DEGENERATE FOLIATIONS IN SEMI-RIEMANNIAN MANIFOLDS

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Abstract: *The main notions and results concerning the linear spaces, semi-Riemannian manifolds and submanifolds have a direct link with the subject. Because the Gram-Schmidt orthogonalization is fundamental we have to proceed at his resumption in the intention to do it applicable for our demarches.*

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Introduction

The theory of Riemannian foliations has been treated during the time under various aspects.

We can cite references like [15], [24], [27], [28] or [34]. Also has been treated particular foliations like totally geodesics ([5], [9], [20], minimals ([14]) or of other types. All these results have been obtained under the generous foundation of the Riemannian geometry. Once with the development of the researches in the field of the Semi-Riemannian geometry ([1], [2], [3], [6], [17], [30], [31]) it is natural to search how we can extend all these results. It is born a new problem that concerns the study of degenerate foliations.

The main notions and results concerning the linear spaces, semi-riemannian manifolds and submanifolds have a direct link with the subject. Because the Gram-Schmidt orthogonalization is fundamental, we have proceeded at a resumption of his in the intention to do it applicable for our demarches. Many works of Semi-Riemannian geometry remind us that this procedure it is applicable also in the case of Semi-Riemannian metrics ([11], [26]). In [11] it is presented the concrete manner of orthonormal vectors construction, but the author ignores the fact that if a Gram determinant is nul all this construction stops even if we try to change the basis. In the sequel we present some aspects concerning Semi-Riemannian manifolds and fibre bundles. Also, we introduce the notions of spacelike, timelike and lightlike vectors following in this direction the paper [26].

The notion of degenerate foliation builds the transversal distribution of a foliation, notion which will substitute that of classical orthogonal distribution. We proceed also at a decomposition of these foliations following [2] in four categories: r-degenerate foliations, coisotropic, isotropic and totally degenerates. On account of specific aspects we shall work permanently with some distributions like the screen distribution, transversal screen distribution and degenerate transversal distribution. After the description of various geometric objects we shall study its behaviour at the change of the screen distribution and the change of the coordinates' neighbourhood of an arbitrary point.

We generalize the tensors presented in [27] and we clarify some problems like the integrability and the totally geodesibility of the null and screen distributions. Moreover, we shall build the Gauss-Weingarten formulae together with all geometrical objects concerned. After this demarche we shall obtain a number of characterisation theorems for the distributions or various introduced geometrical objects.

In this paper it will be defined the total geodesic degenerate foliations and totally umbilical degenerate foliations and we shall obtain some characterisation theorems. The discussion is made on the r-degenerate foliations, the results were modulated for the other types.

The final chapter gives some examples of degenerate foliations on a class of 4-manifolds endowed with a relativistic metric, which generalises the exterior Schwarzschild, Reissner-Weil, de Sitter and Minkowski metrics. There are presented four concrete examples and the last proves that on this type of manifolds does not exist totally degenerate foliations.

1. Preliminaries

Let V a linear space and $g:V\times V\to\mathbf{R}$ a symmetric bilinear form. The form g is called non-degenerate if $g(x,y)=0\ \forall y\in V\Rightarrow x=0$ and degenerate if $\exists x\neq 0$ such that $\forall y\in V\Rightarrow g(x,y)=0$. g is called positive definite (negative definite) if $g(x,x)\geq 0$

$(g(x,x)\leq 0)$ $\forall x\in V$ and $g(x,x)=0\Rightarrow x=0$ and semi-definite if $\exists x,y\in V$ such that $g(x,x)>0$ and $g(y,y)<0$.

We note (V,g) a linear space V provided with a bilinear, symmetric, non-degenerate form g. We note also W<V the fact that W is a subspace of V. The set $W^{\perp}=\{y\in V \mid g(y,x)=0\ \forall x\in W\}$ is called the orthogonal subspace of W. In general $W^{\perp}$ is not a complementary subspace of W.

Theorem 1.1 [26] Let W<(V,g). Then:

$$\dim W+\dim W^{\perp}=\dim V \tag{1.1}$$

$$(W^{\perp})^{\perp}=W \tag{1.2}$$

If g is non-degenerate on V it is not obligatory that she is non-degenerate on any subspace of V.

A subspace W<(V,g) is called non-degenerate (degenerate) subspace if the restriction $g\mid W$ is non-degenerate (degenerate).

Theorem 1.2 [26] A subspace W of (V,g) is non-degenerate if and only if $V=W\oplus W^{\perp}$.

W is non-degenerate if and only if $W\cap W^{\perp}=\{0\}$. By (1.2) and the theorem 1.2 follows that W is non-degenerate if and only if $W^{\perp}$ is non-degenerate.

A basis $B=\{e_1,...,e_n\}$ of a linear space (V,g) is called orthonormal basis if $g(e_i,e_j)=\pm\delta_{ij}$, i,j=1,...,n where δ_{ij} is the Kronecker symbol.

The Gram-Schmidt orthogonalization process.

Let (V,g) a linear space provided with a bilinear, symmetric, non-degenerate form g. Let also $B=\{v_1,...,v_n\}$ an arbitrary basis of V, composed by non-null vectors ($g(v_i,v_i)\neq 0$, i=1,...,n). We shall determine by depart of B an orthonormal basis of V.

Let therefore $w_1=\dfrac{v_1}{\sqrt{|g(v_1,v_1)|}}$. We have $g(w_1,w_1)=\dfrac{g(v_1,v_1)}{|g(v_1,v_1)|}=\varepsilon_1=\pm 1$. Let suppose that we have determined the vectors $w_1,...,w_{p-1}$ such that $g(w_i,w_j)=0$, $i,j=1,...,p-1$, $i\neq j$ and $g(w_i,w_i)=\varepsilon_i\in\{-1,1\}$, $i=1,...,p-1$. Let:

$$(1.3)\qquad w_p = \varepsilon_1 ... \varepsilon_{p-1}\sqrt{\frac{\varepsilon_p}{g(v_p,v_p)-\sum_{i=1}^{p-1}\varepsilon_i g^2(w_i,v_p)}}\left(v_p-\sum_{j=1}^{p-1}\varepsilon_j g(w_j,v_p)w_j\right)$$

if $g(v_p,v_p)\neq \sum_{i=1}^{p-1}\varepsilon_i g^2(w_i,v_p)$ and $\varepsilon_p\in\{-1,1\}$ such that the square root be definite. We have $g(w_p,w_p)=\varepsilon_p$ and $g(w_p,w_i)=0$, i=1,...,p-1. Let now $W_{p-1}^{\perp}=\mathrm{Span}(w_1,...,w_{p-1})^{\perp}$ where Span(...) is the subspace generate by the respective vectors. The subspace $W_{p-1}=\mathrm{Span}(w_1,...,w_{p-1})$ is non-degenerate. Indeed, let suppose that there is $x\in W_{p-1}$, $x\neq 0$ such that $g(x,y)=0$ $\forall y\in W_{p-1}$. Let $x=\sum_{i=1}^{p-1}\alpha_i w_i\neq 0$. We have $g(x,w_k)=g(\sum_{i=1}^{p-1}\alpha_i w_i,w_k)=\alpha_k\varepsilon_k$ therefore $\alpha_k\varepsilon_k=0$ that is $\alpha_k=0$, k=1,...,p-1. Accordingly x=0 therefore contradiction. By the theorem 1.2 we have that $V=W_{p-1}\oplus W_{p-1}^{\perp}$. Let now $v_p=\sum_{i=1}^{p-1}c_i w_i+z_p$ where $z_p\in W_{p-1}^{\perp}$ (the decomposition being unique by the direct sum). We have:

$$g(v_p,v_p)-\sum_{i=1}^{p-1}\varepsilon_i g^2(v_p,w_i)=g(z_p,z_p)$$

If $g(z_p,z_p)\neq 0$ then (1.3) is applicable. If $g(z_p,z_p)=0$ we do a parmutation of the vectors $\{v_p,...,v_n\}$. If $\exists k\in\{p,...,n\}$ such that $g(z_k,z_k)\neq 0$ after a possible renumbering we can apply (1.3). If $\forall k\in\{p,...,n\}\Rightarrow g(z_k,z_k)=0$ where $z_k=pr_{W_{p-1}^{\perp}}v_k$, k=p,...,n (the projection of v_k on $W_{p-1}^{\perp}$) then $\exists k,r=p,...,n$ with $k\neq r$ such that $g(z_k,z_r)\neq 0$. Indeed, if $\forall k,r=p,...,n\Rightarrow g(z_k,z_r)=0$ then how $\{z_p,...,z_n\}$ constitutes a basis of $W_{p-1}^{\perp}$ follows that $W_{p-1}^{\perp}$ is degenerate therefore contradiction. Let therefore, after a possible renumbering, z_p and z_{p+1} such that $g(z_p,z_{p+1})\neq 0$. Let now:

$$\overline{v_p}=av_p+bv_{p+1}\text{ with }a,b\neq 0$$

We have $g(\overline{v_p},\overline{v_p})-\sum_{i=1}^{p-1}\varepsilon_i g^2(\overline{v_p},w_p)=2ab\,g(z_p,z_{p+1})\neq 0$ therefore we can apply (1.3) for $v_p\to\overline{v_p}$.

Finally, for p=n follows trivial $g(z_n,z_n)\neq 0$ because in the opposite case $W_{n-1}^{\perp}=Span(z_n)$ is degenerate therefore contradiction.

If we consider now the orthonormal basis $B=\{e_1,...,e_n\}$ of (V,g) and we note $\varepsilon_i=g(e_i,e_i)$, i=1,...,n follows:

$$x=\sum_{i=1}^{n}\varepsilon_i g(x,e_i)e_i \ \forall x\in V \tag{1.4}$$

Let (V,g) a linear spece. We call the index of g: $\nu=\text{ind } g=\max\{\dim W \mid W<V,\ g\mid W \text{ is negative definite}\}$. We shall write sometimes ν=ind V.

Lemma 1.1 [26] Let (V,g) a linear space and W a non-degenerate subspace of V. Then

$$\text{ind } V=\text{ind } W+\text{ind } W^{\perp} \tag{1.5}$$

Remark In general the inequality holds: ind V≥ind W+ind $W^{\perp}$ $\forall W<V$.

Lemma 1.2 [26] Let (V,g) a linear space. Then there is a subspace W<V of maximal dimension=min {ind g, dim V-ind g} such that $g\mid W=0$.

In what follows we suppose that all the differentiable manifolds have the metrics with constant index on them and all the geometrical objects are of infinite class.

Let a Semi-Riemannian manifold (M,g). A tangent vector $X\in T_pM$, $p\in M$ is called spacelike vector if g(X,X)>0 or X=0, lightlike vector if g(X,X)=0 and $X\neq 0$ and timelike vector if g(X,X)<0. The collection of lightlike vectors of T_pM is called the null cone in $p\in M$.

2. Degenerate foliations of the Semi-Riemannian manifolds

Let (M,g) a Semi-Riemannian manifold, (m+n)-dimensional, m,n≥1, g being the semi-riemannian metric on M.

Let q the index of the metric g which we shall suppose constant on M. If q=0 or q=m+n then the metric is riemannian. How in this case the induced metric on any leaf of the foliation is also riemannian follows that if we want to talk about degeneration we shall suppose that $1\leq q\leq m+n-1$. Therefore M is not a Riemannian manifold.

Definition 2.1 A degenerate foliation of codimension m of M is a decomposition of M into a disjoint union of conected, degenerate submanifolds of codimension m of M, called leafs of the foliation such that for any $p\in M$ there is a neighbourhood U of p in M and a submersion $f_U:U\to\mathbf{R}^m$ with the property: $\forall\, x\in\mathbf{R}^m$, $f_U^{-1}(x)$ is a leaf of the restriction of foliation at U, $F\big|_U$.

We shall consider in what follows like coordinates neighbourhoods of any point $p\in M$ the collections U by the upper definition.

Considering now a degenerate foliation of codimension m of M, let:

$$T(F)=\bigcup_{\substack{p\in M\\ \text{L the leaf of F}\\ \text{which contains p}}} T_pL$$

We shall show now that T(F) is a fibre bundle of rank n on M.

Let $p\in M$ and U a neighbourhood of p in M such there is a submersion $f_U:U\to\mathbf{R}^m$ with the property that $\forall\, x\in\mathbf{R}^m$, $f_U^{-1}(x)$ is a leaf of the restriction of the foliation on U, $F\big|_U$.

Considering now the leaf L passing through $p\in M$ we define:

$\pi:T(F)\to\mathbf{R}^m$, $\pi(T_pL)=f_U(p)\ \forall p\in M$

The map π is correct defined because by any leaf L of F corresponds an unique $x\in\mathbf{R}^m$ such that $L=f_U^{-1}(x)$. Indeed, if we suppose that $\exists x\neq y\in\mathbf{R}^m$ such that $L=f_U^{-1}(x)=f_U^{-1}(y)$ then $x=f_U(f_U^{-1}(x))=f_U(L)=f_U(f_U^{-1}(y))=y$ from where follows contradiction. On the other hand $f_U(p)\in f_U(L)=f_U(f_U^{-1}(x))=\{x\}$ otherwise: $f_U(p)=x$. We have also that the map π does not depend on the coordinates neighbourhood U. Indeed, if we shall consider U and V neighbourhoods of $p\in M$ satifying the definition conditions and the submersions $f_U:U\to\mathbf{R}^m$, $f_V:V\to\mathbf{R}^m$ then $\forall x\in\mathbf{R}^m$ follows that $f_U^{-1}(x)$ is a leaf of the restriction of the foliation on U and $f_V^{-1}(x)$ is a leaf of the restriction of the foliation on V. How through $p\in M$ pass a unique leaf follows that $f_U^{-1}(x)\cap f_V^{-1}(x)$ is a restriction of the foliation on $U\cap V$. But $p\in U\cap V\subset U,V$ follows $f_U(p)=f_V(p)=x$.

We have now that for any $x\in\mathbf{R}^m$: $\pi^{-1}(x)=T_pL$, $p\in M$ such that $f_U(p)=x$ is a real linear space of dimension n.

Let $p\in M$ and L the leaf passing through p. Let consider also a neighbourhood U of p and the submersion $f_U:U\to\mathbf{R}^m$ with the property that $L=f_U^{-1}(x)$ for a fixed point $x\in\mathbf{R}^m$. Let also a basis $\{e_1(p),...,e_n(p)\}$ of T_pL.

We define the diffeomorphism:

$\phi:\pi^{-1}(f_U(U))\to f_U(U)\times\mathbf{R}^n$, $\phi(v)=(f_U(q),v_1,...,v_n)$ $\forall q\in U$ $\forall v=v^1e_1(q)+...+v^ne_n(q)\in T_qL$

If we note with pr_1 the projection on the first component we have $pr_1(\phi(v))=f_U(q)=\pi(v)$ $\forall v\in T_qL$ $\forall q\in U$ and the map $\phi_q:T_qL\to f_U(q)\times\mathbf{R}^n$, $\phi(v)=(v_1,...,v_n)$ $\forall q\in U$ $\forall v=v^1e_1(q)+...+ v^ne_n(q)\in T_qL$ is simply an **R**-isomorphism.

We have therefore proved that $(T(F),\pi,\mathbf{R}^n)$ is a fibre bundle of rank n.

Definition 2.2 Considering the vector bundle which we have build we shall say that T(F) is the fibre bundle tangent to the foliation F on M.

From the definition of F, follows that T(F) is an integrable distribution.

Let now L a leaf of F passing through $p\in M$. Considering

$T_p(L)^\perp=\{X_p\in T_pM \mid g(X_p,Y_p)=0,\ \forall Y_p\in T_pL\}$

we have that $T_p(L)^\perp$ is also degenerate. Let $T(F)^\perp=\bigcup_{L\in F}T_p(L)^\perp$. Like in the preceding construction we can show that $T(F)^\perp$ is a fibre bundle on $\mathbf{R}^m$ called the normal fibre bundle of the foliation F .

Considering now the fibre bundle T(F) of a degenerate foliation F and join to any point $p\in M$ the tangent space $T_p(L)$ at the leaf L passing through p we shall obtain an n-dimensional integrable distribution on M noted in what follows with D_F and called the distribution asociated to the degenerate foliation. Because the distribution D_F is integrable follows that she is involutive that is $\forall X,Y\in D_F \Rightarrow[X,Y]\in D_F$. Considering now D_F let $D_F^\perp$ the orthogonal distribution of D_F in TM. Is obvious that $D_F^\perp$ is obtained also by the association at any point $p\in M$ of the orthogonal space $T_p(L)^\perp$ of $T_p(L)$ relative of the leaf L passing through p.

Let now $p\in M$ and U a coordinates neighbourhood of p in M. Considering a coordinates system in $p\in M$: $(x^1,...,x^{n+m})$ follows by the definition that there is a submersion $f_U:U\to\mathbf{R}^m$ with the property that for any $x=(a^1,...,a^m)\in\mathbf{R}^m$, the leaf of the restriction of the foliation at U is given by the equations:

$$x^{n+1}=a^1,...,x^{n+m}=a^m$$

If we consider another coordinates system $(y^1,...,y^{n+m})$ in U follows:

$$y^i=\frac{\partial y^i}{\partial x^k}x^k\text{ , i=1,...,n+m}$$

How y^{n+i}=constant, i=1,...,m follows $\frac{\partial y^{n+i}}{\partial x^k}=0$, i=1,...,m, k=1,...,n.

The structural group consists by the matrices of the form:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A\in M_n(\mathbf{R})$ and $C\in M_m(\mathbf{R})$ are non-singular and $B\in M_{nm}(\mathbf{R})$.

Let now $T(M)|_U$ the restriction on U of the tangent bundle of the manifold M and $\{X_1,...,X_n,Y_{n+1},...,Y_{n+m}\}$ a basis for the local sections of $T(M)|_U$. If V is another neighbourhood of p in M such that $U\cap V\neq\emptyset$ and $\{X'_1,...,X'_n,Y'_{n+1},...,Y'_{n+m}\}$ is a basis for the local sections of $T(M)|_V$ then according to the structural group we have:

(2.1) $$X_i^{'}=\sum_{k=1}^{n}A_i^kX_k+\sum_{\beta=n+1}^{n+m}B_i^\beta Y_\beta$$

(2.2) $$Y_\alpha^{'}=\sum_{\beta=n+1}^{+m}C_\alpha^\beta Y_\beta$$

$\forall$ i=1,...,n $\forall\alpha$=n+1,...,n+m, A_i^k, B_i^β, C_α^β being arbitrary maps, indefinite differentiable on $U\cap V$ satisfying in addition the condition that the matrices $A=(A_i^k)$ and $C=(C_\alpha^\beta)$ being non-singular.

From now on, if we shall introduce new geometrical objects we shall verify the invariability of them at the transforming (2.1) and (2.2).

From the degeneration of the foliation F follows that the intersection of the distributions D_F and $D_F^{\perp}$ is non-null therefore T_pL and $T_pL^{\perp}$ are orthogonal non-complementary degenerate subspaces in T_pM $\forall p\in M$, L being the leaf passing through $p\in M$.

We define now: $N = D_F \cap D_F^{\perp}$ named accordingly with [2] the null distribution of M appropriate the foliation F .

Let r=dim N . By the lemma 1.2 follows that $r\leq \min\{q,m+n-q\}$ and how $N \subset D_F$, $N \subset D_F^{\perp}$ follows that $r\leq \min\{q,n,m,m+n-q\}$. We can consider always (taking possible -g like metric on M) that we have: $q\leq \left[\frac{m+n}{2}\right]$ (where [a] is the bigest integer less then a). Because $q\leq m+n-q$ follows that $1\leq r\leq \min\{q,n,m\}\leq \min\{n,m\}$ from where:

$$1\leq r\leq \min\{n,m\}$$

Definition 2.3 The foliation F of M is called r-degenerate foliation (or degenerate foliation if the rank r is undercurrent from context) if the null distribution is of dimension r.

If we consider the bracket $[X,Y]$ $\forall X,Y\in N$ follows that the null distribution is not necessary integrable.

From this reason we shall distinguish in what follows two important cases: N is an integrable distribution or N is not integrable.

Let suppose now that N is an integrable distribution. If we consider in M an open neighbourhood U and an adapted basis for the null distribution: $\{\xi_1,\dots,\xi_r,X_{r+1},\dots,X_n\}$ where ξ_i are vector fields defined on U hwo generates N and X_j vector fileds defined on U which complete the basis for D_F follows that N_p $\forall p\in U$ is the tangent space for a submanifold of the leaf L passing through $p\in U$.

The problem is now what is happend at the intersection of two coordinates neighbourhoods of an arbitrary point $p\in M$. If U and V are two such neighbourhoods such that $U\cap V\neq\emptyset$ let consider $\{\xi_1,\dots,\xi_r,X_{r+1},\dots,X_m\}$ a basis for $D_F\big|_U$ and $\{\xi'_1,\dots,\xi'_r,X'_{r+1},\dots,X'_m\}$ a basis for $D_F\big|_V$.

That $\{\xi'_1,...,\xi'_r\}$ be a basis for $N|_V$ it must that $\xi'_i = \sum_{j=1}^{r} \alpha_i^j \xi_j$. In that case $[\xi'_i, \xi'_j] = \sum_{k,p=1}^{r} \{\alpha_i^k \alpha_j^k [\xi_k, \xi_p] + \alpha_i^k \xi_k (\alpha_j^p) \xi_p - \alpha_j^p \xi_p (\alpha_i^k) \xi_k\}$ and with the integrability of $N|_U$ follows that $N|_V$ is also integrable. Therefore the integrability of N in a point $p\in M$ does not depend of its coordinates neighbourhood.

We see upper that for a leaf L passing through $p\in M$ the subspaces T_pL and $T_p(L)^{\perp}$ are not complementary. In order that we can introduce similar notions to the geometry of the nondegenerate foliations it is necessary the construction of a distribution complementary to those of the foliation, called the transversal distribution, hwo is different from that orthogonal.

In order that we can build now the transversal fibre bundle of a degenerate foliation it is necessary to distinguish between four cases: **I.**$1\leq r<\min\{m,n\}$; **II.**$1\leq r=m<n$; **III.**$1\leq r=n<m$; **IV.**$1\leq r=m=n$.

Case I. $1\leq r<\min\{m,n\}$ In that case the foliation is called r-degenerate foliation the danger of confusion being discarded because we shall specify always if it is the general case or those particular.

Let consider now S (F) the complementary distribution orthogonal to N in D_F. We call it, in agreement with [2] S (F)-the screen distribution of the foliation F . We have therefore the direct orthogonal sum:

$$D_F = N \perp S(F) \tag{2.3}$$

The screen distribution S (F) is nondegenerate relative to g. Indeed, if $\exists Z\in S(F)$ such that $g(Z,Y)=0$ $\forall Y\in S(F)$ then like $Z\in D_F$ we have also $g(Z,\xi)=0$ $\forall\xi\in N$. It follows therefore that $g(Z,X)=0$ $\forall X\in D_F$ hwo imply the fact that $Z\in N$. But this fact comes into contradiction with $N\cap S(F)=\{0\}$.

We shall suppose in what follows that ind(g) is constant on S (F).

Remark The screen distribution S (F) is not unique determined by N in D_F therefore from this reason every time we shall obtain a result we shall examine the relationship of this from S (F).

Let now the complementary distribution of N in the orthogonal distribution $D_F^{\perp}$ marked with $S(F^{\perp})$ and called the transversal screen distribution of the foliation F .

Like in the case of the screen distribution this is nondegenerate relative to g. We have therefore the orthogonal decomposition:

(2.4) $$D_F^{\perp} = N \perp S(F^{\perp})$$

Because S (F) is nondegenerate in TM we consider the decomposition:

(2.5) $$TM = S(F) \perp S(F)^{\perp}$$

where $S(F)^{\perp}$ is the complementary distribution, orthogonal to S (F) in TM. We have therefore, finally the following decomposition:

(2.6) $$S(F)^{\perp} = S(F^{\perp}) \perp S(F^{\perp})^{\perp}$$

where $S(F^{\perp})^{\perp}$ is the complementary distribution, orthogonal to $S(F^{\perp})$ in $S(F)^{\perp}$.

Considering now $\xi \in N$ follows from (2.3) $\xi \perp S(F)$ therefore from (2.5) we have: $\xi \in S(F)^{\perp}$. From (2.4) follows $\xi \perp S(F^{\perp})$. Finally, from the decomposition (2.6) follows that $\xi \in S(F^{\perp})^{\perp}$. We have therefore $N \subset S(F^{\perp})^{\perp}$.

We shall note from now on, a r-degenerate foliation with:

$(F, g, S(F), S(F^{\perp}))$.

Remark From the fact that dim N =r we have:

$\dim S(F) = n-r$, $\dim S(F)^{\perp} = m+r$, $\dim S(F^{\perp}) = m-r$, $\dim S(F^{\perp})^{\perp} = 2r$

Lemma 2.1 Let $(F, g, S(F), S(F^{\perp}))$ a r-degenerate foliation of a Semi-Riemannian manifold (M,g). If U is an open set of M and $\{\xi_1,...,\xi_r\}$ is a basis of $N|_U$ then there are vector fields $\{N_1,...,N_r\}$ from $S(F^{\perp})^{\perp}|_U$) such that:

(2.7) $$g(N_i,\xi_j)=\delta_{ij}$$

(2.8) $$g(N_i,N_j)=0$$

$\forall$ i,j=1,...,r.

Proof. Let consider the distribution H complementary to N in S $(F^{\perp})^{\perp}$ and a basis $\{V_1,...,V_r\}$ of $H\,|_U$. Relative to the decomposition S $(F^{\perp})^{\perp}=N \perp H$ the vector fileds N_i have the expressions:

(2.9) $$N_i = \sum_{k=1}^{r}\left(\alpha_i^k \xi_k + \beta_i^k V_k\right),\ i=1,...,r$$

where α_i^k and β_i^k are smooth mapings on U. We shall define the matrices of r-order: $A=(\alpha_i^j)$, $B=(\beta_i^j)$, $C=(g(V_i,\xi_j))$, $D=(g(V_i,V_j))$. In order that N_i satisfy the relations (2.7), (2.8) it must that:

$$\delta_{ij} = g(N_i,\xi_j) = g(\sum_{k=1}^{r}\alpha_i^k\xi_k + \sum_{k=1}^{r}\beta_i^k V_k, \xi_j) = \sum_{k=1}^{r}\alpha_i^k g(\xi_k,\xi_j) + \sum_{k=1}^{r}\beta_i^k g(V_k,\xi_j) =$$
$$\sum_{k=1}^{r}\beta_i^k g(V_k,\xi_j)$$
$$0 = g(N_i,N_j) = g(\sum_{k=1}^{r}\alpha_i^k\xi_k + \sum_{k=1}^{r}\beta_i^k V_k, \sum_{s=1}^{r}\alpha_j^s\xi_s + \sum_{s=1}^{r}\beta_j^s V_s) = \sum_{k=1}^{r}\sum_{s=1}^{r}\alpha_i^k\alpha_j^s g(\xi_k,\xi_s) +$$
$$\sum_{k=1}^{r}\sum_{s=1}^{r}\alpha_i^k\beta_j^s g(\xi_k,V_s) + \sum_{k=1}^{r}\sum_{s=1}^{r}\beta_i^k\alpha_j^s g(V_k,\xi_s) + \sum_{k=1}^{r}\sum_{s=1}^{r}\beta_i^k\beta_j^s g(V_k,V_s) =$$
$$\sum_{k=1}^{r}\sum_{s=1}^{r}\alpha_i^k\beta_j^s g(\xi_k,V_s) + \sum_{k=1}^{r}\sum_{s=1}^{r}\beta_i^k\alpha_j^s g(V_k,\xi_s) + \sum_{k=1}^{r}\sum_{s=1}^{r}\beta_i^k\beta_j^s g(V_k,V_s)$$

With the matrices upper introduced, these relations become:

(2.10) $$BC=I$$

(2.11) $$AC^tB^t+BCA^t+BDB^t=0$$

where I is the identity of $M_r(\mathbf{R})$ and ^{t} describes the transpose of a matrix. Because S $(F^{\perp})^{\perp}$ is nondegenerate follows that C is invertible therefore from (2.10):

(2.12) $$B=C^{-1}$$

From (2.11), (2.12) follows:

(2.13) $$A+A^t= -C^{-1}D(C^{-1})^t$$

therefore:

(2.14) $$A = -\frac{1}{2}C^{-1}D(C^{-1})^t + S$$

for any skew-symmetrical matrix S of r-order. From (2.9) we have:

(2.15) $$N = \left(-\frac{1}{2}C^{-1}D(C^{-1})^t + S\right)\xi + C^{-1}V$$

where we note $N=(N_1,...,N_r)^t$, $\xi=(\xi_1,...,\xi_r)^t$, $V=(V_1,...,V_r)^t$.

From (2.7) and (2.8) follows easy that $\{\xi_1,...,\xi_r,N_1,...,N_r\}$ is a basis of $S(F^{\perp})^{\perp}|_U$.

Remark From (2.15) follows that the vector fields N_i, i=1,...,r are not unique determined, they depending by the arbitrary choice of the matrix S.

Theorem 2.1 Let (F, g, S (F), S (F $^{\perp}$)) a r-degenerate foliation of a Semi-Riemannian manifold (M,g). There is a complementary distribution of N in $S(F^{\perp})^{\perp}$ marked with deg(F) and called the degenerate transversal distribution of the foliation F relative to S (F) and S (F $^{\perp}$) such that the vector fields $\{N_1,...,N_r\}$ defined in lemma 2.1 is a basis for deg(F)).

Proof. From lemma 2.1 let $\{N_1,...,N_r\}$ be definite through (2.15). Considering another open set U' of M such that $U\cap U'\neq\emptyset$ let $\{\xi'_1,...,\xi'_r\}$ and $\{V'_1,...,V'_r\}$ basis of $N|_{U'}$ respectively $H|_{U'}$. If we note like in lemma 2.1: $N'=(N_1',...,N_r')^t$, $\xi'=(\xi_1',...,\xi_r')^t$, $V'=(V_1',...,V_r')^t$ we have $\xi'=E\xi$ and $V'=FV$ where E and V are nonsingular matrices. With the notations $C'=(g(V_i',\xi_j'))$, $D'=(g(V_i',V_j'))$ we have on $U\cap U'$:

(2.16) $$C'=FCE^t$$

(2.17) $$D'=FDF^t$$

From (2.15) we have on U':

(2.18) $$N' = \left(-\frac{1}{2}C'^{-1}D'(C'^{-1})^t + S'\right)\xi' + C'^{-1}V'$$

with S' skew-symmetrical matrix. Using (2.16) and (2.17) in (2.18) we have:

(2.19) $$N' = E^{-1}\left[\left(-\frac{1}{2}C^{-1}D\left(C^{-1}\right)^{t} + E^{t}S'E\right)\xi + C^{-1}V\right]$$

If we choose $S'=(E^{-1})^{t}SE^{-1}$ which is also skew-symmetric, we have

(2.20) $N'=E^{-1}N$

From (2.20) follows that there is the distribution deg(F) generated by $\{N_1,...,N_r\}$ from lemma 2.1.

Let now show that deg(F) is complementary to N in S $(F^{\perp})^{\perp}$. If we suppose the reverse, let $0\neq X\in N \cap \deg(F)$. Considering a basis like upper we have: $X=\sum_{i=1}^{r}a^{i}N_{i}=\sum_{j=1}^{r}b^{j}\xi_{j}$. Using (2.7) and (2.8) we have that:

$$0=g(\sum_{j=1}^{r}b^{j}\xi_{j},\xi_{i})=g(X,\xi_{i})=g(\sum_{j=1}^{r}a^{j}N_{j},\xi_{i})=a^{j}$$

therefore X=0-contradiction with our suppose. How dim deg(F)=r and ind$\{N_1,...,N_r\}$ follows that the set of vector fields $\{N_1,...,N_r\}$ is a basis of (deg(F)).

Remark From the theorem 2.1 we conclude that dim (deg(F))=r.

If we return now to the beginning problem, that is the replacement of the classical orthogonal distribution with a complementary distribution to D_F in TM, let therefore the ortogonal direct sum:

(2.21) $tr(F)=\deg(F)\perp S(F^{\perp})$

where deg(F) is an arbitrary degenerate transversal distribution of F . From (2.21) follows that tr(F) is a distribution on M named the transversal distribution of the foliation F .

The dimension of this distribution is therefore:

dim tr(F)=dim deg(F)+dim S $(F^{\perp})$=r+n-r=n

Finally, we have the decomposition:

$$\text{(2.22)} \qquad TM = D_F \oplus tr(F) = S(F) \perp S(F^{\perp}) \perp (N \oplus deg(F))$$

From the upper considerations we have on TM a local quasi-orthonormal basis along to F in an open neighbourhood U: $\{X_{r+1},...,X_n, W_{r+1},...,W_m,\xi_1,...,\xi_r,N_1,...,N_r\}$ where $X_\alpha \in S(F)|_U$, $\alpha=r+1,...,n$, $W_a \in S(F^{\perp})|_U$, $a=r+1,...,m$, $\xi_i \in N|_U$, $i=1,...,r$, $N_i \in deg(F)|_U$, $i=1,...,r$.

From now on we shall make the understanding about the indexes: $\alpha,\beta,...=r+1,...,n$; $a,b,...=r+1,...,m$; $i,j,...= 1,...,r$.

On $(N \oplus deg(F))|_U$ we have an orthonormal basis:

$\left\{u_i = \frac{\xi_i - N_i}{\sqrt{2}}, v_i = \frac{\xi_i + N_i}{\sqrt{2}}\right\}_{i=1,...,r}$ and how $g(u_i,u_i)=-1$, $g(v_i,v_i)=1$, $i=1,...,n$ follows that the index of $(N \oplus deg(F))|_U=r$. Because $N \oplus deg(F)$ is nondegenerate, by the lemma 1.1 and (2.22) we have:

$$\text{(2.23)} \qquad q=ind(S(F))+ind(S(F^{\perp}))+r$$

Theorem 2.2 Let $(F, g, S(F), S(F^{\perp}))$ a r-degenerate foliation of a Semi-Riemannian manifold (M,g). If the index of the manifold M and those of the null distribution N are equals nule, then $S(F)$ and $S(F^{\perp})$ are Riemannian distributions.

Proof. We consider in (2.23) q=r from where $ind(S(F))+ind(S(F^{\perp}))=0$ therefore $ind(S(F))=ind(S(F^{\perp}))=0$.

Corollary 2.1 Let $(F, g, S(F), S(F^{\perp}))$ a r-degenerate foliation of a Lorentz manifold (M,g). Then $S(F)$ and $S(F^{\perp})$ are Riemannian distributions.

Proof. On a Lorentz manifold we have q=1 and how $1 \le r \le q$ follows r=1. The assertion reduce to the check of the theorem 2.2.

Case II. $1 \le r=m<n$ In this case the foliation is called coisotropic foliation. How $N \subset D_F^{\perp}$ and $\dim N = r = m = \dim D_F^{\perp}$ follows that $N = D_F^{\perp}$ therefore $S(F^{\perp})=\{0\}$. Considering now the screen distribution $S(F)$ we have: $D_F = S(F) \perp D_F^{\perp}$.

We note from now on a coisotropic foliation with $(F, g, S(F))$.

Remark From the fact that dim N =r=m we have:

dim S (F)=n-m, dim S (F $)^{\perp}$=dim S (F $^{\perp})^{\perp}$=2m

Similar to the proofs of lemma 2.1 and of theorem 2.1 follows:

Lemma 2.2 Let (F, g, S (F)) a coisotropic foliation of a Semi-Riemannian manifold (M,g). If U is an open set from M and $\{\xi_1,...,\xi_m\}$ a basis of $D_F{}^{\perp}|_U$ then there is a system of vector fields $\{N_1,...,N_m\}$ of S (F $)^{\perp}|_U$ such that

(2.24) $$g(N_i,\xi_j)=\delta_{ij}$$

(2.25) $$g(N_i,N_j)=0$$

$\forall$ i,j=1,...,m.

Theorem 2.3 Let (F, g, S (F)) a coisotropic foliation of a Semi-Riemannian manifold (M,g). Then there is a complementary distribution of $D_F{}^{\perp}$ in S (F $^{\perp})^{\perp}$ noted with deg(F) and called the degenerate transversal distribution of the foliation F relative to S (F) such that the system of vector fields $\{N_1,...,N_m\}$ introduced in lemma 2.2 is a basis of deg(F).

The transversal distribution of F becomes:

(2.26) $$tr(F)=deg(F)$$

and the decomposition of TM is:

(2.27) $$TM=S(F)\perp D_F{}^{\perp}\oplus deg(F))$$

The local quasi-orthonormal basis along F in an open neighbourhood U is: $\{X_{m+1},...,X_n,\xi_1,...,\xi_m,N_1,...,N_m\}$ where $X_\alpha\in S(F)|_U$, α=m+1,...,n, $\xi_i\in D_F{}^{\perp}|_U$, i=1,...,m and $N_i\in deg(F)|_U$, i=1,...,m.

Relative to the index follows with the same remark like those preceding the theorem 2.2:

(2.28) $$q=m+ind(S(F))$$

From (2.28) follows:

Theorem 2.4 Any screen distribution of a coisotropic foliation in a Semi-Riemannian manifold has constant index q-m.

Corollary 2.2 In a coisotropic foliation of a Semi-Riemannian manifold with the index equal of those of the orthogonal distribution, the screen distribution becomes Riemannian.

Case III. 1≤r=n<m In this case the foliation is called isotropic foliation. How N $\subset D_F^{\perp}$ and dim N =r=n=dim D_F follows that N $=D_F$ therefore S (F)={0}. Considering the transversal screen distribution S (F $^{\perp}$) we have:

$D_F^{\perp}=D_F \perp S(F^{\perp})$.

We shall note from now on an isotropic foliation with (F, g, S (F $^{\perp}$)).

Remark From the fact that dim N =r=n we have:

dim S (F)$^{\perp}$=m+n, dim S (F $^{\perp}$)=m-n, dim S (F $^{\perp}$)$^{\perp}$=2n

Remark In the case of an isotropic foliation N $=D_F$ therefore N is an integrable distribution.

Similar to the proofs of lemma 2.1 and of theorem 2.1 follows:

Lemma 2.3 Let (F, g, S (F $^{\perp}$)) an isotropic foliation of a Semi-Riemannian manifold (M,g). If U is an open set of M and $\{\xi_1,...,\xi_n\}$ is a basis of $D_F|_U$ then there is a system of vector fields $\{N_1,...,N_n\}$ of $S(F^{\perp})^{\perp}|_U$ such that

(2.29) $g(N_i,\xi_j)=\delta_{ij}$

(2.30) $g(N_i,N_j)=0$

$\forall$ i,j=1,...,n.

Theorem 2.5 Let (F, g, S (F $^{\perp}$)) an isotropic foliation of a Semi-Riemannian manifold (M,g). Then there is a complementary distribution to D_F in S (F $^{\perp}$)$^{\perp}$ noted with deg(F) and called the degenerate transversal distribution of the foliation F relative to $S(F^{\perp})$ such that the system of vector fields $\{N_1,...,N_n\}$ defined in lemma 2.3 is a basis of deg(F)).

The transversal distribution of F is now:

(2.31) $$tr(F)=deg(F)\perp S(F^{\perp})$$

and the decomposition of TM becomes:

(2.32) $$TM=D_F\oplus deg(F))\perp S(F^{\perp})$$

The local quasi-orthonormal basis along F in an open neighbourhood U is: $\{\xi_1,...,\xi_n,N_1,...,N_n,W_{n+1},...,W_m\}$ where $W_a\in S(F^{\perp})|_U$, $a=n+1,...,m$, $\xi_i\in D_F|_U$, $i=1,...,n$ and $N_i\in deg(F)|_U$, $i=1,...,n$.

With the same remark like those preceding the theorem 2.2 we have:

(2.33) $$q=n+ind(S(F^{\perp}))$$

From (2.33) follows:

Theorem 2.6 Any transversal screen distribution of an isotropic foliation in a Semi-Riemannian manifold has constant index q-n.

Corollary 2.3 In an isotropic foliation of a Semi-Riemannian manifold with index equal with those of the foliation's distribution, the transversal screen distribution becomes Riemannian.

Case IV. $1\leq r=m=n$ In this case, the foliation is called totally degenerate foliation. How $N\subset D_F$ and $N\subset D_F^{\perp}$ and $\dim N=r=m=n=\dim D_F=\dim D_F^{\perp}$ follows that $N=D_F=D_F^{\perp}$ therefore $S(F)=S(F^{\perp})=\{0\}$.

We note from now on a totally degenerate foliation with (F, g).

Remark From the fact that dim N =r=n=m we have:

$\dim S(F)^{\perp}=2m$, $\dim S(F^{\perp})^{\perp}=2m$

Remark In the case of a totally degenerate foliation $N=D_F$ therefore N is an integrable distribution.

We have now analogously with lemma 2.1 and theorem 2.1:

Lemma 2.4 Let (F, g) a totally degenerate foliation of a Semi-Riemannian manifold (M,g). If U is an open set of M and $\{\xi_1,...,\xi_m\}$ is a basis of $D_F|_U$ then there is a system of vector fields $\{N_1,...,N_m\}$ of $TM|_U$ such that

(2.34) $$g(N_i,\xi_j)=\delta_{ij}$$

(2.35) $$g(N_i,N_j)=0$$

$\forall$ i,j=1,...,m.

Theorem 2.7 Let (F, g) a totally degenerate foliation of a Semi-Riemannian manifold (M,g). Then there is a complementary distribution of D_F in TM noted with deg(F) and called the degenerate transversal distribution of the foliation F such that the system of vector fields $\{N_1,...,N_m\}$ defined in lemma 2.4 is a basis of deg(F).

The transversal distribution of F is now:

(2.36) $$tr(F)=deg(F)$$

and the decomposition of TM becomes:

(2.37) $$TM=D_F\oplus deg(F)$$

The local quasi-orthonormal basis along F in an open neighbourhood U of M is: $\{\xi_1,...,\xi_m,N_1,...,N_m\}$ where $\xi_i\in D_F|_U)$, i=1,...,m and $N_i\in deg(F)|_U)$, i=1,...,m.

With the same remark like those preceding the theorem 2.2 we have:

(2.38) $$q=m$$

From (2.38) we have:

Theorem 2.8 A degenerate foliation of a Semi-Riemannian manifold can be totally degenerate only if the codimension of the foliation is equal with the index of the manifold.

Finally in this section we shall investigate two problems:

- Which are the conversion formulae of a local quasi-orthonormal basis along F in a coordinate neighbourhood U when we change the screen distribution?
- Which are the conversion formulae of a local quasi-orthonormal basis along F at the change of the coordinate neighbourhood?

Before the beginning we make the following:

Remark Let $X=(X_1,...,X_n)^t$ and $Y=(Y_1,...,Y_m)^t$, $n,m\geq 1$ two systems of vector fields where X_i, $Y_j\in TM$, $i=1,...,n$, $j=1,...,m$. Let consider also $X'=(X_1',...,X_n')^t$ and $Y'=(Y_1',...,Y_m')^t$ another two systems of vector fields with X_i', $Y_j'\in TM$, $i=1,...,n$, $j=1,...,m$. Let $A=(a_{ij})\in M_n(\mathbf{R})$ and $B=(b_{ij})\in M_m(\mathbf{R})$ the passing matrices from X at X' respectively from Y at Y'. We have therefore $X'=AX$ and $Y'=BY$. Let consider now the matrices $G(X,Y)=(g(X_i,Y_j))\in M_{nm}(\mathbf{R})$ and $G(X',Y')=(g(X_i',Y_j'))\in M_{nm}(\mathbf{R})$. We have:

$$g(X_i',Y_j')=g(\sum_{k=1}^{n}a_{ik}X_k,\sum_{p=1}^{m}b_{jp}Y_p)=\sum_{k=1}^{n}\sum_{p=1}^{m}a_{ik}g(X_k,Y_p)b_{jp}\ ,\ i=1,...,n, j=1,...,m$$

from where we obtain the relation:

$$G(X',Y')=AG(X,Y)B^t \qquad (2.39)$$

where G(X,Y) is the Gram determinat of X and Y.

For the first question, let consider for the beginning the case of r-degenerate foliations with $1\leq r<\min\{m,n\}$. Let U a coordinates neighbourhood of M and $\{\xi_1,...,\xi_r,X_{r+1},...,X_n,W_{r+1},...,W_m,N_1,...,N_r\}$ a local quasi-orthonormal basis along F in U and $\{\xi_1,...,\xi_r,X'_{r+1},...,X'_n,W'_{r+1},...,W'_m, N'_1,...,N'_r\}$ a local quasi-orthonormal basis along F in U relative to the decompositions $TM=S(F)\perp S(F^{\perp})\perp(N\oplus deg(F))$ respectively
$TM=S'(F)\perp S'(F^{\perp})\perp(N\oplus deg'(F))$.

Let therefore:

$$\begin{pmatrix} \xi \\ X' \\ W' \\ N' \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ A_1 & A_2 & 0 & 0 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \end{pmatrix} \begin{pmatrix} \xi \\ X \\ W \\ N \end{pmatrix} \tag{2.40}$$

where we note with ξ,X,W,N,X',W',N' the matrices who have like components the vector fields with the same name and A_i,B_j,C_j, i=1,2, j=1,2,3,4 are matrices of corresponding dimensions, A_2 and B_3 being nonsingular.

The conditions for the first basis are:

(2.41) $G(\xi,\xi)=0,\ G(\xi,X)=0,\ G(\xi,W)=0,\ G(\xi,N)=I$

$G(X,X)=G_{X},\ G(X,W)=0,\ G(X,N)=0$

$G(W,W)=G_{W},\ G(W,N)=0$

$G(N,N)=0$

and for the second:

(2.42) $G(\xi,\xi)=0,\ G(\xi,X')=0,\ G(\xi,W')=0,\ G(\xi,N')=I$

$G(X',X')=G'_{X},\ G(X',W')=0,\ G(X',N')=0$

$G(W',W')=G'_{W},\ G(W',N')=0$

$G(N',N')=0$

where we have note: $G(X,X)=G_X\in M_{n-r}(\mathbf{R})$, $G(W,W)=G_W\in M_{m-r}(\mathbf{R})$, $G(X',X')=G'_X\in M_{n-r}(\mathbf{R})$, $G(W',W')=G'_W\in M_{m-r}(\mathbf{R})$. Is obvious that G_X,G_W,G'_X and G'_W are nonsingular matrices.

From (2.40)-(2.42) we have after the notation: A_2=A, B_3=B, C_1=E, C_2=C, C_3=D:

(2.43) $$\begin{pmatrix}\xi\\ X'\\ W'\\ N'\end{pmatrix}=\begin{pmatrix}I & 0 & 0 & 0\\ -AG_XC^t & A & 0 & 0\\ -BG_WD^t & 0 & B & 0\\ E & C & D & I\end{pmatrix}\begin{pmatrix}\xi\\ X\\ W\\ N\end{pmatrix}$$

where

(2.44) $$G_XA^t=A^{-1}G'_X$$

$$G_WB^t=B^{-1}G'_W$$

$$E+E^t+CG_XC^t+DG_WD^t=0$$

If we coinsider now the case of coisotropic foliations, let U a coordinates neighbourhood of M and $\{\xi_1,...,\xi_m,X_{m+1},...,X_n,N_1,...,N_m\}$ a local quasi-orthonormal basis along F in U and $\{\xi_1,...,\xi_m,X'_{m+1},...,X'_n,N'_1,...,N'_m\}$ a local quasi-orthonormal basis along F in U relative to the decompositions TM=S (F)⊥D $_F{}^{\perp}$⊕ deg(F) and TM=S '(F)⊥D $_F{}^{\perp}$⊕ deg'(F) respectively. Let therefore:

(2.45) $$\begin{pmatrix}\xi\\ X'\\ N'\end{pmatrix}=\begin{pmatrix}I & 0 & 0\\ A_1 & A_2 & 0\\ B_1 & B_2 & B_3\end{pmatrix}\begin{pmatrix}\xi\\ X\\ N\end{pmatrix}$$

with the same notations like upper.

The conditions for the first basis are:

(2.46) $$G(\xi,\xi)=0,\ G(\xi,X)=0,\ G(\xi,N)=I$$

$$G(X,X)=G_X,\ G(X,N)=0$$

$$G(N,N)=0$$

and for the second:

(2.47) $G(\xi,\xi)=0,\ G(\xi,X')=0,\ G(\xi,N')=I$

$G(X',X')=G'_X,\ G(X',N')=0$

$G(N',N')=0$

where we have note: $G(X,X)=G_X\in M_{n-m}(\mathbf{R})$, $G(X',X')=G'_X\in M_{n-m}(\mathbf{R})$. It is obvious that G_X and G'_X are nonsingular matrices.

From (2.45)-(2.47) we have with the notations: $A_2=A$, $B_1=E$, $B_2=C$:

(2.48)
$$\begin{pmatrix}\xi\\ X'\\ N'\end{pmatrix}=\begin{pmatrix}I & 0 & 0\\ -AG_XC^t & A & 0\\ E & C & I\end{pmatrix}\begin{pmatrix}\xi\\ X\\ N\end{pmatrix}$$

where:

(2.49) $G_XA^t=A^{-1}G'_X$

$E+E^t+CG_XC^t=0$

Let now the case of the isotropic foliations and U a coordinates neighbourhood of M, $\{\xi_1,\dots,\xi_n,W_{n+1},\dots,W_m,N_1,\dots,N_n\}$ a local quasi-orthonormal basis along F in U and $\{\xi_1,\dots,\xi_n,W'_{n+1},\dots,W'_m,N'_1,\dots,N'_n\}$ a local quasi-orthonormal basis along F in U relative to the decompositions $TM=(D_F\oplus deg(F))\perp S(F^\perp)$ and $TM=(D_F\oplus deg'(F))\perp S'(F^\perp)$ respectively. Let therefore:

(2.50)
$$\begin{pmatrix}\xi\\ W'\\ N'\end{pmatrix}=\begin{pmatrix}I & 0 & 0\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{pmatrix}\begin{pmatrix}\xi\\ W\\ N\end{pmatrix}$$

with the same notations like upper.

The conditions for the first basis are:

(2.51) $G(\xi,\xi)=0$, $G(\xi,W)=0$, $G(\xi,N)=I$

$G(W,W)=G_W$, $G(W,N)=0$

$G(N,N)=0$

and for the second:

(2.52) $G(\xi,\xi)=0$, $G(\xi,W')=0$, $G(\xi,N')=I$

$G(W',W')=G'_W$, $G(W',N')=0$

$G(N',N')=0$

where we have note: $G(W,W)=G_W\in M_{m-n}(\mathbf{R})$, $G(W',W')= G'_W\in M_{m-n}(\mathbf{R})$.

Is obvious that G_W and G'_W are nonsingular matrices.

From (2.50)-(2.52) we have with the notations: $A_2=B$, $B_1=E$, $B_2=D$:

(2.53) $$\begin{pmatrix} \xi \\ W' \\ N' \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ -BG_WD^t & B & 0 \\ E & D & I \end{pmatrix} \begin{pmatrix} \xi \\ W \\ N \end{pmatrix}$$

where:

(2.54) $$G_WB^t=B^{-1}G'_W$$

$E+E^t+DG_WD^t=0$

Finally, let consider now the case of totally degenerate foliations and let U a coordinates neighbourhood of M, $\{\xi_1,...,\xi_m,N_1,...,N_m\}$ a local quasi-orthonormal basis along F in U and $\{\xi_1,...,\xi_m, N'_1,...,N'_m\}$ a local quasi-orthonormal basis along F in U relative to the decompositions $TM=D_F \oplus deg(F)$ and $TM=D_F \oplus deg'(F)$ respectively.

Let therefore:

(2.55) $$\begin{pmatrix} \xi \\ N' \end{pmatrix} = \begin{pmatrix} I & 0 \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} \xi \\ N \end{pmatrix}$$

with the same notations like upper.

The conditions for the first basis are:

(2.56) $G(\xi,\xi)=0,\ G(\xi,N)=I$

$G(N,N)=0$

and for the second:

(2.57) $G(\xi,\xi)=0,\ G(\xi,N')=I$

$G(N',N')=0$

From (2.55)-(2.57) we have with the notation: $A_1=E$:

(2.58) $$\begin{pmatrix} \xi \\ N' \end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} \xi \\ N \end{pmatrix}$$

where E is a matrix of n-order satisfying the relation:

(2.59) $E+E^t=0$

being therefore skew-symmetric.

Let consider now the second question. We treat this in the case of the integrability of the null distribution N.

For the beginning we shall analyse the case of r-degenerate foliations with $1 \le r < \min\{m,n\}$.

If we have U and V two coordinates neighbourhoods such that $U \cap V \neq \emptyset$ let consider $\{\xi_1,\ldots,\xi_r,X_{r+1},\ldots,X_n,W_{r+1},\ldots,W_m,N_1,\ldots,N_r\}$ a local quasi-orthonormal basis along F in U and $\{\xi'_1,\ldots,\xi'_r,X'_{r+1},\ldots,X'_n,W'_{r+1},\ldots,W'_m, N'_1,\ldots,N'_r\}$ a local quasi-orthonormal basis along F in V. From (2.1), (2.2) and the integrability of N on $U \cap V$ we have:

$$\begin{pmatrix} \xi' \\ X' \\ W' \\ N' \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ B & C & D & E \\ 0 & 0 & F & G \\ 0 & 0 & H & J \end{pmatrix} \begin{pmatrix} \xi \\ X \\ W \\ N \end{pmatrix} \tag{2.60}$$

where A,B,C,D,E,F,G,H,J are matrices of appropriate dimensions. Let note also that A,C and $\begin{pmatrix} F & G \\ H & J \end{pmatrix}$ are nonsingular matrices. If we proceed similar like in the first problem we have finally:

Theorem 2.9 Let F a r-degenerate foliation with integrable null distribution of a Semi-Riemannian manifold (M,g). If we have U and V two coordinates neighbourhoods in an arbitrary point $p \in M$ such that $U \cap V \neq \emptyset$ and if we shall consider $\{\xi_1,...,\xi_r,X_{r+1}, ...,X_n,W_{r+1},...,W_m,N_1,...,N_r\}$ a local quasi-orthonormal basis along F in U and $\{\xi'_1,...,\xi'_r,X'_{r+1},...,X'_n,W'_{r+1},...,W'_m, N'_1,...,N'_r\}$ a local quasi-orthonormal basis along F in V, follows:

$$\begin{pmatrix} \xi' \\ X' \\ W' \\ N' \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & (A^{-1})^t \end{pmatrix} \begin{pmatrix} \xi \\ X \\ W \\ N \end{pmatrix} \tag{2.61}$$

where A is a nonsingular matrix of r-order and B and C orthogonal matrices of n-r respectively m-r orders satisfying in addition the conditions:

(2.62) $BG_XB^t=G'_X$

$CG_WC^t=G'_W$

In the cases of coisotropic, isotropic and totally degenerate foliations we have analogously:

Theorem 2.10 Let F a coisotropic foliation with integrable null distribution of a Semi-Riemannian manifold (M,g). If we have U and V two coordinates neighbourhoods in an arbitrary point $p\in M$ such that $U\cap V\neq\varnothing$ and if we shall consider $\{\xi_1,...,\xi_m,X_{m+1},...,X_n,N_1,...,N_m\}$ a local quasi-orthonormal basis along F in U and $\{\xi'_1,...,\xi'_m,X'_{m+1},...,X'_n,N'_1,...,N'_m\}$ a local quasi-orthonormal basis along F in V, follows:

(2.63) $$\begin{pmatrix}\xi'\\X'\\N'\end{pmatrix}=\begin{pmatrix}A&0&0\\0&B&0\\0&0&(A^{-1})^t\end{pmatrix}\begin{pmatrix}\xi\\X\\N\end{pmatrix}$$

where A is a nonsingular matrix of m-order and B an orthogonal matrix of n-m-order satisfying in addition the condition:

(2.64) $$BG_XB^t=G'_X$$

Theorem 2.11 Let F an isotropic foliation of a Semi-Riemannian manifold (M,g). If we have U and V two coordinates neighbourhoods in an arbitrary point $p\in M$ such that $U\cap V\neq\varnothing$ and if we shall consider $\{\xi_1,...,\xi_n,W_{n+1},...,W_m,N_1,...,N_n\}$ a local quasi-orthonormal basis along F in U and $\{\xi'_1,...,\xi'_n,W'_{n+1},...,W'_m,N'_1,...,N'_n\}$ a local quasi-orthonormal basis along F in V, follows:

(2.65) $$\begin{pmatrix}\xi'\\W'\\N'\end{pmatrix}=\begin{pmatrix}A&0&0\\0&B&0\\0&0&(A^{-1})^t\end{pmatrix}\begin{pmatrix}\xi\\W\\N\end{pmatrix}$$

where A is a nonsingular matrix of n-order and B an orthogonal matrix of m-n-order satisfying in addition the condition:

(2.66) $$BG_WB^t=G'_W$$

Theorem 2.12 Let F a totally degenerate foliation of a Semi-Riemannian manifold (M,g). If we have U and V two coordinates neighbourhoods in an arbitrary point $p\in M$ such that $U\cap V\neq\varnothing$ and if we shall consider $\{\xi_1,...,\xi_m,N_1,...,N_m\}$ a local quasi-orthonormal basis along F in U and $\{\xi'_1,...,\xi'_m,N'_1,...,N'_m\}$ a local quasi-orthonormal basis along F in V, follows:

(2.67) $$\begin{pmatrix}\xi'\\N'\end{pmatrix}=\begin{pmatrix}A&0\\0&(A^{-1})^t\end{pmatrix}\begin{pmatrix}\xi\\N\end{pmatrix}$$

where A is a nonsingular matrix of m-order.

The end of this section consists in five examples of various kind of degenerate foliations with integrable null distribution. We shall see in the next chapters that particular types of foliations come into this hypotehesis.

In what follows in examples on $\mathbf{R}^n_\mu(\underbrace{--\ldots-}_{\mu\text{ times}}\underbrace{++\ldots+}_{n-\mu\text{ times}})$ with the coordinates $(x^1,\ldots,x^n)$ we shall note $\partial_i=\frac{\partial}{\partial x^i}$, i=1,...,n. We shall note also the Semi-Riemannian metric on $\mathbf{R}^n_\mu$ with g.

2.1. Let the smooth map f:$\mathbf{R}\to\mathbf{R}$, $f^2(x)>1$ $\forall x\in\mathbf{R}$. Such an example is $f(x)=x^n+2$, $x\in\mathbf{R}$, $n\in\mathbf{N}$. Let also $\alpha\in\mathbf{R}-\left\{\frac{\pi}{2}+n\pi \middle| n\in\mathbf{Z}\right\}$. Let consider now the map $\varphi:\mathbf{R}^2\to\mathbf{R}$, $\varphi(x^1,x^2)=f(x^1\cos\alpha-x^2\sin\alpha)$ $\forall(x^1,x^2)\in\mathbf{R}^2$.

On the Semi-Riemannian manifold $M=\mathbf{R}^4_2(-,-,+,+)$ let the vector fields:

$$\xi=\sin\alpha\cos\alpha\,(\varphi^2(x^1,x^2)-1)\partial_1+\cos^2\alpha(\varphi^2(x^1,x^2)-1)\partial_2+\cos\alpha(\varphi^2(x^1,x^2)-1)\partial_3$$

$$X=\frac{1}{\cos\alpha\sqrt{\varphi^2(x^1,x^2)-1}}\partial_1+\frac{\sin\alpha}{\cos\alpha\sqrt{\varphi^2(x^1,x^2)-1}}\partial_2+\frac{\varphi(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}}\partial_4$$

We have now $g(\xi,\xi)=g(\xi,X)=0$ and $g(X,X)=1$. On the other hand:

$$[\xi,X]=-\frac{2\varphi(x^1,x^2)\varphi'(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}^{\,3}}\xi$$

therefore ξ and X defined a 1-degenerate foliation F on M.

A local quasi-orthonormal basis along the 1-degenerate foliation is given by: $\{X,W,\xi,N\}$ where:

$$W=\frac{\varphi(x^1,x^2)\cos\alpha}{\sqrt{\varphi^2(x^1,x^2)-1}}\partial_1-\frac{\varphi(x^1,x^2)\sin\alpha}{\sqrt{\varphi^2(x^1,x^2)-1}}\partial_2+\frac{1}{\sqrt{\varphi^2(x^1,x^2)-1}}\partial_4,$$

$$N=\frac{1}{2\cos^3\alpha\left(\varphi^2(x^1,x^2)-1\right)^2}[-\sin\alpha\,(1+\varphi^2(x^1,x^2)\cos^2\alpha)\partial_1+$$

$\cos\alpha\,(1-\varphi^2(x^1,x^2)\cos^2\alpha)\partial_2+(\varphi^2(x^1,x^2)\cos^2\alpha-1)\partial_3-2\sin\alpha\cos\alpha\,\varphi(x^1,x^2)\partial_4]$

2.2. Let $M=\mathbf{R}^4{}_2-\mathbf{R}\times\mathbf{R}\times\{-1,0,1\}\times\mathbf{R}$ and the vector fields:

$\xi=\sin x^4\,\partial_1+\cos x^4\,\partial_2+\partial_3$, $X=\cos x^4\,\partial_1-\sin x^4\,\partial_2+\frac{1}{x^3}\partial_4$

We have now $g(\xi,\xi)=g(\xi,X)=0$ and $g(X,X)=\frac{1}{(x^3)^2}-1\neq 0$ because $x^3\neq\pm1$. On the other hand: $[\xi,X]=-\frac{1}{x^3}X$ therefore ξ and X defined a 1-degenerate foliation F on M.

A local quasi-orthonormal basis along the 1-degenerate foliation is given by: $\{X,W,\xi,N\}$ where:

$W=\frac{1}{\sqrt{|(x^3)^2-1|}}(\cos x^4\,\partial_1-\sin x^4\,\partial_2+x^3\partial_4)$,

$N=\frac{1}{2}(-\sin x^4\,\partial_1-\cos x^4\,\partial_2+\partial_3)$

2.3. Let the Semi-Riemannian manifold $M=\{(x,y,z)\in\mathbf{R}^3 \mid y\neq0, z\neq2k\pi, z\neq\frac{\pi}{2}+2k\pi, k\in\mathbf{Z}\}$ endowed with the metric g defined through:

$ds^2=dx^2+y^2dz^2+2(\sin z+\cos z-1)dxdy+2y(\cos z-\sin z)dxdz$

We have $\det g=-y^2(\sin z+\cos z-1)^2=-8y^2\sin^2\frac{z}{2}\sin^2\left(\frac{\pi}{4}-\frac{z}{2}\right)<0$.

If we apply the Jacobi theorem we have that g is a Semi-Riemannian metric of index 1.

We note in what follows $\partial_x=\frac{\partial}{\partial x},\partial_y=\frac{\partial}{\partial y},\partial_z=\frac{\partial}{\partial z}$. Let now the vector fields:

$\xi=-2(\sin z+\cos z-1)\partial_x+\partial_y$ and $X=-2y(\cos z-\sin z)\partial_x+\partial_z$

We have $g(\xi,\xi)=0$, $g(\xi,X)=0$ and $g(X,X)=y^2>0$. On the other hand: $[\xi,\xi]=[X,X]=0$ and $[\xi,X]=[\partial_y-2(\sin z+\cos z-1)\partial_x,-2y(\cos z-\sin z)\partial_x+\partial_z]=0$.

We have therefore a foliation F generated by the vector fields ξ and X. We have $D_F=Span(\xi,X)$, N =Span(ξ), S (F)=Span(X). In order that the foliation be coisotropic it is necessary that $N=D_F^{\perp}$. We have:

$D_F^{\perp}=\{4\lambda(1+\cos z\sin z-\cos z-\sin z)\partial_x+\lambda(1-\cos z-\sin z)\partial_y \mid \forall\lambda\in F(M)\}$ = Span(ξ)=N therefore the foliation F is coisotropic.

A local quasi-orthonormal basis along the coisotropic foliation is given by: $\{X,\xi,N\}$ where:

$$N=\frac{1}{9y(\sin z-\cos z)^2(\sin z+\cos z-1)^2}\big[y(\sin z+\cos z-1)\partial_x+$$
$$y(3\sin z\cos z-2)\partial_y+3(\sin z-\cos z)(\sin z+\cos z-1)\partial_z\big]$$

2.4. Let $M=\mathbf{R}^4_2(-,-,+,+)$ and the vector field: $\xi=\sin u\,\partial_1+\cos u\,\partial_2+\partial_3$ where u is an arbitrary smooth map on M. We have now: $g(\xi,\xi)=0$ and how $[\xi,\xi]=0$ follows that ξ defined an isotropic foliation.

A local quasi-orthonormal basis along the isotropic foliation is given by: $\{\xi,N,W_1,W_2\}$ where:

$$N=\frac{1}{2}(-\sin u\,\partial_1-\cos u\,\partial_2+\partial_3),$$

$$W_1=\partial_4,\ W_2=\cos u\,\partial_1-\sin u\,\partial_2$$

2.5. Let $M=\mathbf{R}^4_2(-,-,+,+)$ and the vector fields: $\xi_1=f\partial_1+f\partial_3$, $\xi_2=h\partial_2+h\partial_4$ where f and h are smooth mapings on M, everywhere non-null. We have: $g(\xi_1,\xi_1)=0$, $g(\xi_2,\xi_2)=0$, $g(\xi_1,\xi_2)=0$ and $[\xi_1,\xi_2]=-\frac{h}{f}\left(\frac{\partial f}{\partial x^2}+\frac{\partial f}{\partial x^4}\right)\xi_1+\frac{f}{h}\left(\frac{\partial h}{\partial x^1}+\frac{\partial h}{\partial x^3}\right)\xi_2\in Span(\xi_1,\xi_2)$, $[\xi_1,\xi_1]=[\xi_2,\xi_2]=0$ therefore they generate a totally degenerate foliation.

A local quasi-orthonormal basis along the isotropic foliation is given by: $\{\xi_1,\xi_2,N_1,N_2\}$ where:

$$N_1 = \frac{1}{2f}(-\partial_1 + \partial_3),\ N_2 = \frac{1}{2h}(\partial_2 + \partial_4)$$

3. Fundamental tensors of a degenerate foliation

Let F a degenerate foliation of a Semi-Riemannian manifold (M,g). We shall note with ∇ the Levi-Civita connection on M corresponding to g. For the sake of simplicity we shall consider the decomposition of TM given by (2.22):

(3.1) $$TM = S(F) \perp S(F^{\perp}) \perp (N \oplus deg(F))$$

where in the case of a coisotropic foliation we have that $S(F^{\perp})=\{0\}$ and $N = D_F^{\perp}$, in the case of an isotropic foliation having $S(F)=\{0\}$ and $N = D_F$ and in the case of totally degenerate foliations having: $S(F)= S(F^{\perp})=\{0\}$ and $N = D_F = D_F^{\perp}$.

We shall define four projectors relative to the decomposition (3.1):

(3.2) $P_1:TM \to N$, $P_2:TM \to S(F)$, $P_3:TM \to S(F^{\perp})$, $P_4:TM \to deg(F)$

We have:

(3.3) $P_1+P_2+P_3+P_4=I$, $P_iP_j=\delta_{ij}P_i$

$\forall$ i,j=1,...,4, I being the identity.

From (3.1), (3.2) follows:

(3.4) $g(P_iX,P_jY)=0\ \forall (i,j)\in(\{1,2,3,4\}\times\{1,2,3,4\})-\{(1,4),(2,2),(3,3)\}\ \forall\ X,Y\in TM$

In what follows we shall note also:

$S_1=N$, $S_2=S(F)$, $S_3=S(F^{\perp})$, $S_4=deg(F)$

We shall define a tensors family of type (1,2):

(3.5) $$A^k:TM \times TM \to TM,\ A^k_X Y = \nabla_{P_kX}Y - \sum_{i=1}^{4} P_i \nabla_{P_kX} P_i Y = \sum_{\substack{i,j=1\\ i\neq j}}^{4} P_i \nabla_{P_kX} P_j Y$$

$\forall$ k=1,2,3,4 $\forall$ X,Y$\in$TM.

From (3.5) follows that if Y$\in$S $_q$: $A^k_X Y = \sum_{\substack{i=1 \\ i\neq q}}^{4} P_i \nabla_{P_k X} P_q Y \quad \forall X\in TM \ \forall k,q=1,2,3,4$

and how in the upper sum i≠q follows:

(3.6) $g(A^k{}_X Y,Z)=0 \ \forall Y,Z\in S_q \ \forall X\in TM \ \forall k,q=1,2,3,4$

From the definition we have also:

(3.7) $A^k_{P_k X} Y = A^k_X Y \quad \forall X,Y\in TM \ \forall k=1,2,3,4$

Theorem 3.1 The tensors A^k, k=1,2,3,4 are skew-symmetric with g that is:

(3.8) $g(A^k{}_X Y,Z)+g(Y,A^k{}_X Z)=0 \ \forall X,Y,Z\in TM$

Proof. Let X,Y,Z$\in$TM and 1≤k≤4 fixed. We have:

$$g(A^k_X Y, Z) + g(Y, A^k_X Z) = g(\nabla_{P_k X} Y - \sum_{i=1}^{4} P_i \nabla_{P_k X} P_i Y, Z) + g(Y, \nabla_{P_k X} Z - \sum_{j=1}^{4} P_j \nabla_{P_k X} P_j Z) =$$
$$\nabla_{P_k X} g(P_4 Y, P_1 Z) + \nabla_{P_k X} g(P_2 Y, P_2 Z) + \nabla_{P_k X} g(P_3 Y, P_3 Z) + \nabla_{P_k X} g(P_1 Y, P_4 Z) -$$
$$g(\nabla_{P_k X} P_4 Y, P_1 Z) - g(\nabla_{P_k X} P_2 Y, P_2 Z) - g(\nabla_{P_k X} P_3 Y, P_3 Z) - g(\nabla_{P_k X} P_1 Y, P_4 Z) -$$
$$g(P_4 Y, \nabla_{P_k X} P_1 Z) - g(P_2 Y, \nabla_{P_k X} P_2 Z) - g(P_3 Y, \nabla_{P_k X} P_3 Z) - g(P_1 Y, \nabla_{P_k X} P_4 Z) =$$
$$(\nabla_{P_k X} g)(P_4 Y, P_1 Z) + (\nabla_{P_k X} g)(P_2 Y, P_2 Z) + (\nabla_{P_k X} g)(P_3 Y, P_3 Z) + (\nabla_{P_k X} g)(P_1 Y, P_4 Z) = 0$$

Theorem 3.2 The distribution S $_k$, k=1,2 is integrable if and only if $A^k{}_X Y=A^k{}_Y X$, $\forall$X,Y$\in$S $_k$.

Proof. Let k=1,2-fixed. We have for any X,Y$\in$S $_k$:

$$A^k_X Y - A^k_Y X = \sum_{\substack{i=1 \\ i\neq k}}^{4} P_i \nabla_{P_k X} P_k Y - \sum_{\substack{i=1 \\ i\neq k}}^{4} P_i \nabla_{P_k Y} P_k X = \sum_{\substack{i=1 \\ i\neq k}}^{4} P_i (\nabla_{P_k X} P_k Y - \nabla_{P_k Y} P_k X) =$$
$$\sum_{\substack{i=1 \\ i\neq k}}^{4} P_i [P_k X, P_k Y] = [P_k X, P_k Y] - P_k [P_k X, P_k Y]$$

If $A^k{}_X Y = A^k{}_Y X$ follows $[P_k X, P_k Y] = P_k[P_k X, P_k Y] \in S_k$ therefore S_k is integrable. Reciprocal, if S_k is integrable then $[P_k X, P_k Y] \in S_k$ therefore $[P_k X, P_k Y] = P_k[P_k X, P_k Y]$. But this means that $A^k{}_X Y = A^k{}_Y X \ \forall\ X, Y \in S_k$.

Remarks

Let a degenerate foliation F of a Semi-Riemannian manifold (M,g).

1. The null distribution N is integrable if and only if the tensor A^1 is symmetric (k=1 in the theorem 3.2);
2. The screen distribution S (F) is integrable if and only if the tensor A^2 is symmetric (k=2 in the theorem 3.2);
3. If the foliation F is isotropic or totally degenerate then the tensors A^1 and A^2 are symmetrics. Indeed, in these cases we have: $N = D_F$ and S (F)={0}. We have therefore that the null distribution N is integrable from where follows that the tensor A^1 is symmetric. Also on the screen distribution S (F) the bracket identically vanishes and therefore the tensor A^2 is symmetric.

In the case of the integrability of S_k, k=1,2 we have the following:

Theorem 3.3 The integral manifold of the distribution S_k, k=1,2 is totally geodesic if and only if $A^k{}_X Y = 0 \ \forall\ X, Y \in S_k$.

Proof. For any $X, Y \in S_k$ we have that $A^k_X Y = \sum_{\substack{i=1 \\ i \neq k}}^{4} P_i \nabla_{P_k X} P_k Y$. Let S the integral manifold of S_k for a fixed k. S is totally geodesic if and only if $\nabla_{P_k X} P_k Y \in S_k$. But this is equivalent with $A^k{}_X Y = 0$.

In what follows we shall determine the Gauss-Weingarten formulae for the degenerate foliations.

Considering $X, Y \in D_F$ we have: $P_3 X = P_4 X = 0$, $P_3 Y = P_4 Y = 0$. By the fact that $X = P_1 X + P_2 X$, $Y = P_1 Y + P_2 Y$ follows from (3.5):

(3.9) $A^1_{P_1 X} P_1 Y = \nabla_{P_1 X} P_1 Y - P_1 \nabla_{P_1 X} P_1 Y$

(3.10) $A^1_{P_1 X} P_2 Y = \nabla_{P_1 X} P_2 Y - P_2 \nabla_{P_1 X} P_2 Y$

(3.11) $A^2_{P_2 X} P_1 Y = \nabla_{P_2 X} P_1 Y - P_1 \nabla_{P_2 X} P_1 Y$

(3.12) $A^2_{P_2X}P_2Y = \nabla_{P_2X}P_2Y - P_2\nabla_{P_2X}P_2Y$

The Levi-Civita connection becomes:

$$\nabla_X Y = \nabla_{P_1X}P_1Y + \nabla_{P_1X}P_2Y + \nabla_{P_2X}P_1Y + \nabla_{P_2X}P_2Y = A^1_{P_1X}P_1Y + P_1\nabla_{P_1X}P_1Y + A^1_{P_1X}P_2Y + P_2\nabla_{P_1X}P_2Y + A^2_{P_2X}P_1Y + P_1\nabla_{P_2X}P_1Y + A^2_{P_2X}P_2Y + P_2\nabla_{P_2X}P_2Y \quad (3.13)$$

From (3.13) decomposing after $D_F = N \perp S(F), S(F^\perp)$ and deg(F) we have:

$$\nabla^F_x Y = P_1A^1_{P_1X}P_1Y + P_1A^1_{P_1X}P_2Y + P_1A^2_{P_2X}P_1Y + P_1A^2_{P_2X}P_2Y + P_2A^1_{P_1X}P_1Y + P_2A^1_{P_1X}P_2Y + P_2A^2_{P_2X}P_1Y + P_2A^2_{P_2X}P_2Y + P_1\nabla_{P_1X}P_1Y + P_1\nabla_{P_2X}P_1Y + P_2\nabla_{P_1X}P_2Y + P_2\nabla_{P_2X}P_2Y \quad (3.14)$$

(3.15) $h^S(X,Y) = P_3A^1_{P_1X}P_1Y + P_3A^1_{P_1X}P_2Y + P_3A^2_{P_2X}P_1Y + P_3A^2_{P_2X}P_2Y$

(3.16) $h^L(X,Y) = P_4A^1_{P_1X}P_1Y + P_4A^1_{P_1X}P_2Y + P_4A^2_{P_2X}P_1Y + P_4A^2_{P_2X}P_2Y$

Considering now $X \in D_F$ and $V \in tr(F)$ we have: $P_3X=P_4X=0$, $P_1V=P_2V=0$. By the fact that $X=P_1X+P_2X$, $V=P_3V+P_4V$ follows from (3.5):

(3.17) $A^1_{P_1X}P_3V = \nabla_{P_1X}P_3V - P_3\nabla_{P_1X}P_3V$

(3.18) $A^1_{P_1X}P_4V = \nabla_{P_1X}P_4V - P_4\nabla_{P_1X}P_4V$

(3.19) $A^1_{P_2X}P_3V = \nabla_{P_2X}P_3V - P_3\nabla_{P_2X}P_3V$

(3.20) $A^1_{P_2X}P_4V = \nabla_{P_2X}P_4V - P_4\nabla_{P_2X}P_4V$

The Levi-Civita connection becomes:

$$\nabla_X V = \nabla_{P_1X}P_3V + \nabla_{P_1X}P_4V + \nabla_{P_2X}P_3V + \nabla_{P_2X}P_4V = A^1_{P_1X}P_3V + P_3\nabla_{P_1X}P_3V + A^1_{P_1X}P_4V + P_4\nabla_{P_1X}P_4V + A^1_{P_2X}P_3V + P_3\nabla_{P_2X}P_3V + A^1_{P_2X}P_4V + P_4\nabla_{P_2X}P_4V \quad (3.21)$$

From (3.21) decomposing after $D_F = N \perp S(F)$, $S(F^{\perp})$ and deg(F) we have:

$$(3.22)\ \ -A_V X = P_1 A^1_{P_1X} P_3 V + P_1 A^1_{P_1X} P_4 V + P_1 A^1_{P_2X} P_3 V + P_1 A^1_{P_2X} P_4 V + P_2 A^1_{P_1X} P_3 V + P_2 A^1_{P_1X} P_4 V + P_2 A^1_{P_2X} P_3 V + P_2 A^1_{P_2X} P_4 V$$

$$(3.23)\ \ D^S_X V = P_3 A^1_{P_1X} P_3 V + P_3 A^1_{P_1X} P_4 V + P_3 A^1_{P_2X} P_3 V + P_3 A^1_{P_2X} P_4 V + P_3 \nabla_{P_1X} P_3 V + P_3 \nabla_{P_2X} P_3 V$$

$$(3.24)\ \ D^L_X V = P_4 A^1_{P_1X} P_3 V + P_4 A^1_{P_1X} P_4 V + P_4 A^1_{P_2X} P_3 V + P_4 A^1_{P_2X} P_4 V + P_4 \nabla_{P_1X} P_4 V + P_4 \nabla_{P_2X} P_4 V$$

From the tensorial character of A^1 respectively A^2 and from the fact that ∇ is **R**-bilinear in both terms and is F (M)-linear in the first term follows that all the geometrical objects introduced through (3.14), (3.15), (3.16), (3.22), (3.23) and (3.24) are **R**-bilinear and F (M)-linear in the first term. The fact that ∇^F is linear connection on D_F is easy to proven therefore we shall name ∇^F the linear connection induced on F . From (3.15), (3.16) and the tensorial character of A^1 respectively A^2 follows that h^S and h^L are tensors of type (1,2) defined by: $h^L: D_F \times D_F \to \deg(F)$, $h^S: D_F \times D_F \to S(F^{\perp})$. We shall name h^L the second degenerate fundamental form of F and h^S the second screen fundamental form of F . From (3.22) follows from the same tensorial character of A^1, A^2 that A is a tensor of type (1,2) defined by: $A: D_F \times tr(F) \to D_F$. We shall name A_V the Weingarten operator of F relative to V.

From (3.23), (3.24) follows:

$$(3.25)\ \ D^S_X fV = fD^S_X V + X(f) P_3 V$$

$$(3.26)\ \ D^L_X fV = fD^L_X V + X(f) P_4 V$$

$\forall f \in F(M)$.

Remarks

1. From (3.25) follows: $D^S_X fP_3 V = f\nabla^S_X P_3 V + X(f) P_3 V$, $D^S_X fP_4 V = f\nabla^S_X P_4 V$
2. From (3.26) follows: $D^L_X fP_3 V = fD^L_X P_3 V$, $D^L_X fP_4 V = fD^L_X P_4 V + X(f) P_4 V$
3. From (3.25) and (3.26) follows:

$$D^S_X fV + D^L_X fV = fD^S_X V + X(f)P_3V + fD^L_X V + X(f)P_4V = f(D^S_X V + D^L_X V) + X(f)V$$

Because D^S and D^L are not linear connections, we shall consider theirs restrictions at S ($F^\perp$) respectively at deg(F). Let therefore:

(3.27) $\nabla^S : D_F \times S(F^\perp) \to S(F^\perp)$, $\nabla^S_X(P_3V) = D^S_X P_3V$

(3.28) $\nabla^L : D_F \times \deg(F) \to \deg(F)$, $\nabla^L_X(P_4V) = D^L_X P_4V$

(3.29) $D^S : D_F \times \deg(F) \to S(F^\perp)$, $D^S(X, P_4V) = D^S_X P_4V$

(3.30) $D^L : D_F \times S(F^\perp) \to \deg(F)$, $D^L(X, P_3V) = D^L_X P_3V$

$\forall X \in D_F\ \forall V \in tr(F)$.

From the first remark and the preceding considerations, follows that ∇^S is a linear connection on $D_F \times S(F^\perp)$ and D^S is a tensor of type (1,2) on $D_F \times \deg(F)$. Also from the second remark follows that ∇^L is a linear connection on $D_F \times \deg(F)$ and D^L is a tensor of type (1,2) on $D_F \times S(F^\perp)$. We have therefore from (3.27)-(3.30):

(3.31) $D^S_X V = D^S_X P_3V + D^S_X P_4V = \nabla^S_X P_3V + D^S(X, P_4V)$

(3.32) $D^L_X V = D^L_X P_3V + D^L_X P_4V = D^L(X, P_3V) + \nabla^L_X P_4V$

We define now:

(3.33) $h : D_F \times D_F \to tr(F)$, $h(X,Y) = h^S(X,Y) + h^L(X,Y)\ \forall X, Y \in D_F$

and we shall call the second fundamental form of F relative to tr(F).

Let also:

(3.34) $\nabla^t : D_F \times tr(F) \to tr(F)$, $\nabla^t_X V = D^S_X V + D^L_X V\ \ \forall X \in D_F\ \forall V \in tr(F)$

By the third remark follows that ∇^t is a linear connection on $D_F \times tr(F)$ named the transversal linear connection of F .

We can write now:

(3.35) $$\nabla_X Y=\nabla^F{}_X Y+h(X,Y)$$

(3.36) $$\nabla_X V= -A_V X+\nabla^t{}_X V$$

$\forall\ X,Y\in D_F\ \forall V\in tr(F)$.

Because the distribution D_F is integrable follows that ∇^F is a linear connection whitout torsion on D_F.

From (3.33), (3.34) follows that the formulae (3.35), (3.36) become:

(3.37) $$\nabla_X Y=\nabla^F{}_X Y+h^L(X,Y)+h^S(X,Y)$$

(3.38) $$\nabla_X V= -A_V X+D^L{}_X V+ D^S{}_X V$$

$\forall\ X,Y\in D_F\ \forall V\in tr(F)$.

Analogously, using (3.31), (3.32) follows:

(3.39) $$\nabla_X V= -A_V X+\nabla^L{}_X P_4 V+D^L(X,P_3 V)+\nabla^S{}_X P_3 V+D^S(X,P_4 V)$$

$\forall\ X\in D_F\ \forall V\in tr(F)$.

From (3.39) follows the particular cases:

(3.40) $$\nabla_X W= -A_W X+D^L(X,W)+\nabla^S{}_X W$$

(3.41) $$\nabla_X N= -A_N X+\nabla^L{}_X N+D^S(X,N)$$

$\forall\ X\in D_F\ \forall W\in S(F^{\perp})\ \forall N\in deg(F)$.

Remark In the cases of coisotropic or totally degenerate foliations we have: $S(F^{\perp})=\{0\}$ therefore $P_3=0$ from where $h^S=0$, $\nabla^S=0$, $D^S=0$ and $D^L=0$. The formulae (3.37) and (3.41) become:

(3.42) $$\nabla_X Y=\nabla^F{}_X Y+h^L(X,Y)$$

(3.43) $$\nabla_X N= -A_N X+\nabla^L{}_X N$$

$\forall$ X,Y$\in D_F$ $\forall$N$\in$deg(F).

We shall call the formulae (3.35), (3.37), (3.42) the Gauss formulae and (3.36), (3.38), (3.39), (3.40), (3.41), (3.43) the Weingarten formulae for the degenerate foliation F.

Theorem 3.4 Let a degenerate foliation F of a Semi-Riemannian manifold (M,g). We have:

(3.44) $g(h^S(X,Y),W)+g(Y,D^L(X,W))=g(A_WX,Y)$

(3.45) $g(h^L(X,Y),\xi)+g(h^L(X,\xi),Y)+g(Y,\nabla^F{}_X\xi)=0$

(3.46) $g(D^S(X,N),W)=g(A_WX,N)$

(3.47) $g(A_NX,N')+g(A_{N'}X,N)=0$

(3.48) $g(A_NX,P_2Y)=g(N,\nabla_XP_2Y)$

$\forall$ X,Y$\in D_F$ $\forall\xi\in$N $\forall$W$\in$S (F $^\perp$)) $\forall$N,N'$\in$deg(F)).

Proof. Let X,Y$\in D_F$, $\xi\in$N, W$\in$S (F $^\perp$)), N,N'$\in$deg(F)). Then:

- $g(A_WX,Y)=g(-\nabla_XW+D^L(X,W),Y)=-g(\nabla_XW,Y)+g(D^L(X,W),Y)=g(W,\nabla_XY)+g(D^L(X,W),Y)=g(W,h^S(X,Y))+g(D^L(X,W),Y)$
- $0=\nabla_Xg(Y,\xi)=g(\nabla_XY,\xi)+g(Y,\nabla_X\xi)=g(h^L(X,Y),\xi)+g(Y,\nabla^F{}_X\xi+h^L(X,\xi))=g(h^L(X,Y),\xi)+g(h^L(X,\xi),Y)+g(Y,\nabla^F{}_X\xi)$
- $g(A_WX,N)=g(-\nabla_XW,N)=g(W,\nabla_XN)=g(W,D^S(X,N))$
- $g(A_NX,N')+g(A_{N'}X,N)=g(-\nabla_XN,N')+g(-\nabla_XN',N)= -\nabla_Xg(N,N')=0$
- $g(A_NX,P_2Y)=g(-\nabla_XN,P_2Y)=g(N,\nabla_XP_2Y)$

If we have now $\{N_1,...,N_r\}$ a basis for deg(F) and $\{W_{r+1},...,W_m\}$ a basis for S (F $^\perp$)) (the last in the case of r-degenerate foliations or of those isotropic) for a given screen distribution S (F) we define:

(3.49) $$h^L(X,Y)=\sum_{i=1}^{r}h_i^L(X,Y)N_i$$

(3.50) $$h^S(X,Y)=\sum_{a=r+1}^{m} h^S_a(X,Y)W_a$$

in the case of coisotropic or totally degenerate foliations defining $h^S{}_a=0$ $\forall$ a=r+1,..,m.

We call $h^L{}_i$ the degenerate local second fundamental forms and $h^S{}_a$ the screen local second fundamental forms of F .

Theorem 3.5 In a degenerate foliation (F, g, S (F),S (F $^{\perp}$)) of a Semi-Riemannian manifold (M,g) the degenerate local second fundamental forms are independent by the screen distribution and by the transversal distribution.

Proof. From (3.37) follows:

(3.51) $$h^L{}_i(X,Y)=g(h^L(X,Y),\xi_i)=g(\nabla_X Y,\xi_i)$$

$\forall$ X,Y$\in$ D $_F$ $\forall$i=1,...,r.

From (2.61), (2.63), (2.65), (2.67), (3.51) follows that at a change of coordinates neighbourhood of a point p$\in$F where F is a foliation with integrable null distribution we have:

(3.52) $$h^{L\prime}(X,Y)=Ah^L(X,Y)$$

$\forall$ X,Y$\in$ D $_F$, $h^{L\prime}$, h^L being column vectors with the components $h^L{}_i{}'$ respectively $h^L{}_i$ relative to the two bases. After this remark we have immediately:

Theorem 3.6 In a degenerate foliation (F, g, S (F),S (F $^{\perp}$)) with integrable null distribution of a Semi-Riemannian manifold (M,g) the vanishing of the degenerate local second fundamental forms does not depend by the coordinates neighbourhood of an arbitrary point p of F .

From (3.51) we have now $h^L{}_i(X,\xi_j)=g(\nabla_X\xi_j,\xi_i)=-g(\nabla_X\xi_i,\xi_j)=-h^L{}_j(X,\xi_i)$ therefore:

(3.53) $$h^L{}_i(X,\xi_j)+h^L{}_j(X,\xi_i)=0$$

$\forall X \in D_F \; \forall$ i,j=1,...,r and for i=j:

(3.54) $h^L_i(X,\xi_i)=0$

$\forall X \in D_F \; \forall$ i=1,...,r.

If we make a circular permutation in (3.53) we have:

(3.55) $h^L_i(\xi_j,\xi_k)=0$

$\forall$ i,j,k=1,...,r.

From (3.54) and (3.55) we have:

Theorem 3.7 In a degenerate foliation (F, g, S (F),S (F $^\perp$)) of a Semi-Riemannian manifold (M,g) the degenerate local second fundamental forms are degenerate and they identically vanish on the null distribution N of F .

In the cases of isotropic or totally degenerate foliations we have that N $=D_F$ therefore:

Corrolary 3.1 In an isotropic or totally degenerate foliation (F, g, S (F $^\perp$)) of a Semi-Riemannian manifold (M,g) the degenerate local second fundamental forms identically vanish on D_F .

The problem is now how the induced connection ∇^F will transform on F at a change of the screen distribution?

For the beginning we shall analyse the case of r-degenerate foliation with 0<r<min{m,n}.

Let U a coordinates neighbourhood of M and $\{\xi_1,...,\xi_r,X_{r+1},...,X_n, W_{r+1},...,W_m,N_1,...,N_r\}$ a local quasi-orthonormal basis along F in U and $\{\xi_1,...,\xi_r,X'_{r+1},...,X'_n,W'_{r+1},...,W'_m,N'_1,...,N'_r\}$ a local quasi-orthonormal basis along F in U relative to the decompositions TM=S (F)⊥S (F $^\perp$)⊥(N ⊕ deg(F)) respectively TM=S '(F)⊥S '(F $^\perp$)⊥(N ⊕ deg'(F)). From (2.43), (2.44), (3.37), (3.49)-(3.51) we have:

$$(3.56)\ \nabla^{F}_{X}Y=\nabla'^{F}_{X}Y+\sum_{i=1}^{r}\left(E^{t}h^{L}(X,Y)-DG_{W}{}^{t}B^{t}h'^{S}(X,Y)\right)^{i}\xi_{i}+\sum_{a=r+1}^{n}\left(C^{t}h^{L}(X,Y)\right)^{a}X_{a}$$

$\forall$ X,Y$\in D_F$, ( )^i and ( )^a being the coordinates in corresponding bases.

Analogously, we have:

$$(3.57)\qquad \nabla^{F}_{X}Y=\nabla'^{F}_{X}Y+\sum_{i=1}^{m}\left(E^{t}h^{L}(X,Y)\right)^{i}\xi_{i}+\sum_{a=m+1}^{n}\left(C^{t}h^{L}(X,Y)\right)^{a}X_{a}$$

for o coisotropic foliation,

$$(3.58)\qquad \nabla^{F}_{X}Y=\nabla'^{F}_{X}Y-\sum_{i=1}^{n}\left(DG_{W}{}^{t}B^{t}h'^{S}(X,Y)\right)^{i}\xi_{i}$$

for an isotropic foliation and

$$(3.59)\qquad \nabla^{F}_{X}Y=\nabla'^{F}_{X}Y$$

for a totally degenerate foliation.

Theorem 3.8 In a r-degenerate foliation (F, g, S (F), S (F $^{\perp}$)) of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is independent by the screen distribution if and only if

$$(3.60)\qquad E^{t}h^{L}(X,Y)=DG_{W}{}^{t}B^{t}h'^{S}(X,Y)$$

$$(3.61)\qquad C^{t}h^{L}(X,Y)=0$$

$\forall$ X,Y$\in D_F$ and $\forall$ B a non-singular matrix of m-r-order, C,D,E being arbitrary matrices of types r$\times$(n-r), r$\times$(m-r) respectively r$\times$r which satisfy in addition the relations (2.44).

Theorem 3.9 In a coisotropic foliation (F, g,S (F)) of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is independent by the screen distribution if and only if

$$(3.62)\qquad E^{t}h^{L}(X,Y)=0$$

$$(3.63)\qquad C^{t}h^{L}(X,Y)=0$$

$\forall$ X,Y$\in D_F$ and $\forall$ C,E m$\times$(n-m) and m$\times$m-orders matrices, which satisfy in addition, the relations (2.49).

Theorem 3.10 In an isotropic foliation (F,g,S $(F^{\perp})$) of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is independent by the screen distribution if and only if

(3.64) $$DG_W{}^tB^th'^S(X,Y)=0$$

$\forall$ X,Y$\in D_F$ and $\forall$ B a non-singular matrix of m-n-order, D an arbitrary matrix of n$\times$(m-n)-order, satisfying in addition the relations (2.54).

Theorem 3.11 In a totally degenerate foliation (F, g) of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is independent by the screen distribution.

We shall study in what follows the manner in which the induced connection ∇^F depend on the coordinates neigbourhood. From (2.61), (2.63), (2.65), (2.67) follows:

Theorem 3.12 In a degenerate foliation F, with integrable null distribution, of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is independent by the coordinates neighbourhood of an arbitrary point p$\in$M.

We define now a system of 1-local differential forms:

(3.65) $$\eta_i(X)=g(X,N_i),\ i=1,...,r$$

$\forall$ X$\in D_F$. We have from (3.65):

(3.66) $$X = P_2X + \sum_{i=1}^{r}\eta_i(X)\xi_i$$

$\forall$ X$\in D_F$. We remark from (3.66) that the screen distribution is defined locally by η_i=0, i=1,...,r.

We have define in (3.14) and (3.34) two linear connections ∇^F and ∇^t where the first is symmetrical. The problem is now is if these are metric connextions. From (3.37), (3.49), (3.66) and the condition that ∇ is metric we have:

(3.67) $(\nabla^F{}_X g)(Y,Z)=g(h^L(X,Y),Z)+g(h^L(X,Z),Y)=$

$$\sum_{i=1}^{r}\left[h_i^L(X,Y)\eta_i(Z)+h_i^L(X,Z)\eta_i(Y)\right]\ \forall\ X,Y,Z\in D_F.$$

From (3.36) we have also:

(3.68) $(\nabla^t{}_X g)(V,V')=-[g(A_V X,V')+g(A_{V'}X,V)]\ \forall\ X\in D_F\ \forall V,V'\in tr(F).$

Theorem 3.13 In a degenerate foliation F of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is metric if and only if the local degenerate second fundamental forms identically vanishes on D_F.

Proof. From (3.65) and (3.67) we have for any $X,Y,Z\in D_F$:

(3.69) $$(\nabla^F{}_X g)(P_2Y,P_2Z)=\sum_{i=1}^{r}\left[h_i^L(X,PY)\eta_i(PZ)+h_i^L(X,PZ)\eta_i(PY)\right]=0$$

From (3.53) follows:

(3.70) $$(\nabla^F{}_x g)(\xi_i,\xi_j)=\sum_{k=1}^{r}\left[h_k^L(X,\xi_i)\eta_k(\xi_j)+h_k^L(X,\xi_j)\eta_k(\xi_i)\right]=h_j^L(X,\xi_i)+h_i^L(X,\xi_j)=0$$

$\forall$ i,j=1,...,r and finally:

(3.71) $$(\nabla^F{}_X g)(P_2Y,\xi_i)=\sum_{k=1}^{r}\left[h_k^L(X,PY)\eta_k(\xi_i)+h_k^L(X,\xi_i)\eta_k(PY)\right]=h^L{}_i(X,PY)$$

$\forall$ i=1,...,r.

The vanishing of $\nabla^F g$ is therefore equivalent with $h^L{}_i(X,P_2Y)=0\ \forall X,Y\in D_F$. From the theorem 3.7 follows that it is equivalent with $h^L=0$.

From the corrolary 3.1 and the theorem 3.13 we have:

Corrolary 3.2 In an isotropic or totally degenerate foliation F of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is metric.

Theorem 3.14 In a coisotropic or totally degenerate foliation F of a Semi-Riemannian manifold (M,g) the transversal connection ∇^t is metric.

Proof. Because A_V has values in D_F and in the case of coisotropic or totally degenerate foliations we have that tr(F)=deg(F) follows from (3.43) and (3.68):

$$(\nabla^t{}_X g)(V,V')=-[g(A_V X,V')+g(A_{V'}X,V)]=g(\nabla_X V,V')+g(V,\nabla_X V')=\nabla_X g(V,V')=0$$

$\forall X\in D_F$ $\forall V,V'\in tr(F))$.

Theorem 3.15 In a r-degenerate or isotropic foliation F of a Semi-Riemannian manifold (M,g) the next statements are equivalents:

a) ∇^t is a linear metric connection;
b) The degenerate transversal distribution deg(F) is parallel with respect to ∇^t;
c) A_W takes values in S (F)) $\forall$ $W\in S(F^{\perp})$;
d) $D^S(X,N)=0$ $\forall$ $X\in D_F$ $\forall N\in deg(F)$.

Proof. From (3.47), (3.68) we have:

(3.72) $(\nabla^t{}_X g)(N,N')=-g(A_N X,N')-g(A_{N'}X,N)=0$

(3.73) $(\nabla^t{}_X g)(W,W')=-g(A_W X,W')-g(A_{W'}X,W)=0$

(3.74) $(\nabla^t{}_X g)(W,N)=-g(A_W X,N)-g(A_N X,W)=-g(A_W X,N)$

$\forall$ $X\in D_F$ $\forall W,W'\in S(F^{\perp}))$ $\forall N,N'\in deg(F)$. In (3.74) we have use the fact that $A_N X\in D_F$ therefore $g(A_N X,W)=0$.

a)$\Rightarrow$c) From (3.74) follows that if ∇^t is metric connection then $g(A_W X,N)=0$ therefore $A_W X\in S(F))$ $\forall X\in D_F$.

c)$\Rightarrow$a) From (3.74) follows that $(\nabla^t{}_X g)(W,N)=-g(A_W X,N)=0$ and together with (3.72) and (3.73) imply a).

a)$\Rightarrow$d) From (3.46) and (3.74) we have $0=(\nabla^t{}_X g)(W,N)=-g(A_W X,N)=-g(D^S(X,N),W)$ $\forall W\in S(F^{\perp})$ and how $S(F^{\perp})$ is nondegenerate follows $D^S(X,N)=0$.

d)$\Rightarrow$a) From (3.46) and (3.74) we have $(\nabla^t{}_X g)(W,N)=-g(A_W X,N)=-g(D^S(X,N),W)=0$ and with (3.62) and (3.73) imply a).

a)⇒b) From (3.36) we obtain now:

(3.75) $$g(\nabla^t_X N,W)=-(\nabla^t_X g)(W,N)$$

From a) and (3.75) follows therefore $g(\nabla^t_X N,W)=0$ and how $S(F^{\perp})$ is nondegenerate we have $\nabla^t_X N=0$ $\forall X\in D_F$ $\forall N\in deg(F)$. But this nothing means else that deg(F) is parallel with respect to ∇^t.

b)⇒a) From (3.75) we have $(\nabla^t_X g)(W,N)=0$ and together with (3.72) and (3.73) imply a).

We have seen that the screen distribution is fundamental in the study of degenerate foliations. On the other hand all the introduced geometrical objects does not put in obviousness properties of this. This is the reason for we shall proceed at a refinement of the Gauss formula with respect to the decomposition $D_F = S(F)\perp N$. Let therefore $X,Y\in D_F$. From (3.14) we have:

(3.76) $$\nabla^F_x P_2Y = P_1A^1_{P_1X}P_2Y + P_1A^2_{P_2X}P_2Y + P_2A^1_{P_1X}P_2Y + P_2A^2_{P_2X}P_2Y + P_2\nabla_{P_1X}P_2Y + P_2\nabla_{P_2X}P_2Y$$

(3.77) $$\nabla^F_x P_1Y = P_1A^1_{P_1X}P_1Y + P_1A^2_{P_2X}P_1Y + P_1\nabla_{P_1X}P_1Y + P_1\nabla_{P_2X}P_1Y + P_2A^1_{P_1X}P_1Y + P_2A^2_{P_2X}P_1Y$$ We define:

(3.78) $$\nabla^*_x P_2Y = P_2A^1_{P_1X}P_2Y + P_2A^2_{P_2X}P_2Y + P_2\nabla_{P_1X}P_2Y + P_2\nabla_{P_2X}P_2Y$$

(3.79) $$\nabla^F_x P_2Y = P_1A^1_{P_1X}P_2Y + P_1A^2_{P_2X}P_2Y + P_2A^1_{P_1X}P_2Y + P_2A^2_{P_2X}P_2Y + P_2\nabla_{P_1X}P_2Y + P_2\nabla_{P_2X}P_2Y$$

(3.80) $$A^*_{P_1Y}X = -P_2A^1_{P_1X}P_1Y - P_2A^2_{P_2X}P_1Y$$

(3.81) $$\nabla^{*t}_x P_1Y = P_1A^1_{P_1X}P_1Y + P_1A^2_{P_2X}P_1Y + P_1\nabla_{P_1X}P_1Y + P_1\nabla_{P_2X}P_1Y$$

Like in the preceeding discussion, follows immediately that h^* and A are tensors of type (1,2) defined thus:

$h^*: D_F\times S(F)\to N$, $A: D_F\times N\to S(F)$

We shall call h^* the second fundamental form of S (F) and A^*_ξ the Weingarten operator of S (F) with respect to ξ $\forall\xi\in N$.

Also, ∇^* and ∇^{*t} are linear connections on S (F) respectively N named the induced connection on S (F) respectively the induced connection on N . From (3.76), (3.78) and (3.79) follows:

(3.82) $$\nabla^F_X P_2Y=\nabla^*_X P_2Y+h^*(X,P_2Y)\ \forall X,Y\in D_F$$

From (3.77), (3.80) and (3.81) follows:

(3.83) $\nabla^F{}_X\xi = -A^*{}_\xi X + \nabla^{*t}{}_X\xi \ \forall X\in D_F \ \forall\xi\in N$

Remark In the case of isotropic or totally degenerate foliations we have $D_F = N$ therefore ∇^* and A^* vanish.

Theorem 3.16 Let a degenerate foliation F of a Semi-Riemannian manifold (M,g). The following relations hold:

(3.84) $g(A^*{}_\xi X, P_2Y) = g(\xi, h^L(X,P_2Y))$

(3.85) $(\nabla^{*t}{}_X g)(\xi,\xi') = g(\xi, h^L(X,\xi')) + g(\xi', h^L(X,\xi))$

(3.86) $(\nabla^*{}_X g)(P_2Y, P_2Z) = 0$

(3.87) $g(h^*(X,P_2Y),N) = g(A_N X, P_2Y)$

$\forall X,Y\in D_F \ \forall\xi,\xi'\in N \ \forall N\in deg(F)$.

Proof. Let $X,Y\in D_F$, $\xi,\xi'\in N$, $N\in deg(F)$. Using (3.37), (3.45), (3.69), (3.82), (3.83) we have:

- From (3.37), (3.83) follow: $g(A^*{}_\xi X,P_2Y) = -g(\nabla^F{}_X\xi,P_2Y) = -g(\nabla_X\xi,P_2Y) = g(\xi,\nabla_X P_2Y) = g(\xi,h^L(X,P_2Y)$;
- From (3.37), (3.83) follow: $(\nabla^{*t}{}_X g)(\xi,\xi') = X(g(\xi,\xi')) - g(\nabla^{*t}{}_X\xi,\xi') - g(\xi,\nabla^{*t}{}_X\xi') = X(g(\xi,\xi')) - g(\nabla^F{}_X\xi,\xi') - g(\xi,\nabla^F{}_X\xi') = X(g(\xi,\xi')) - g(\nabla_X\xi,\xi') + g(h^L(X,\xi),\xi') - g(\nabla_X\xi',\xi) + g(h^L(X,\xi'),\xi) = (\nabla_X g)(\xi,\xi') + g(\xi,h^L(X,\xi')) + g(\xi',h^L(X,\xi)) = g(\xi,h^L(X,\xi')) + g(\xi',h^L(X,\xi))$;
- From (3.37), (3.82) follow: $(\nabla^*{}_X g)(P_2Y,P_2Z) = X(g(P_2Y,P_2Z)) - g(\nabla^*{}_X P_2Y,P_2Z) - g(P_2Y,\nabla^*{}_X P_2Z) = X(g(P_2Y,P_2Z)) - g(\nabla^F{}_X P_2Y,P_2Z) - g(P_2Y,\nabla^F{}_X P_2Z) = X(g(P_2Y,P_2Z)) - g(\nabla_X P_2Y,P_2Z) - g(P_2Y,\nabla_X P_2Z) = 0$;
- From (3.37), (3.41), (3.82) follow: $g(h^*(X,P_2Y),N) = g(\nabla^F{}_X P_2Y,N) = g(\nabla_X P_2Y,N) = -g(P_2Y,\nabla_X N) = g(P_2Y,A_N X)$.

Theorem 3.17 Let a degenerate foliation F of a Semi-Riemannian manifold (M,g). Then the operator $A^*{}_\xi$ is self-adjoint on S (F) $\forall\xi\in N$.

Proof. From (3.84) using the fact that h^L is symmetric follows:

(3.88) $g(A^*_\xi P_2X,P_2Y)=g(\xi,h^L(P_2X,P_2Y))=g(\xi,h^L(P_2Y,P_2X))=g(P_2X,A^*_\xi P_2Y)$

Theorem 3.18 Let a degenerate foliation F with the null distribution of rank 1 of a Semi-Riemannian manifold (M,g). Then ∇^{*t} is metric connection on N .

Proof. If the null distribution is of rank 1 then from (3.54) and (3.85) we have: $(\nabla^{*t}_X g)(\xi,\xi)=2g(\xi,h^L(X,\xi))=2g(\xi,h^L_1(X,\xi)N)=2h^L_1(X,\xi)=0$.

Theorem 3.19 Let a degenerate foliation F of a Semi-Riemannian manifold (M,g). Then ∇^{*t} is metric connection on S (F).

Proof. Follows from (3.86).

From (3.45) when $Y=\xi'$, $X\to P_2X$ and (3.84) we have that:

$$0=g(h^L(P_2X,\xi'),\xi)+g(h^L(P_2X,\xi),\xi')+g(\xi',\nabla^F_{P_2X}\xi)=g(A^*_\xi\xi'+A^*_{\xi'}\xi,P_2X)$$

How S (F) is nondegenerate, follows:

(3.89) $A^*_\xi\xi'+A^*_{\xi'}\xi=0\ \forall\ \xi,\xi'\in N$

We shall suppose now that the null distribution N is integrable.

Theorem 3.20 The Weingarten operator of the screen distribution S (F) corresponding to the degenerate foliation F, with integrable null distribution, in a Semi-Riemannian manifold (M,g) vanishes on the null distribution.

Proof. Because N is integrable we have: $\forall\xi,\xi'\in N$: $[\xi,\xi']\in N$. Let $X\in S$ (F), arbitrary. Then:

$$0=g([\xi,\xi'],X)=g(\nabla_\xi\xi',X)-g(\nabla_{\xi'}\xi,X)=g(A^*_\xi\xi',X)-g(A^*_{\xi'}\xi,X)=g(A^*_\xi\xi'-A^*_{\xi'}\xi,X)$$

How S (F) is nondegenerate follows:

(3.90) $A^*_\xi\xi'=A^*_{\xi'}\xi\ \forall\ \xi,\xi'\in N$

From (3.89) and (3.90) follow:

(3.91) $A^{*}_{\xi}\xi'=0 \ \forall\ \xi,\xi'\in N$

Theorem 3.21 The second degenerate fundamental form of a degenerate foliation F, with integrable null distribution, in a Semi-Riemannian manifold (M,g) vanishes on $N \times D_F$.

Proof. From (3.84), (3.91) follow: $g(\xi',h^L(\xi,P_2X))=g(A^{*}_{\xi'}\xi,P_2X)=0\ \forall\xi'\in N$ from where:

(3.92) $h^L(\xi,P_2X)=0\ \forall\ \xi\in N\ \ \forall X\in D_F$

From (3.92) and the theorem 3.7 we have:

(3.93) $h^L(\xi,X)=0\ \forall\ \xi\in N\ \ \forall X\in D_F$

Before the next theorem let do the remark that from (3.69)-(3.71) and (3.92) follow:

(3.94) $\nabla^F_{\xi}g=0\ \forall\ \xi\in N$

Theorem 3.22 Let a degenerate foliation F, with integrable null distribution, in a Semi-Riemannian manifold (M,g). The next assertions are equivalent:

a) The induced connection ∇^F is metric;
b) A^{*}_{ξ} vanishes on S (F) $\forall\ \xi\in N$;
c) N is a Killing distribution;
d) N is a parallel distribution with respect to ∇^F.

Proof. From the corrolary 3.2 follows that for isotropic or totally degenerate foliations the connection ∇^F is metric. We shall consider therefore that F is r-degenerate or coisotropic. From the theorem 3.13 follows that ∇^F is metric if and only if the degenerate second fundamental forms vanish identically on F . On the other hand from the theorem 3.7 and (3.92) follow that ∇^F is metric if and only if $h^L(P_2X,P_2Y)=0\ \forall\ X,Y\in D_F$.

a)$\Rightarrow$b) From (3.84) and the nondegenerate character of S (F) we have: $g(A^{*}_{\xi}P_2X,P_2Y)=g(\xi,h^L(P_2X,P_2Y))=0$ therefore $A^{*}_{\xi}P_2X=0\ \forall X,Y\in D_F\ \ \forall\xi\in N$.

b)$\Rightarrow$a) From (3.84) follows: $g(\xi,h^L(P_2X,P_2Y))=g(A^*_\xi P_2X,P_2Y)=0$ therefore $h^L(P_2X,P_2Y)=0\ \forall X,Y\in D_F$ that is ∇^F is metric connection.

N is a Killing distribution if and only if $g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)=0\ \ \forall\xi\in N$ $\forall X,Y\in D_F$. Using (3.37), (3.83) and (3.93) we have:

$$g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)=-g(A^*_\xi X,Y)-g(A^*_\xi Y,X)$$

From (3.91) follows that for $X,Y\in N$ the upper expresion vanishes. Also If $Y=\xi'$ then $-g(A^*_\xi X,\xi')-g(A^*_\xi\xi',X)=0$. It follows therefore that N is Killing distribution if and only if $g(A^*_\xi P_2X,P_2Y)+g(A^*_\xi P_2Y,P_2X)=0\ \forall X,Y\in D_F$. But from (3.88) we have that N is Killing if and only if $g(A^*_\xi P_2X,P_2Y)=0\ \forall X,Y\in D_F$.

a)$\Rightarrow$c) From (3.84) follows $g(A^*_\xi P_2X,P_2Y)=g(\xi,h^L(P_2X,P_2Y))=0$.

c)$\Rightarrow$a) From (3.84) follows $0=g(A^*_\xi P_2X,P_2Y)=g(\xi,h^L(P_2X,P_2Y))\ \ \forall\xi\in N$ therefore $h^L(P_2X,P_2Y)=0\ \forall\ X,Y\in D_F$.

b)$\Rightarrow$d) If $A^*_\xi P_2X=0\ \forall X\in D_F\ \ \forall\xi\in N$ then from (3.91) follows $\nabla^F{}_X\xi\in N\ \ \forall X\in D_F\ \forall\xi\in N$ therefore N is parallel with respect to ∇^F.

d)$\Rightarrow$b) If N is parallel with respect to ∇^F then $A^*_\xi X=0\ \forall X\in D_F\ \forall\xi\in N$.

If we consider now foliations with arbitrary null distribution we can proove other general results.

From (3.89) we have like a particular case:

$$A^*_\xi\xi=0\ \forall\ \xi\in N \qquad (3.95)$$

It is easy to show that in this case the theorem 3.22 becomes:

Theorem 3.23 Let a degenerate foliation F in a Semi-Riemannian manifold (M,g). The next assertions are equivalent:

a) The induced connection ∇^F is metric;
b) A^*_ξ vanishes on $D_F\ \forall\ \xi\in N$;
c) N is a Killing distribution;
d) N is a parallel distribution with respect to ∇^F.

Theorem 3.24 Let a degenerate foliation F in a Semi-Riemannian manifold (M,g). The next assertions are equivalent:

(i) The screen distribution S (F) is integrable;
(ii) The second fundamental form of S (F) h^* is symmetric on S (F);
(iii) The Weingarten operator A_N is self-adjoint on S (F)) with respect to g $\forall N \in deg(F)$.

Proof. (i)$\Leftrightarrow[P_2X,P_2Y]\in S(F)\ \forall X,Y\in D_F \Leftrightarrow(\nabla^*_{P_2X} PY-\nabla^*_{P_2Y} PX)+(h^*(P_2X,P_2Y)-h^*(P_2Y,P_2X))\in S(F))\Leftrightarrow h^*(P_2X,P_2Y)=h^*(P_2Y,P_2X)\ \forall X,Y\in D_F \Leftrightarrow$(ii). From (3.86) we have that (ii)$\Leftrightarrow g(h^*(P_2X,P_2Y),N)=g(h^*(P_2X,P_2Y),N)\Leftrightarrow g(A_NP_2X,P_2Y)=$

$g(P_2X,A_NP_2Y)\Leftrightarrow$(iii).

Theorem 3.25 Let F a degenerate foliation in a Semi-Riemannian manifold (M,g). The next assertions are equivalent:

(i) The screen distribution S (F) is parallel with respect to ∇^F;

(ii) The second fundamental form of S (F) h^* identically vanishes;

(iii) The Weingarten operator A_N takes values in N .

Proof. From (3.82) we have that (i)$\Leftrightarrow$(ii) and from (3.48) that (i)$\Leftrightarrow$(iii).

In the final of this section it is interesting to see when the null distribution is integrable (from the point of view of the new geometrical objects).

Theorem 3.26 Let F a degenerate foliation in a Semi-Riemannian manifold (M,g). The next assertions are equivalent:

(i) N is integrable;
(ii) $h^L(\xi,P_2X)=0\ \forall\xi\in N\ \ \forall X\in D_{F;}$
(iii) A^*_ξ identically vanishes on N .

Proof. From (3.84) we have $g(A^*_\xi\xi',P_2X)=g(\xi,h^L(\xi',P_2X))\ \forall\xi,\xi'\in N\ \ \forall X\in D_F$. If (ii) holds then $g(A^*_\xi\xi',P_2X)=0\ \forall\xi,\xi'\in N\ \ \forall X\in D_F$ therefore (iii) and reciprocally if (iii) is true then $g(\xi,h^L(\xi',P_2X))\ \forall\xi,\xi'\in N\ \forall X\in D_F$ from where (ii). From (3.91) follows that if (i) is true that is N is integrable then $A^*_\xi\xi'=0\ \forall\xi,\xi'\in N$ therefore (iii). If (iii) is true then $\forall\xi,\xi'\in N\ \ \forall X\in D_F$ follows:

$g([\xi,\xi'],P_2X)=g(\nabla_\xi\xi',P_2X)-g(\nabla_{\xi'}\xi,P_2X)=-g(A^*_{\xi'}\xi,P_2X)+g(A^*_\xi\xi',P_2X)=0$

therefore $[\xi,\xi']\in N\ \forall\ \xi,\xi'\in N$ which is the same thing with (i).

4. Totally geodesic degenerate foliations

Definition We call a degenerate foliation (F, g, S (F), S (F $^\perp$)) of codimension m of a (m+n)-dimensional Semi-Riemannian manifold (M,g) totally geodesic degenerate foliation if any geodesic of an arbitrary leaf of F is a geodesic of M.

Theorem 4.1 Let (F, g, S (F), S (F $^\perp$)) a degenerate foliation of a Semi-Riemannian manifold (M,g). F is totally geodesic if and only if one of the following statements is true:

(i) $h^L=h^S=0$;

(ii) ii$_1$) $A^*_\xi X=0\ \forall\ \xi\in N\ \ \forall X\in D_{F;}$

ii$_2$) $A_W X\in N\ \ \forall W\in S(F^\perp))\ \forall X\in D_{F;}$

ii$_3$) $D^L(X,P_3V)=0\ \forall X\in D_F\ \forall V\in tr(F)$

Proof. The condition that F is totally geodesic is equivalent with $\nabla_X X\in D_F$ $\forall\ X\in D_F$. From (3.37) we see that this is equivalent with $h^L(X,X)=h^S(X,X)=0$ and from the symmetry of h^L and h^S we have (i). Let prove now that (i)$\Rightarrow$(ii). If $h^L=h^S=0$ from (3.84) follows that $g(A^*_\xi X,P_2Y)=0\ \forall\xi\in N\ \forall X,Y\in D_F$ therefore: $A^*_\xi X=0$ that is ii$_1$). From (3.83) and ii$_1$) follows that $\nabla^F_X\xi\in N$ and from (3.37): $\nabla_X\xi\in N$. We have now with (3.40):

$$0=\nabla_X g(\xi,P_3V)=g(\nabla_X\xi,P_3V)+g(\xi,\nabla_X P_3V)=g(\xi,\nabla_X P_3V)=g(\xi,D^L(X,P_3V))$$

$\forall\xi\in N\ \forall X\in D_F\ \forall V\in tr(F)$ therefore ii$_3$). Finally, from (3.44) and ii$_3$) we have: $g(A_W X,Y)=g(Y,D^L(X,W))=0\ \forall X,Y\in D_F\ \ \forall W\in S(F^\perp)$ therefore ii$_2$).

If we shall suppose now that (ii) is true then from (3.84) and ii$_1$) follows: $g(\xi,h^L(X,P_2Y))=g(A^*_\xi X,P_2Y)=0$ therefore $h^L(X,P_2Y)=0\ \ \forall X,Y\in D_F$. From the theorem 3.7 follows that $h^L(\xi,\xi')=0\ \forall\xi,\xi'\in N$. We have finally that $h^L=0$. From (3.44), ii$_2$) and ii$_3$) we have now: $g(h^S(X,Y),W)=0\ \ \forall X,Y\in D_F$ $\forall W\in S(F^\perp)$. Because $S(F^\perp)$ is nondegenerate follows that $h^S=0$ therefore finally (i).

Corrolary 4.1 Let (F, g, S (F)) a coisotropic foliation of a Semi-Riemannian manifold (M,g). The foliation F is totally geodesic if and only if one of the following statements is true:

(i) $h^L=0$;

(ii) $A^*_\xi X=0\ \forall\ \xi\in N\ \ \forall X\in D_F$

Proof. In this case $h^S=0$, $S(F^\perp)=\{0\}$, $P_3=0$ and the statement reduces to the theorem 4.1.

Corrolary 4.2 Let (F, g, $S(F^\perp)$) an isotropic foliation of a Semi-Riemannian manifold (M,g). The foliation F is totally geodesic if and only if one of the following statements is true:

(i) $h^S=0$;

(ii) $D^L(X,P_3V)=0\ \forall X\in D_F\ \ \forall V\in tr(F)$.

Proof. In the case of isotropic foliations, from the corrolary 3.1 follows $h^L=0$ and how S (F)={0} and ii_2) is trivial follows the conclusions of the corrolary.

Corrolary 4.3 If (F, g) is a totally degenerate foliation of a Semi-Riemannian manifold (M,g) then the foliation F is totally geodesic.

Proof. From the corrolary 3.1 we have that $h^L=0$ and how $P_3=0$ we have $h^S=0$ therefore from the theorem 4.1 follows that F is totally geodesic.

Corrolary 4.4 If (F, g, S (F), $S(F^\perp)$) is a totally geodesic degenerate foliation of a Semi-Riemannian manifold (M,g) then the null distribution N is integrable.

Proof. From the theorem 3.26 we see that N is integrable if and only if $h^L(\xi,P_2X)=0\ \forall\ \xi\in N\ \ \forall X\in D_F$. From the theorem 4.1.i) the condition is satisfied by the totally geodesic degenerate foliation.

From the theorems 3.8, 3.9, 3.10 and 3.11 we have:

Theorem 4.2 In a totally geodesic degenerate foliation (F, g, S (F), $S(F^\perp)$) of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is independent of the screen distribution

Remark From (3.37) follows that ∇^F coincides with the restriction of ∇ on D_F.

From the theorem 3.13 we have the following:

Theorem 4.3 In a totally geodesic degenerate foliation (F, g, S (F), S ($F^{\perp}$)) of a Semi-Riemannian manifold (M,g) the induced connection ∇^F on F is metric.

5. Totally umbilical degenerate foliations

Definition We call a degenerate foliation (F, g, S (F), S ($F^{\perp}$)) of codimension m of a (m+n)-dimensional Semi-Riemannian manifold (M,g) totally umbilical degenerate foliation if $\exists H_L \in deg(F)$, $H_S \in S(F^{\perp})$ with the property that:

(5.1) $$h^L(X,Y)=g(X,Y)H_L$$

(5.2) $$h^S(X,Y)=g(X,Y)H_S$$

$\forall X,Y \in D_F$.

Remark From the theorem 4.1 follows that a totally umbilical degenerate foliation is totally geodesic if and only if $H_L=0$ and $H_S=0$.

Remark In the cases of coisotropic or totally degenerate foliations, because $P_3=0$, only the axiom (5.1) is necessary for totally umbilicality.

If we consider now the totally umbilical degenerate foliation F, the formula (3.37) becomes:

(5.3) $$\nabla_X Y=\nabla^F{}_X Y+g(X,Y)H_L+g(X,Y)H_S \ \forall X,Y \in D_F$$

Also, the formula (3.44) becomes:

(5.4) $$g(H_S,W)g(X,Y)+g(Y,D^L(X,W))=g(A_W X,Y) \ \forall \ X,Y \in D_F \ \forall W \in S(F^{\perp})$$

From (3.45) we have:

(5.5) $$g(Y,\nabla^F{}_X \xi)=-g(X,Y)g(H_L,\xi) \ \forall X,Y \in D_F \ \forall \xi \in N$$

If we shall note $(D\xi)(X)=\nabla^F{}_X\xi$ we have from (5.5):

(5.6) $$g(Y,(D\xi)(X))=g(X,(D\xi)(Y)) \ \forall \ X,Y\in D_F$$

From (5.6) follows therefore:

Theorem 5.1 On a totally umbilical degenerate foliation for any $\xi\in N$ the operator $D\xi$ is self-adjoint on D_F with respect to g.

From the definition, we have also

(5.7) $$h^L(X,\xi)=0$$

(5.8) $$h^S(X,\xi)=0$$

$\forall \ X\in D_F \ \forall\xi\in N$.

Theorem 5.2 If (F, g, S (F), S (F $^{\perp}$)) is a totally umbilical degenerate foliation of a Semi-Riemannian manifold (M,g) then the null distribution N is integrable.

Proof. From the theorem 3.26 we have that N is integrable if and only if $h^L(\xi,P_2X)=0 \ \forall\xi\in N \ \forall X\in D_F$. From (5.7) follows this type of foliations satisfies that this.

Theorem 5.3 A totally umbilical isotropic foliation is totally geodesic degenerate.

Proof. If F is isotropic then $N = D_F$. From (5.7) and (5.8) follows that $h^L=h^S=0$ and from the theorem 4.1.i) follows that the foliation is totally geodesic.

Because the totally degenerate foliations are totally geodesic and after the theorem 5.3 the isotropic are also totally geodesic from this moment we shall consider only the cases of r-degenerate with $r<\min\{m,n\}$ or coisotropic foliations.

From (5.7) and the theorem 3.26 we have therefore:

Theorem 5.4 On a totally umbilical r-degenerate with $r<\min\{m,n\}$ or coisotropic foliation we have that $\forall\xi\in N$ the operator A^*_ξ of S (F) vanishes identically on N .

From (3.67) we have now:

(5.9) $(\nabla^F{}_X g)(Y,Z)=g(H_L,Z)g(X,Y)+g(H_L,Y)g(X,Z)$

$\forall$ X,Y,Z$\in D_F$.

From (5.9) follows:

(5.10) $\nabla^F{}_\xi g=0$

Theorem 5.5 On a totally umbilical degenerate foliation ∇^F is a linear connection metric on S (F).

Proof. From (5.9) for $Y\to P_2Y$ and $Z\to P_2Z$ we have:

(5.11) $(\nabla^F{}_X g)(P_2Y,P_2Z)=0$

$\forall X,Y,Z\in D_F \forall\xi\in N$.

Theorem 5.6 On a totally umbilical r-degenerate with r<min{m,n} or coisotropic foliation the induced connection ∇^F is metric if and only if $H_L=0$ (or $h^L=0$).

Proof. From (5.9) we have:

(5.12) $(\nabla^F{}_X g)(P_2Y,\xi)=g(H_L,\xi)g(X,P_2Y)+g(H_L,P_2Y)g(X,\xi)=g(H_L,\xi)g(X,P_2Y)$

$\forall X,Y\in D_F \forall\xi\in N$. If F is r-degenerate with r<min{m,n} or coisotropic then S (F) does not coincides with the null distribution, therefore we can choose a non-null vector field $X\in S$ (F)). If ∇^F is a metric connection then from (5.12) follows: $0=(\nabla^F{}_X g)(P_2X,\xi)=g(H_L,\xi)g(P_2X,P_2X)$ therefore $g(H_L,\xi)=0$ $\forall\xi\in N$ that is $H_L=0$. Reciprocally, from (5.12) follows that if $H_L=0$ then $(\nabla^F{}_X g)(PY,\xi)=0$ $\forall X,Y\in D_F$ $\forall\xi\in N$. Also, from (5.9) follows: $(\nabla^F{}_X g)(\xi,\xi')= g(H_L,\xi')g(X,\xi)+g(H_L,\xi)g(X,\xi')=0$ $\forall\xi,\xi'\in N$. From (5.11) we have:

$(\nabla^F{}_X g)(P_2Y,P_2Z)=0$ $\forall X,Y,Z\in D_F$ therefore ∇^F is a metric connection.

Corrolary 5.2 On a totally umbilical coisotropic foliation the induced connection ∇^F is metric if and only if it is totally geodesic.

Proof. On coisotropic foliations we have $P_3=0$ and therefore from the theorem 5.6 follows that ∇^F is a metric connection if and only if $h^L=0$. But this does not means else that the foliation is degenerate totally geodesic.

Theorem 5.7 On a totally umbilical foliation r-degenerate with r<min{m,n} or coisotropic any vector field of the screen distribution is proper for the Weingarten operator of S (F): A^*_ξ $\forall\xi\in N$.

Proof. From (3.84) we have:

(5.13) $$g(A^*_\xi P_2X,P_2Y)=g(P_2X,P_2Y)g(H_L,\xi)$$

$\forall X,Y\in D_F\ \forall\xi\in N$.

Because $A^*_\xi P_2X\in S(F))$ $\forall\xi\in N$ $\forall X\in D_F$ and S (F) is nondegenerate, from (5.13) follows:

(5.14) $$A^*_\xi P_2X=g(H_L,\xi)P_2X\ \forall\xi\in N\ \ \forall X\in D_F$$

Theorem 5.8 On a totally umbilical foliation (F, g, S (F), S (F $^\perp$)) r-degenerate with r<min{m,n} the following statements are equivalents:

a) ∇^t is a linear metric connection relative to S (F);
b) $A_WP_2X=g(H_S,W)P_2X\ \forall X\in D_F\ \ \forall W\in S(F^\perp)$

Proof. a)⇒b) From (3.72)-(3.74) follows:

(5.15) $$g(A_WP_2X,N)=0$$

$\forall X\in D_F\ \forall W\in S(F^\perp)\ \forall N\in deg(F)$.

From (5.15) we have that $A_WP_2X\in S(F))\ \forall X\in D_F\ \ \forall W\in S(F^\perp)$.

From (5.4) we have also:

(5.16) $$g(A_WP_2X,P_2Y)=g(H_S,W)g(P_2X,P_2Y)+g(P_2Y,D^L(P_2X,W))=g(H_S,W)g(P_2X,P_2Y)$$

therefore:

(5.17) $g(A_W P_2 X - g(H_S,W) P_2 X, P_2 Y)=0$

$\forall$ X,Y$\in D_F$ $\forall W\in S(F^{\perp})$. How S (F) is nondegenerate follows from this:

(5.18) $A_W P_2 X = g(H_S,W) P_2 X$ $\forall X\in D_F$ $\forall W\in S(F^{\perp})$

b)$\Rightarrow$a) If (5.18) holds then $A_W P_2 X\in S(F)$) $\forall X\in D_F$ $\forall W\in S(F^{\perp})$. From (3.72)-(3.74) follows $(\nabla^t_{P_2X} g)(V,V')=0$ $\forall V,V'\in tr(F)$).

Theorem 5.9 Let (F, g, S (F), S $(F^{\perp})$) a foliation r-degenerate with r<min{m,n} or coisotropic of (M,g). Then F is degenerate totally umbilical if and only if the following statements hold:

(i) $h^L(X,\xi)=h^S(X,\xi)=0$ $\forall X\in D_F$ $\forall \xi\in N$;

(ii) $\exists\alpha\in\Lambda^1(S(F^{\perp}))$ such that $g(A_W P_2 X,P_2 Y)=\alpha(W) g(P_2 X,P_2 Y)$ $\forall X,Y\in D_F$ $\forall W\in S(F^{\perp})$;

(iii) $\exists\beta\in\Lambda^1(N)$ such that $A^*_\xi P_2 X=\beta(\xi) P_2 X$ $\forall X\in D_F$ $\forall\xi\in N$.

Proof. If F is totally umbilical then (i) follows from (5.7) and (5.8). From (5.16) defining $\alpha(W)=g(H_S,W)$ $\forall W\in S(F^{\perp})$) follows (ii). Finally, from (5.14) defining $\beta(\xi)=g(H_L,\xi)$ $\forall\ \xi\in N$ follows (iii).

Reciprocally, let suppose that (i), (ii), (iii) are true. We define now $H_S\in S(F^{\perp})$ such that:

(5.19) $g(H_S,W)=\alpha(W)$ $\forall W\in S(F^{\perp})$

and $H_L\in deg(F)$ such that

(5.20) $g(H_L,\xi)=\beta(\xi)$ $\forall\ \xi\in N$

From (3.44) we have $g(h^S(P_2X,P_2Y),W)=g(A_W P_2 X,P_2 Y)=g(H_S,W) g(P_2 X,P_2 Y)$ and because S $(F^{\perp})$ is nondegenerate follows that $h^S(P_2X,P_2Y)=g(P_2X,P_2Y)H_S$. From (i) we have now (5.2). From (3.84) follows that $g(h^L(P_2X,P_2Y),\xi)= g(A^*_\xi P_2X,P_2Y)=g(H_L,\xi)g(P_2X,P_2Y)$ therefore $h^L(P_2X,P_2Y)=g(P_2X,P_2Y)H_L$ and with

(i) we have (5.1). From (5.1) and (5.2) follows that F is totally umbilical degenerate foliation.

Let see now some examples that illustrate the phenomenon of totally geodesibility or umbilicality.

5.1. If we go back to the example 2.1 we have:

$$\overline{\nabla}_X X = -\frac{\varphi'(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}^3} W - \frac{\varphi(x^1,x^2)\varphi'(x^1,x^2)\sin\alpha}{\cos^2\alpha\left(\varphi^2(x^1,x^2)-1\right)^3}\xi$$

$$\overline{\nabla}_X \xi = \frac{2\varphi(x^1,x^2)\varphi'(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}^3}\xi$$

$$\overline{\nabla}_\xi \xi = 0$$

therefore:

$$\nabla_X X = -\frac{\varphi(x^1,x^2)\varphi'(x^1,x^2)\sin\alpha}{\cos^2\alpha\left(\varphi^2(x^1,x^2)-1\right)^3}\xi$$

$$\nabla_X \xi = \frac{2\varphi(x^1,x^2)\varphi'(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}^3}\xi$$

$$\nabla_\xi \xi = 0$$

Finally we have:

$h^L(X,X)=0$, $h^L(X,\xi)=0$, $h^L(\xi,\xi)=0$, $h^S(X,X)=-\frac{\varphi'(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}^3}W$, $h^S(X,\xi)=0$, $h^S(\xi,\xi)=0$. If we consider now $H_S=-\frac{\varphi'(x^1,x^2)}{\sqrt{\varphi^2(x^1,x^2)-1}^3}W$ follows $h^S(X,X)=g(X,X)H_S$ therefore the foliation is totally umbilical 1-degenerate.

5.2. We shall present now an example from [2]. Let the 1-degenerate foliation in $\mathbf{R}^4_2(-,-,+,+)$ with a quasi-orthonormal basis given by:

$$\xi = \partial_1 + \partial_2 + \sqrt{2}\partial_3$$
$$X = \sqrt{2}\left[1+(x^1-x^2)^2\right]\partial_2 + \left[1+(x^1-x^2)^2\right]\partial_3 - \sqrt{2}(x^1-x^2)\partial_4$$
$$W = 2\left(x^2-x^1\right)\partial_2 + \sqrt{2}(x^2-x^1)\partial_3 + \left[1+(x^1-x^2)^2\right]\partial_4$$
$$N = -\frac{1}{2}\partial_1 + \frac{1}{2}\partial_2 + \frac{1}{\sqrt{2}}\partial_3$$

where N =Span(ξ),S (F)=Span(X),S (F $^{\perp}$)=Span(W) and deg(F)=Span(N).

The fact that it is a foliation follows from:

$$[\xi, X] = 2\sqrt{2}(x^1-x^2)\partial_2 + 2(x^1-x^2)\partial_3 - \sqrt{2}\partial_4 - 2\sqrt{2}(x^1-x^2)\partial_2 - 2(x^1-x^2)\partial_3 + \sqrt{2}\partial_4 = 0$$

We have now easy that $h^L=0$, $h^S(X,\xi)=h^S(\xi,\xi)=0$ and

$$h^S(X,X) = \frac{2\left(1-(x^1-x^2)^4\right)}{1+(x^1-x^2)^4}W$$

Because $g(X,X)=-(1+(x^1-x^2)^4)$ we have that $h^S(X,X)=g(X,X)H_S$ where $H_S=\frac{2\left((x^1-x^2)^4-1\right)}{\left(1+(x^1-x^2)^4\right)^2}W$. We have therefore that F is a totally umbilical 1-degenerate foliation in $\mathbf{R}^4_2$.

5.3. Let consider now the example 2.3. Because $h^L(\xi,X)=h(\xi,\xi)=0$ we have $h^L_1(X,X)=g(\nabla_X X,\xi)=-2y(\sin z+\cos z)(\sin z+\cos z-1)$. Let therefore:

$$H_L = \frac{-2(\sin z+\cos z)}{9y^2(\sin z-\cos z)^2(\sin z+\cos z-1)}\left[y(\sin z+\cos z-1)\partial_x + y(3\sin z\cos z-2)\partial_y + 3(\sin z-\cos z)(\sin z+\cos z-1)\partial_z\right]$$

We have $H_L \in \deg(F)$ and $h^L(X,X)=g(X,X)H_L$. Because $0=h^L(X,\xi)=g(X,\xi)H_L$ and $0=h^L(\xi,\xi)=g(\xi,\xi)H_L$ follows that the foliation is coisotropic, totally umbilical.

5.4. From the corrolary 4.3 follows that the example 2.5 is a totally degenerate and totally geodesic foliation.

6. Examples of degenerate foliations

on manifolds provided with relativistic metrics

Let therefore the manifold M with the metric:

(6.1) $$ds^2 = V^2(r)dt^2 - \frac{1}{V^2(r)}dr^2 - r^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right]$$

where $V \neq 0$.

Remark We have the following particular cases:

1) $V^2(r) = 1 - \frac{2m}{r}$ correspond to the exterior Schwarzschild metric;

2) $V^2(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$ correspond to the Reissner - Weil metric;

3) $V^2(r) = 1 - \frac{r^2}{R^2}$ correspond to the de Sitter metric;

4) $V^2(r) = 1$ correspond to the Minkowski metric;

5) $V(r) = Cr$, $C \in \mathbf{R}^*$

We shall note for the simplicity: $\partial_t = \frac{\partial}{\partial t}, \partial_r = \frac{\partial}{\partial r}, \partial_\theta = \frac{\partial}{\partial \theta}, \partial_\varphi = \frac{\partial}{\partial \varphi}$.

Theorem 6.1 Let the Semi-Riemannian manifold M endowed with the metric:

$$ds^2 = V^2(r)dt^2 - \frac{1}{V^2(r)}dr^2 - r^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right], \ V \neq 0$$

If ∇ is the Levi-Civita connection on M then the following relations hold:

$$\nabla_{\partial_t}\partial_t = V^3 V'_r\,\partial_r \quad \nabla_{\partial_t}\partial_r = \nabla_{\partial_r}\partial_t = \frac{V'_r}{V}\partial_t \quad \nabla_{\partial_r}\partial_r = -\frac{V'_r}{V}\partial_r \quad \nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta$$

$$\nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r = \frac{1}{r}\partial_\varphi \quad \nabla_{\partial_\theta}\partial_\theta = -rV^2\partial_r \quad \nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \frac{\cos\theta}{\sin\theta}\partial_\varphi$$

$$\nabla_{\partial_\varphi}\partial_\varphi = -rV^2\sin^2\theta\,\partial_r - \sin\theta\cos\theta\,\partial_\theta$$

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Proof. Through direct calculus.

6.1. Let the foliation F generated by the vector fields: $\xi=\frac{1}{V}\partial_t+\frac{1}{r}\partial_\theta$ and $X=e^{tV}\partial_\varphi$. We have $[\xi,X]=[\frac{1}{V}\partial_t+\frac{1}{r}\partial_\theta, e^{tV}\partial_\varphi]=e^{tV}\partial_\varphi=X$, $[\xi,\xi]=[X,X]=0$ therefore F is a foliation. Because $g(\xi,\xi)=g(\frac{1}{V}\partial_t+\frac{1}{r}\partial_\theta, \frac{1}{V}\partial_t+\frac{1}{r}\partial_\theta)=0$, $g(\xi,X)=0$, $g(X,X)=-r^2e^{2tV}\sin^2\theta\neq 0$ follows that the foliation is degenerate. We have therefore N =Span(ξ) and S (F)=Span(X). Considering $D_F=\{\alpha\xi+\beta X \mid \alpha,\beta\in$ F (M)} we have: $D_F^{\perp}=\{ar\partial_t+b\partial_r+aV\partial_\theta \mid a,b\in$ F (M)}$=Span(r\partial_t+V\partial_\theta,\partial_r)= Span(\xi,\partial_r)$. We obtain therefore S (F $^\perp$)=Span(W) where $W=\partial_r$. If consider now $N=\frac{1}{V}\partial_t+V\partial_r$ we have deg(F)=Span(N) and therefore the foliation is 1-degenerate with a local quasi-orthonormal basis given by {X,W,ξ,N}.

If we compute the principal geometrical objects we have:

$h^L(\xi,\xi)=h^L(X,\xi)=0,\ h^L(X,X)=re^{2tV}\sin\theta\cos\theta N$

$h^S(\xi,\xi)=(VV'_r-\frac{V^2}{r})W,\ h^S(\xi,X)=0,\ h^S(X,X)= -rVe^{2tV}\sin\theta\cos\theta W$

$$\nabla^F{}_\xi\xi=0, \nabla^F{}_\xi X=\left(1+\frac{\cos\theta}{r\sin\theta}\right)X, \nabla^F{}_X X=-re^{2tV}\sin\theta\cos\theta\xi, \nabla^F{}_X\xi=\frac{\cos\theta}{r\sin\theta}X$$

$$\nabla^*{}_\xi X=\left(1+\frac{\cos\theta}{r\sin\theta}\right)X, \nabla^*{}_X X=0$$

$$\nabla^{*t}{}_\xi\xi=\nabla^{*t}{}_X\xi=0$$

$$h^*(\xi,X)=0, h^*(X,X)=-re^{2tV}\sin\theta\cos\theta\xi$$

$$A^*_\xi\xi=0, A^*_\xi X=-\frac{\cos\theta}{r\sin\theta}X$$

$$A_W\xi=-\frac{1}{r}\xi, A_W X=-\frac{1}{r}X, A_N\xi=-\frac{V}{r}\xi, A_N X=-\frac{V}{r}X$$

$$D^L(\xi,W)=\left(\frac{V'_r}{V}-\frac{1}{r}\right)N, D^L(X,W)=0, D^S(\xi,N)=\frac{V^2}{r}W, D^S(X,N)=0$$

From these and the theorem 4.1 follows that F is not totally geodesic. In order that F be totally umbilical it must that: $h^S(\xi,\xi)=0 \Leftrightarrow VV'_r - \frac{V^2}{r}=0$ herefore $V(r)=Cr$, $C\in \mathbf{R}^*$. Reciprocally, if $V(r)=Cr$, $C\in \mathbf{R}^*$ then $h^S(\xi,\xi)=0$. If we define:

$$H_L = -\frac{\cos\theta}{r\sin\theta} N \text{ and } H_S = \frac{C\cos\theta}{\sin\theta} W$$

we have $h^L(X,Y)=g(X,Y)H_L$ and $h^S(X,Y)=g(X,Y)H_S$ $\forall X,Y\in D_F$ therefore the foliation is totally umbilical.

6.2. Let the foliation F generated by the vector fields $\xi = \frac{1}{V}\partial_t + V\partial_r$]i $X = e^{\alpha\int\frac{1}{V(r)}dr}\partial_\theta$, $\alpha\in \mathbf{R}$. We have $[\xi,X]=\alpha X$, $[\xi,\xi]=[X,X]=0$ therefore F is really a foliation. Because $g(\xi,\xi)=0$, $g(\xi,X)=0$, $g(X,X)=-r^2e^{2\alpha\int\frac{1}{V(r)}dr}\neq 0$ follows that the foliation is degenerate. We have therefore: N = Span(ξ) and S (F)=Span(X). Considering now $D_F=\{\alpha\xi+\beta X \mid \alpha,\beta\in F(M)\}$ we have: $D_F^\perp=\{a\partial_t+aV^2\partial_r+b\partial_\varphi \mid a,b\in F(M)\}=\mathrm{Span}(\xi,\partial_\varphi)$. We obtain therefore S $(F^\perp)$=Span(W) where $W=\partial_\varphi$. Considering $N=\frac{1}{2V}\partial_t-\frac{V}{2}\partial_r$ we have deg(F)=Span(N) and therefore the foliation is 1-degenerate with a local quasi-orthonormal basis given by {X,W,ξ,N}.

If we compute the principal geometrical objects we have:

$$h^L(\xi,\xi)=0,\ h^L(\xi,X)=0,\ h^L(X,X)=rVe^{2\alpha\int\frac{1}{V(r)}dr}N$$
$$h^S(\xi,\xi)=0,\ h^S(\xi,X)=0,\ h^S(X,X)=0$$

$$\nabla^F{}_\xi\xi=V'(r)\xi,\ \nabla^F{}_\xi X=\left(\alpha+\frac{V}{r}\right)X,\ \nabla^F{}_X X=-\frac{rV}{2}e^{2\alpha\int\frac{1}{V(r)}dr}\xi,\ \nabla^F{}_X\xi=\frac{V}{r}X$$
$$\nabla^*{}_\xi X=\left(\alpha+\frac{V}{r}\right)X,\ \nabla^*{}_X X=0$$

$$\nabla^{*t}{}_\xi\xi=V'(r)\xi,\ \nabla^{*t}{}_X\xi=0$$
$$h^*(\xi,X)=0,\ h^*(X,X)=-\frac{rV}{2}e^{2\alpha\int\frac{1}{V(r)}dr}\xi$$

$$A^*_\xi \xi = 0, A^*_\xi X = \frac{V}{r} X$$

$$A_W \xi = 0, A_W X = 0, A_N \xi = 0, A_N X = -\frac{V}{r^2} X$$

$$D^L(\xi, W) = 0, D^L(X, W) = 0, D^S(\xi, N) = 0, D^S(X, N) = 0$$

From these and the theorem 4.1 follows that the foliation F is not totally geodesic because $h^L(X,X)\neq 0$. If we define now $H_L=-\frac{V}{r}N$ and $H_S=0$ we have $h^L(X,Y)=H_L g(X,Y)$ and $h^S(X,Y)=H_S g(X,Y)$ $\forall X,Y\in D_F$ therefore the foliation is totally umbilical.

6.3. Let the foliation F generated by the vector fields $\xi=f(r)\partial_t+V^2f(r)\partial_r$, $X_1=\frac{1}{r}\partial_\theta$, $X_2=\frac{1}{r\sin\theta}\partial_\varphi$ where f:**R**→**R** is a smooth map non-null everywhere.

We have $[\xi,X_1]=-\frac{V^2f(r)}{r}X_1$, $[\xi,X_2]=-\frac{V^2f(r)}{r^2\sin\theta}X_2$, $[X_1,X_2]=-\frac{\cos\theta}{r\sin\theta}X_2$ therefore F is really a foliation. Because $g(\xi,\xi)=g(\xi,X_1)=g(\xi,X_2)=0$, $g(X_1,X_1)=-1$, $g(X_2,X_2)=-1$ follows that F is a degenerate foliation.

We have therefore: N = Span(ξ) and S (F)=Span(X_1,X_2). Like upper we have: $N=\frac{1}{2V^2f(r)}\partial_t-\frac{1}{2f(r)}\partial_r$ therefore deg(F)=Span(N). The foliation is therefore coisotropic 1-codimensional with a local quasi-orthonormal basis given by $\{X_1,X_2,\xi,N\}$.

If we compute now the degenerate second fundamental form of F , we have:

$$h^L(\xi,\xi)=0\,,\,h^L(\xi,X_1)=0\,,\,h^L(\xi,X_2)=0\,,\,h^L(X_1,X_1)=h^L(X_2,X_2)=\frac{V^2f(r)}{r}N$$

Defining now: $H_L=\frac{V^2f(r)}{r}N$ follows that $h^L(X,Y)=H_L g(X,Y)$ $\forall X,Y\in D_F$ therefore the foliation F is coisotropic totally umbilical.

6.4. Let the foliation F generated by the vector field $\xi=f(r)\partial_t+V^2f(r)\partial_r$ where f:**R**→**R** is a smooth map non-null everywhere. Because $g(\xi,\xi)=0$ and $[\xi,\xi]=0$

follows that the foliation F is degenerate. If we shall proceed like in the first example, we have:

$$W_1 = \frac{1}{r}\partial_t + \frac{V^2}{r}\partial_r + \frac{1}{r}\partial_\theta,\ W_2 = \frac{1}{r\sin\theta}\partial_t + \frac{V^2}{r\sin\theta}\partial_r + \frac{1}{r\sin\theta}\partial_\varphi$$

where $g(W_1,W_1)=g(W_2,W_2)=-1$.

Also:

$$N = \frac{(V^2+r^2)\sin^2\theta + V^2}{2f(r)r^2\sin^2\theta V^2}\partial_t + \frac{(V^2-r^2)\sin^2\theta + V^2}{2f(r)r^2\sin^2\theta}\partial_r - \frac{1}{f(r)r^2}\partial_\theta - \frac{1}{f(r)r^2\sin^2\theta}\partial_\varphi$$

where $g(N,N)=1$.

The foliation is therefore isotropic 1-codimensional with a local quasi-orthonormal basis given by $\{\xi, W_1, W_2, N\}$.

If we compute the degenerate second fundamental form of F and the screen second fundamental form we have: $h^L=0$, $h^S=0$ therefore the foliation F is isotropic degenerate totally geodesic.

6.5. Because dim N =min {1,3}=1 we have therefore that on manifolds endowed with relativistic metrics does not exists totally degenerate foliations.

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