Mathematical and Quantitative Methods

Stackelberg Model for Linear Marginal Costs

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Abstract: The paper treats the Stackelberg where marginal costs corresponding to two companies are linear. It also examines the profitability of the merger of the two companies in order to maximize the profit.

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1. Introduction

Let two companies A and B. Consider first that the company A is a leading quantity. If it will produce good Q_A units of a good, then the company B will adjust its production at $Q_B=f(Q_A)$units of the same good (f being called the reaction function).

The sale price is dependent on the total quantity of goods reached the market. Be so:

$p=p(Q_A+Q_B)$

the price per unit of good.

The A company must establish a level of production related to the reaction of B, because through its production realized will determine the selling price of the product. Similarly, firm B will adjust its production according to A, because at a higher or lower production, the price will change and therefore the profit of the company.

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2 The Analysis

Consider the function of price (the inverse demand function) of the form:

p(Q)=a-bQ, a,b>0

Also, consider that the marginal costs for A and B are linear:

$$Cm_A(Q)=\alpha_A Q+\beta_A,\ Cm_B(Q)=\alpha_B Q+\beta_B,\ \alpha_A,\alpha_B\geq 0$$

Let also the profit of the leader:

$$\Pi_A(Q_A)=p(Q_A+Q_B)Q_A-CT_A(Q_A)=aQ_A-b(Q_A+Q_B)Q_A-CT_A(Q_A)=$$

$$-bQ_A^2+(a-bQ_B)Q_A-CT_A(Q_A)$$

Because Q_B=f(Q_A) we have:

$$\Pi_A(Q_A)=p(Q_A+f(Q_A))Q_A-CT_A(Q_A)=-bQ_A^2+aQ_A-bQ_Af(Q_A)-CT_A(Q_A)$$

Let consider now, also, the profit of the satellite:

$$\Pi_B(Q_B)=p(Q_A+Q_B)Q_B-CT_B(Q_B)=aQ_B-b(Q_A+Q_B)Q_B-CT_B(Q_B)=$$

$$-bQ_B^2+(a-bQ_A)Q_B-CT_B(Q_B)$$

The extreme condition for the profit of A is:

$$\frac{\partial\Pi_A(Q_A)}{\partial Q_A}=-2bQ_A+a-bf(Q_A)-bQ_Af'(Q_A)-Cm_A(Q_A)=0$$

or other, taking into account of the marginal cost expression:

$$-2bQ_A+a-bf(Q_A)-bQ_Af'(Q_A)-\alpha_AQ_A-\beta_A=0$$

and those for the satellite B:

$$\frac{\partial\Pi_B(Q_B)}{\partial Q_B}=-2bQ_B+a-bQ_A-Cm_B(Q_B)=0$$

or, taking into account of the marginal cost expression:

$$-2bQ_B+a-bQ_A-\alpha_BQ_B-\beta_B=0$$

Considering therefore the production of the leader Q_A being given, it follows that the production of the satellite satisfies the condition:

$$Q_B = \frac{a - \beta_B - bQ_A}{2b + \alpha_B}$$

Varying now the production Q_A we have that $Q_B=f(Q_A)=\frac{a - \beta_B - bQ_A}{2b + \alpha_B}$ from where, the question of leader's profit maximization is:

$$-2bQ_A + a - b\frac{a - \beta_B - bQ_A}{2b + \alpha_B} + \frac{b^2 Q_A}{2b + \alpha_B} - \alpha_A Q_A - \beta_A = 0$$

or other:

$$Q_A^* = \frac{ab + a\alpha_B - 2b\beta_A - \alpha_B\beta_A + b\beta_B}{2b^2 + 2b(\alpha_A + \alpha_B) + \alpha_A\alpha_B}$$

We obtain now for the satellite B:

$$Q_B^* = \frac{a - \beta_b - bQ_A}{2b + \alpha_B} =$$

$$\frac{(b^2 + (2\alpha_A + \alpha_B)b + \alpha_A\alpha_B)a + (2\beta_A - 3\beta_B)b^2 - (2\alpha_A\beta_b + 2\alpha_B\beta_B - \alpha_B\beta_A)b - \alpha_A\alpha_B\beta_B}{(2b + \alpha_B)(2b^2 + 2b(\alpha_A + \alpha_B) + \alpha_A\alpha_B)}$$

The condition for the leader to have a higher production than the satellite returns to $Q_A^* > Q_B^*$ which is equivalent to:

$$(a - 6\beta_A + 5\beta_B)b^2 + [(2a + 3\beta_B)(\alpha_B - \alpha_A) + 5(\alpha_A\beta_B - \alpha_B\beta_A)]b + a\alpha_B(\alpha_B - \alpha_A) + \alpha_B(\alpha_A\beta_B - \alpha_B\beta_A) > 0$$

From $Cm_A(Q) = \alpha_A Q + \beta_A$, $Cm_B(Q) = \alpha_B Q + \beta_B$ we obtain after a simple integration:

$CT_A(Q)=\frac{\alpha_A}{2}Q^2 + \beta_A Q + \gamma_A$, $CT_B(Q)=\frac{\alpha_B}{2}Q^2 + \beta_B Q + \gamma_B$ with $\gamma_A, \gamma_B \geq 0$

Returning to the profits of both firms A and B we have:

$$\Pi_A(Q_A^*) = p(Q_A^* + Q_B^*)Q_A^* - CT_A(Q_A^*) =$$

$$-bQ_A^{*2} + (a - bQ_B^*)Q_A^* - \frac{\alpha_A}{2}Q_A^{*2} - \beta_A Q_A^* - \gamma_A =$$

$$\frac{\left(-2b^2-2b(\alpha_A+\alpha_B)-\alpha_A\alpha_B\right)Q_A^{*2}+2\left(ab+a\alpha_B+b\beta_B-2b\beta_A-\alpha_B\beta_A\right)Q_A^*-4\gamma_A b-2\alpha_B\gamma_A}{2(2b+\alpha_B)}$$

$$\Pi_B\left(Q_B^*\right)=p\left(Q_A^*+Q_B^*\right)Q_B^*-CT_B\left(Q_B^*\right)=aQ_B^*-b\left(Q_A^*+Q_B^*\right)Q_B^*-\frac{\alpha_B}{2}Q_B^{*2}-\beta_B Q_B^*-\gamma_B=$$

$$\frac{\left(a-\beta_B-bQ_A^*\right)^2}{2(2b+\alpha_B)}-\gamma_B$$

Suppose now that the two firms merge to form a monopoly with the same total production:

$Q^*=Q_A^*+Q_B^*$

the price $p=p\left(Q_A^*+Q_B^*\right)=p\left(Q^*\right)$ keeping also constant.

The profit of the monopoly is:

$\Pi(Q^*)=(p-CTM(Q^*))\cdot Q^*$

where $CTM(Q^*)=\frac{CT_A(Q_A^*)+CT_B(Q_B^*)}{Q_A^*+Q_B^*}$ is the average cost of the production of the monopoly.

We have therefore:

$$\Pi(Q^*)=\left(p-\frac{CT_A(Q_A^*)+CT_B(Q_B^*)}{Q_A^*+Q_B^*}\right)\left(Q_A^*+Q_B^*\right)=p\left(Q_A^*+Q_B^*\right)-CT_A\left(Q_A^*\right)-CT_B\left(Q_B^*\right)=$$

$$a-b\left(Q_A^*+Q_B^*\right)-\frac{\alpha_A}{2}Q_A^{*2}-\beta_A Q_A^*-\gamma_A-\frac{\alpha_B}{2}Q_B^{*2}-\beta_B Q_B^*-\gamma_B=$$

$$-\frac{\alpha_A}{2}Q_A^{*2}-(b+\beta_A)Q_A^*+a-\frac{\alpha_B}{2}\frac{\left(a-\beta_b-bQ_A^*\right)^2}{(2b+\alpha_B)^2}-(b+\beta_B)\frac{a-\beta_b-bQ_A^*}{2b+\alpha_B}-\gamma_A-\gamma_B$$

But:

$\Pi_A\left(Q_A^*\right)+\Pi_B\left(Q_B^*\right)=$

$$\frac{\left(-b^2-2b\left(\alpha_A+\alpha_B\right)-\alpha_A\alpha_B\right)Q_A^{*2}+2\left(a\alpha_B+2b\beta_B-2b\beta_A-\alpha_B\beta_A\right)Q_A^*+a^2+\beta_B^2-2a\beta_B}{2\left(2b+\alpha_B\right)}-\gamma_A-\gamma_B$$

from where:

$$\Pi\left(Q_A^*+Q_B^*\right)-\Pi_A\left(Q_A^*\right)-\Pi_B\left(Q_A^*\right)=$$

$$\frac{b\left(b+\alpha_B\right)^2Q_A^{*2}-\left(b+\alpha_B\right)\left(2b^2+2b\beta_B+a\alpha_B+b\alpha_B\right)Q_A^*+\left(a\alpha_B+2ab+b\beta_B-a^2\right)\left(b+\alpha_B\right)+\beta_B\left(b\beta_B+b^2+a\alpha_B\right)}{\left(2b+\alpha_B\right)^2}$$

where:

$$Q_A^*=\frac{ab+a\alpha_B-2b\beta_A-\alpha_B\beta_A+b\beta_B}{2b^2+2b\left(\alpha_A+\alpha_B\right)+\alpha_A\alpha_B}$$

If $\Pi\left(Q_A^*+Q_B^*\right)-\Pi_A\left(Q_A^*\right)-\Pi_B\left(Q_A^*\right)>0$ then, when the two companies will merge, the profit will increase.

3 Conclusions

The assumptions made in the given hypothesis is plausible because in a reasonable time, the marginal cost can be assumed linear. Also, the terms of the merger of two companies in order to increase profitability are necessary to be known, because in a highly competitive market conditions and strong competing firms, an atomization of the production leading to low profits or even the disappearance of firms from the market.

4 References

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