i$$

and the corresponding marginal utility of the unit k of good i is:

$$U_{m,i}(k) = U_i(k) - U_i(k-1) = \int_{k-1}^{k} U_{m,i}(x)dx_i$$

In the general case of the variation in consumption of all existing goods, for k_1 units of good 1,...,k_n units of good n, we will consider first the simple way $\gamma:[0,1]\to\mathbf{R}^n$, $\gamma(t)=(tk_1,...,tk_n)$. This is nothing more than the large diagonal of the n-dimensional parallelepiped: $[0,k_1]\times...\times[0,k_n]$. Let also the differential form:

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \ldots + \frac{\partial U}{\partial x_n} dx_n$$

that is continuous everywhere after the C^2 character of U. Along the path γ, the integral of dU is defined by:

$$\int_\gamma dU = \int_0^1 \left(\frac{\partial U}{\partial x_1}(\gamma_1(t),...,\gamma_n(t))\gamma'_1(t) + \ldots + \frac{\partial U}{\partial x_n}(\gamma_1(t),...,\gamma_n(t))\gamma'_n(t) \right) dt$$

where $\gamma_1,...,\gamma_n$ are the components of γ. The Leibniz-Newton's theorem for exact differential forms (forms with property $\exists$U such that ω=dU) states that: $\int_\gamma dU$ $=U(\gamma(1))-U(\gamma(0))$.

In the present case:

$$U(k_1,...,k_n)-U(0,...,0)=\int_0^1\left(\frac{\partial U}{\partial x_1}(k_1t,...,k_nt)k_1+...+\frac{\partial U}{\partial x_n}(k_1t,...,k_nt)k_n\right)dt=$$

$$k_1\int_0^1 U_{m,1}(k_1t,...,k_nt)dt+...+k_n\int_0^1 U_{m,n}(k_1t,...,k_nt)dt$$

Because U(0)=0, resulting the final formula:

$$U(k_1,...,k_n)=k_1\int_0^1 U_{m,1}(k_1t,...,k_nt)dt+...+k_n\int_0^1 U_{m,n}(k_1t,...,k_nt)dt$$

6. The Marginal Rate of Substitution

Let consider, first the case of two variable goods, the other remains fixed. Let the goods i and j with $i \neq j$. We define the space restriction of consumption: $G_{ij}=\{(x_1,...,x_n) \mid x_k=a_k=\text{const}, k=\overline{1,n}, k\neq i,j, x_i,x_j\in\mathbf{R}_+\}$ relative to the two goods where the others remain fixed. Also be: $D_{ij}=\{(x_i,x_j) \mid (x_1,...,x_n)\in G_{ij}\}$ - the consumption domain corresponding only to goods i and j.

We define: $u_{ij}:D_{ij}\rightarrow\mathbf{R}_+$ - the restriction of the utility function at goods i and j, i.e.:

$$u_{ij}(x_i,x_j)=U(a_1,...,a_{i-1},x_i,a_{i+1},...,a_{j-1},x_j,a_{j+1},...,a_n)$$

The functions u_{ij} define a surface in $\mathbf{R}^3$ for any pair of goods (i,j).

We will call **partial marginal rate of substitution** between goods i and j, relative to G_{ij} (caeteris paribus), the variation of the amount of good j in order to substitute an amount of the good i in the hypothesis of utility conservation.

We will note in what follows:

$$RMS(i,j,G_{ij})=\frac{dx_j}{dx_i}$$

Since $u_{ij}(x_i,x_j)=\bar{u}$ =const, we obtain by differentiation: $du_{ij}(x_i,x_j)=0$ i.e.: $\frac{\partial u_{ij}}{\partial x_i}dx_i+\frac{\partial u_{ij}}{\partial x_j}dx_j=0$ therefore: $\frac{dx_j}{dx_i}=-\frac{\frac{\partial u_{ij}}{\partial x_i}}{\frac{\partial u_{ij}}{\partial x_j}}=-\frac{\left.\frac{\partial U}{\partial x_i}\right|_{G_{ij}}}{\left.\frac{\partial U}{\partial x_j}\right|_{G_{ij}}}=-\frac{\left.U_{m,i}\right|_{G_{ij}}}{\left.U_{m,j}\right|_{G_{ij}}}$.

We can write: $RMS(i,j,G_{ij})=-\frac{\left.U_{m,i}\right|_{G_{ij}}}{\left.U_{m,j}\right|_{G_{ij}}}$ which is a function of x_i and x_j. In a fixed point $\bar{x}=(\bar{x}_1,...,\bar{x}_n)$ we have:

$$RMS(i,j,\bar{x})=-\frac{U_{m,i}(\bar{x})}{U_{m,j}(\bar{x})}$$

Let now consider the case when the consumption of all goods vary. Let therefore an arbitrary point $\bar{x}\in SC$ such that $U(\bar{x})=U_0$=const and $U_{m,k}(\bar{x})\neq 0$, $k=\overline{1,n}$. Differentiating in $\bar{x}$ we obtain: $0=dU=\sum_{j=1}^{n}\frac{\partial U}{\partial x_j}dx_j$ therefore: $\frac{\partial U}{\partial x_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial U}{\partial x_j}\frac{dx_j}{dx_i}=0$ or, in terms of marginal utility: $U_{m,i}+\sum_{\substack{j=1\\ j\neq i}}^{n}U_{m,j}\frac{dx_j}{dx_i}=0$. If we note $\frac{dx_j}{dx_i}=y_j$, $j=\overline{1,n}$, $j\neq i$, we get: $U_{m,i}+\sum_{\substack{j=1\\ j\neq i}}^{n}U_{m,j}y_j=0$. With the aid of the partial substitution marginal rate introduced above, by dividing at $U_{m,i}$, we get:

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\bar{x})}=1$$

The above relationship is nothing but the equation of a hyperplane in $\mathbf{R}^{n-1}$ of coordinates $(y_1,...,\hat{y}_i,...,y_n)$ (the sign ^ means that that term is missing) that intersects the coordinate axes in $RMS(i,j,\bar{x})$. This hyperplane is the locus of consumption goods changes relative to a change in the consumption good "i" such that the utility remain constant.

For this reason, we will call the locus: **the marginal substitution hyperplane** between the good "i" and the other goods (note below $H_{mi,j}$).

In particular, for two goods, the marginal substitution hyperplane between the good i and the good j, of **R**, is reduced to: $\frac{y_j}{RMS(i,j,\bar{x})}=1$ where $y_j=\frac{dx_j}{dx_i}$. We have therefore $\frac{dx_j}{dx_i}=y_j=RMS(i,j,\bar{x})$ which is consistent with the definition of marginal rate of substitution.

We will define now the **overall marginal rate of substitution** between good i and the other as the opposite distance from the origin to the marginal substitution hyperplane, namely:

$$RMS(i,\bar{x})=-\frac{1}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{RMS^2(i,j,\bar{x})}}}=-\frac{U_{m,i}(\bar{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}U_{m,j}^2(\bar{x})}}$$

We note that for the particular case of two goods, we get as above:

$$RMS(i,\bar{x})=-\frac{U_{m,i}(\bar{x})}{U_{m,j}(\bar{x})}$$

Considering now $v=(y_1,...,\hat{y}_i,...,y_n)\in H_{mi,j}$ we have: $\|v\|=\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}y_j^2}$ and from the Cauchy-Schwarz inequality:

$$\frac{\|v\|}{|RMS(i,\bar{x})|}=\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}y_j^2}\cdot\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{RMS^2(i,j,\bar{x})}}\geq\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\bar{x})}=1$$

that is: $\|v\|\geq|RMS(i,\bar{x})|$. By these results, the overall marginal rate of substitution is the minimum (in the meaning of norm) changes in consumption so that total utility remains unchanged.

Considering now the marginal substitution hyperplane: $\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{y_j}{RMS(i,j,\bar{x})}=1$ the equation of the normal line from the origin to it is:

$$\frac{y_1}{\frac{1}{RMS(i,1,\bar{x})}}=...=\frac{y_{i-1}}{\frac{1}{RMS(i,i-1,\bar{x})}}=\frac{y_{i+1}}{\frac{1}{RMS(i,i+1,\bar{x})}}=...=\frac{y_n}{\frac{1}{RMS(i,n,\bar{x})}}$$

therefore:

$$\begin{cases} y_1 = \dfrac{\lambda}{RMS(i,1,\overline{x})} \\ \dots \\ y_{i-1} = \dfrac{\lambda}{RMS(i,i-1,\overline{x})} \\ y_{i+1} = \dfrac{\lambda}{RMS(i,i+1,\overline{x})} \\ \dots \\ y_n = \dfrac{\lambda}{RMS(i,n,\overline{x})} \end{cases}, \lambda \in \mathbf{R}$$

The intersection of the normal line with the hyperplane, represents the coordinates of the point of minimum norm. We therefore have: $\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\lambda}{RMS^2(i,j,\overline{x})} = 1$ and $\lambda = \frac{1}{\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{RMS^2(i,j,\overline{x})}}$. The point of minimum norm has the coordinates:

$$RMS^2(i,\overline{x})\left(\frac{1}{RMS(i,1,\overline{x})},\dots,\frac{\hat{1}}{RMS(i,i,\overline{x})},\dots,\frac{1}{RMS(i,n,\overline{x})}\right) =$$

$$-\frac{U_{m,i}(\overline{x})}{\sum_{\substack{j=1 \\ j\neq i}}^{n} U_{m,j}^2(\overline{x})}\left(U_{m,1}(\overline{x}),\dots,\hat{U}_{m,i}(\overline{x}),\dots,U_{m,n}(\overline{x})\right)$$

which norm is nothing else than $|RMS(i,\overline{x})|$.

The above coordinates of the point is no more than minimal vector (in the meaning of norm) of the consumption changes such that the utility remain unchanged. We will say briefly that this is **the minimal vector** of the i-th good **substitution**.

For the discrete case, we will define the marginal rate of substitution between goods “i” and “j”, caeteris paribus, the quantity of good “j” required for replacement a unit of “i” in the situation of the utility conservation.

Let us recall that, in the case of left discretized:

$$U_{ml,i}(\overline{x}) = \frac{U(\overline{x}_1,\dots,\overline{x}_{i-1},\overline{x}_i,\overline{x}_{i+1},\dots,\overline{x}_n) - U(\overline{x}_1,\dots,\overline{x}_{i-1},\overline{x}_i - \Delta x_i,\overline{x}_{i+1},\dots,\overline{x}_n)}{\Delta x_i} \quad \forall i=\overline{1,n}$$

from where:

$$U_{ml,i}(\overline{x})\,\Delta x_i + U_{ml,j}(\overline{x})\,\Delta x_j =$$
$$2U(\overline{x}_1,...,\overline{x}_n) - U(\overline{x}_1,...,\overline{x}_{i-1},\overline{x}_i - \Delta x_i,\overline{x}_{i+1},...,\overline{x}_n) - U(\overline{x}_1,...,\overline{x}_{j-1},\overline{x}_j - \Delta x_j,\overline{x}_{j+1},...,\overline{x}_n)$$

For very small variations of x_i and x_j, respectively, therefore $\Delta x_i \approx 0$, $\Delta x_j \approx 0$ we get:

$$U_{ml,i}(\overline{x})\,\Delta x_i + U_{ml,j}(\overline{x})\,\Delta x_j \approx 0$$

or:

$$U_{ml,i}(\overline{x})\,\Delta x_i \approx - U_{ml,j}(\overline{x})\,\Delta x_j$$

We have therefore the **left partial marginal rate of substitution** between goods "i" and "j":

$$RMS_l(i,j,\overline{x}) = \frac{\Delta x_j}{\Delta x_i} \approx -\frac{U_{ml,i}(\overline{x})}{U_{ml,j}(\overline{x})}$$

Also in the case of right discretized:

$$U_{mr,i}(\overline{x}) = \frac{U(\overline{x}_1,...,\overline{x}_{i-1},\overline{x}_i + \Delta x_i,\overline{x}_{i+1},...,\overline{x}_n) - U(\overline{x}_1,...,\overline{x}_{i-1},\overline{x}_i,\overline{x}_{i+1},...,\overline{x}_n)}{\Delta x_i} \quad \forall i=\overline{1,n}$$

from where:

$$U_{mr,i}(\overline{x})\,\Delta x_i + U_{mr,j}(\overline{x})\,\Delta x_j =$$
$$U(\overline{x}_1,...,\overline{x}_{i-1},\overline{x}_i + \Delta x_i,\overline{x}_{i+1},...,\overline{x}_n) + U(\overline{x}_1,...,\overline{x}_{j-1},\overline{x}_j + \Delta x_j,\overline{x}_{j+1},...,\overline{x}_n) - 2U(\overline{x}_1,...,\overline{x}_n)$$

At very small variations of x_i and x_j:

$$U_{mr,i}(\overline{x})\,\Delta x_i + U_{mr,j}(\overline{x})\,\Delta x_j \approx 0$$

therefore:

$$U_{mr,i}(\overline{x})\,\Delta x_i \approx - U_{mr,j}(\overline{x})\,\Delta x_j$$

We can coclude that the **right partial marginal rate of substitution** between goods "i" and "j" is:

$$RMS_r(i,j,\overline{x}) = \frac{\Delta x_j}{\Delta x_i} \approx -\frac{U_{mr,i}(\overline{x})}{U_{mr,j}(\overline{x})}$$

In the bilateral discretized case:

$$U_{mb,i}(\overline{x}) = \frac{U(\overline{x}_1,...,\overline{x}_{i-1},\overline{x}_i + \Delta x_i,\overline{x}_{i+1},...,\overline{x}_n) - U(\overline{x}_1,...,\overline{x}_{i-1},\overline{x}_i - \Delta x_i,\overline{x}_{i+1},...,\overline{x}_n)}{2\Delta x_i}$$
$$\forall i=\overline{1,n}$$

therefore:

$$U_{mb,i}(\overline{x})\,\Delta x_i + U_{mb,j}(\overline{x})\,\Delta x_j =$$

$$\frac{U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)-U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i-\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)}{2}+$$
$$\frac{U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j+\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)-U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j-\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)}{2}$$

or:

$$\begin{aligned}&2\left(U_{mb,i}(\bar{x})\Delta x_i+U_{mb,j}(\bar{x})\Delta x_j\right)=\\&U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i+\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)+U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j+\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)\\&-U(\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_i-\Delta x_i,\bar{x}_{i+1},...,\bar{x}_n)-U(\bar{x}_1,...,\bar{x}_{j-1},\bar{x}_j-\Delta x_j,\bar{x}_{j+1},...,\bar{x}_n)\end{aligned}$$

At very small variations of x_i and x_j:

$$U_{mb,i}(\bar{x})\,\Delta x_i+U_{mb,j}(\bar{x})\,\Delta x_j\approx 0$$

or, equivalent:

$$U_{mb,i}(\bar{x})\,\Delta x_i\approx -U_{mb,j}(\bar{x})\,\Delta x_j$$

We will define therefore the **bilateral partial marginal rate of substitution** between goods “i” and “j” is:

$$RMS_b(i,j,\bar{x})=\frac{\Delta x_j}{\Delta x_i}\approx-\frac{U_{mb,i}(\bar{x})}{U_{mb,j}(\bar{x})}$$

Let us note now for simplicity:

$$\alpha_k=\frac{U(\bar{x}_1,...,\bar{x}_{k-1},\bar{x}_k+\Delta x_k,\bar{x}_{k+1},...,\bar{x}_n)}{\Delta x_k}$$
$$\beta_k=\frac{U(\bar{x}_1,...,\bar{x}_{k-1},\bar{x}_k,\bar{x}_{k+1},...,\bar{x}_n)}{\Delta x_k}\qquad \forall k=\overline{1,n}$$
$$\gamma_k=\frac{U(\bar{x}_1,...,\bar{x}_{k-1},\bar{x}_k-\Delta x_k,\bar{x}_{k+1},...,\bar{x}_n)}{\Delta x_k}$$

With these notations, we get then:

$$U_{mb,k}(\bar{x})=\frac{\alpha_k-\gamma_k}{2}=\frac{(\alpha_k-\beta_k)+(\beta_k-\gamma_k)}{2}=\frac{U_{ml,k}(\bar{x})+U_{mr,k}(\bar{x})}{2}\quad \forall k=\overline{1,n}$$

For the three cases above, we therefore:

$$RMS_b(i,j,\bar{x})=-\frac{U_{mb,i}(\bar{x})}{U_{mb,j}(\bar{x})}=-\frac{U_{ml,i}(\bar{x})+U_{mr,i}(\bar{x})}{U_{ml,j}(\bar{x})+U_{mr,j}(\bar{x})}$$

After this formula, we get:

$$\left(\mathrm{RMS}_b(i,j,\bar{x})-\mathrm{RMS}_l(i,j,\bar{x})\right)\left(\mathrm{RMS}_b(i,j,\bar{x})-\mathrm{RMS}_r(i,j,\bar{x})\right)=$$
$$-\frac{\left(U_{ml,i}(\bar{x})U_{mr,j}(\bar{x})-U_{ml,j}(\bar{x})U_{mr,i}(\bar{x})\right)^2}{U_{ml,j}(\bar{x})U_{mr,j}(\bar{x})\left(U_{ml,j}(\bar{x})+U_{mr,j}(\bar{x})\right)^2}<0$$

by virtue of the fact that marginal utilities are positive.

Following this result, we get a not surprising result: $\mathrm{RMS}_b(i,j,\bar{x})$ is situated between $\mathrm{RMS}_l(i,j,\bar{x})$ and $\mathrm{RMS}_r(i,j,\bar{x})$ so it is the best approximation for the partial marginal rate of substitution.

A better approximation for the partial marginal rates of substitution can be taken given the fact that at a decrease in consumption of a product, the left marginal utility of consumption meaning this direction, while the right marginal utility is much more useful if the consumption growth. Therefore, to estimate the changes in consumer in the direction of the decreasing for the good "i" and, obviosly for the increasing in the case of "j", we will compute **the adjusted decreasing partial marginal rate of substitution**:

$$\mathrm{RMS}_{aj,dec}(i,j,\bar{x})=-\frac{U_{ml,i}(\bar{x})}{U_{mr,j}(\bar{x})}$$

and analogously, to estimate changes in consumer in the direction of the increasing for the good "i" and for the decreasing in the case of "j", we will compute **the adjusted increasing partial marginal rate of substitution**:

$$\mathrm{RMS}_{aj,inc}(i,j,\bar{x})=-\frac{U_{mr,i}(\bar{x})}{U_{ml,j}(\bar{x})}$$

For arbitrary evolutions in consumption, we will use or (for simplicity) **the bilateral partial marginal rate of substitution**:

$$\mathrm{RMS}_b(i,j,\bar{x})=-\frac{U_{mb,i}(\bar{x})}{U_{mb,j}(\bar{x})}$$

which is between the two adjusted rates because:

$$(\mathrm{RMS}_b(i,j,\bar{x})-\mathrm{RMS}_{aj,dec}(i,j,\bar{x}))(\mathrm{RMS}_b(i,j,\bar{x})-\mathrm{RMS}_{aj,inc}(i,j,\bar{x}))=$$
$$-\frac{\left(U_{mr,j}(\bar{x})U_{mr,i}(\bar{x})-U_{ml,i}(\bar{x})U_{ml,j}(\bar{x})\right)^2}{\left(U_{ml,j}(\bar{x})+U_{mr,j}(\bar{x})\right)^2U_{mr,j}(\bar{x})U_{ml,j}(\bar{x})}<0$$

Similarly, for the discretized case, the overall marginal rate of substitution will be:

$$RMS_l(i,\bar{x}) = \frac{1}{\Delta x_i \sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} \frac{1}{(\Delta x_j)^2}}} \approx -\frac{U_{ml,i}(\bar{x})}{\sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} U_{ml,j}^2(\bar{x})}}$$

$$RMS_r(i,\bar{x}) = \frac{1}{\Delta x_i \sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} \frac{1}{(\Delta x_j)^2}}} \approx -\frac{U_{mr,i}(\bar{x})}{\sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} U_{mr,j}^2(\bar{x})}}$$

$$RMS_b(i,\bar{x}) = \frac{1}{\Delta x_i \sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} \frac{1}{(\Delta x_j)^2}}} \approx -\frac{U_{mb,i}(\bar{x})}{\sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} U_{mb,j}^2(\bar{x})}}$$

7. Exemples of Preferences

7.1. Perfectly Substitutable Goods

We say that n goods are perfectly substitutable if the utility function is linear, i.e.:

$$U(x_1,...,x_n)=a_1x_1+...+a_nx_n,\ a_i>0,\ i=\overline{1,n}$$

In this case, we have: $U_{m,i}=\frac{\partial U}{\partial x_i}=a_i$, $i=\overline{1,n}$ and the partial substitution marginal rate is:

$$RMS(i,j,\bar{x})=-\frac{a_i}{a_j}$$

whereas the overall marginal rate of substitution is:

$$RMS(i,\bar{x})=-\frac{a_i}{\sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n} a_j^2}}$$

Also the marginal substitution hyperplane between good "i" and the other goods has the equation:

$$\sum_{\substack{j=1\\j\neq i}}^{n} a_j y_j + a_i = 0$$

We can see in this case that both the marginal rate of substitution and the overall are constant. This implies that whatever is the level of consumption, the substitutability between any two goods, caeteris paribus, has the same factor. The minimal vector of the i-th good substitution is:

$$-\frac{a_i}{\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2}\left(a_1,...,\hat{a}_i,...,a_n\right)$$

7.2. Independent Goods from the Utility Point of View

We will say that n goods are independent from utility the point of view if the utility function is:

$$U(x_1,...,x_n)=f_1(x_1)+...+f_n(x_n),\ \text{with } f_i\in C^2(0,\infty),\ f_i''\le 0,\ i=\overline{1,n}\ \text{and } f_1(0)+...+f_n(0)=0$$

In this case we have:

$$U_{m,i}=\frac{\partial U}{\partial x_i}=f_i',\ i=\overline{1,n},\ \frac{\partial^2 U}{\partial x_i^2}=f_i'',\ i=\overline{1,n},\ \frac{\partial^2 U}{\partial x_i \partial x_j}=0\ \ \forall i\neq j$$

Because $d^2U=\sum_{i,j=1}^{n} f_i'' dx_i^2$ follows that U is concave.

Before proceeding further, let note that in the case of the linearity of functions f_i ($f_i(x_i)=a_ix_i$, $a_i>0$) the goods become perfectly substitutable.

The partial substitution marginal rate is: $RMS(i,j,\overline{x})=-\frac{f_i'(\overline{x}_i)}{f_j'(\overline{x}_j)}$ and the overall marginal rate of substitution: $RMS(i,\overline{x})=-\frac{f_i'(\overline{x}_i)}{\sqrt{\sum_{\substack{j=1\\j\neq i}}^{n} f_j'^2(\overline{x}_j)}}$.

Also the marginal substitution hyperplane between good "i" and the other goods has the equation:

$$\sum_{\substack{j=1\\j\neq i}}^{n} f_j'(\overline{x}_j)y_j + f_i'(\overline{x}_i)=0$$

The minimal vector of the i-th good substitution is therefore:

$$-\frac{f_i'(\overline{x}_i)}{\sum_{\substack{j=1\\j\neq i}}^{n} f_j'^2(\overline{x}_j)}\left(f_1'(\overline{x}_1),...,\hat{f}_i'(\overline{x}_i),...,f_n'(\overline{x}_n)\right)$$

7.3. Separable Goods from the Utility Point of View

We will say that n goods are separable from the utility point of view if the utility function is:

$U(x_1,...,x_n)=f_1(x_1)\cdot...\cdot f_n(x_n)$, cu $f_i\in C^2(0,\infty)$, $f_i(x)>0$ $\forall x>0$, $f_i''\leq 0$, $i=\overline{1,n}$, $f_1(0)\cdot...\cdot f_n(0)=0$

and the quadratic form:

$$H=\sum_{i=1}^{n}\frac{f_i''}{f_i}\xi_i^2+\sum_{i,j=1}^{n}\frac{f_i'f_j'}{f_if_j}\xi_i\xi_j$$

is negatively defined.

In this case, we have:

$$U_{m,i}=\frac{\partial U}{\partial x_i}=f_1...f_i'...f_n=U\frac{f_i'}{f_i},\ i=\overline{1,n},\ \frac{\partial^2 U}{\partial x_i^2}=f_1...f_i''...f_n=U\frac{f_i''}{f_i},\ i=\overline{1,n},$$

$$\frac{\partial^2 U}{\partial x_i\partial x_j}=f_1...f_i'...f_j'...f_n=U\frac{f_i'f_j'}{f_if_j}\quad\forall i\neq j$$

From the fact that H is a negative defined quadratic form, it follows that U is concave.

The partial substitution marginal rate is: $RMS(i,j,\overline{x})=-\frac{f_i'(\overline{x}_i)f_j(\overline{x}_j)}{f_j'(\overline{x}_j)f_i(\overline{x}_i)}$ and the overall marginal rate of substitution is: $RMS(i,\overline{x})=-\frac{f_i'(\overline{x}_i)}{f_i(\overline{x}_i)\sqrt{\sum_{\substack{j=1\\j\neq i}}^{n}\frac{f_j'^2(\overline{x}_j)}{f_j^2(\overline{x}_j)}}}$.

Also the marginal substitution hyperplane between the i-th good and the others has the equation:

$$\sum_{\substack{j=1\\j\neq i}}^{n}\frac{f_j'(\overline{x}_j)}{f_j(\overline{x}_j)}y_j+\frac{f_i'(\overline{x}_i)}{f_i(\overline{x}_i)}=0$$

and the minimal vector of goods substitution:

$$-\frac{f_i'(\overline{x}_i)}{f_i(\overline{x}_i)\sum\limits_{\substack{j=1\\j\neq i}}^{n}\frac{f_j'^2(\overline{x}_j)}{f_j^2(\overline{x}_j)}}\left(\frac{f_1'(\overline{x}_1)}{f_1(\overline{x}_1)},...,\frac{\hat{f}_i'(\overline{x}_i)}{f_i(\overline{x}_i)},...,\frac{f_n'(\overline{x}_n)}{f_n(\overline{x}_n)}\right)$$

Let consider, as a particular example, the Cobb-Douglas function:

$$U(x_1,x_2,...,x_n)=x_1^{\alpha_1}...x_n^{\alpha_n}\ \text{cu}\ \alpha_i>0,\ \sum_{i=1}^{n}\alpha_i\leq 1$$

Computing the marginal partial rate of substitution, we get: $RMS(i,j,\overline{x})=-\frac{\alpha_i\overline{x}_j}{\alpha_j\overline{x}_i}$

and the overall marginal rate of substitution: $RMS(i,\overline{x})=-\frac{\alpha_i}{\overline{x}_i\sqrt{\sum\limits_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha_j^2}{\overline{x}_j^2}}}$. Also the marginal substitution hyperplane between the i-th good and the others has the equation:

$$\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha_j}{\overline{x}_j}y_j+\frac{\alpha_i}{\overline{x}_i}=0$$

8. Conclusions

The onset of the notions of preference in selecting baskets of goods and the utility on the other hand impose a number of precautions both conceptual and technical. Even if such an axiomatic constraint will lead to “loss” of some important cases, the axiomatisation gives, on the one hand, rigor to the theory and, on the other hand, generates new situations by using norms more or less exotic (1-norm, ∞-norm, p-norms).

On the other hand, the analysis of the concurrent consumption of n goods variance, decontrols the theory from “caeteris paribus” constraints, getting interesting conclusions and making a first step towards the overall analysis of the microeconomic phenomenon. The n-dimensional approach to the basic phenomena, even if it is based on a number of notions of n-dimensional Euclidean geometry or, in the overall treatment of the utility function, a series of differential geometry results, adds generality and more accurate simulation of microeconomic reality.

Also the treatment of the marginal utility to the left, right or bilateral as well as marginal rates of substitution of different types, introduced above, allows enrichment practical conclusions, eliminating the classical mono-directional variations like the discretized derivative derived as the average of discretized left and right derivatives gives more precise information on the behavior of a function.

9. References

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