Miscellaneous

On Certain Conditions for Generating Production Functions - I

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Abstract: The article is the first in a series that will treat underlying conditions to generate a production function. The importance of production functions is fundamental to analyze and forecast the various indicators that highlights different aspects of the production process. How often forgets that these functions start from some premises, the article comes just meeting these challenges, analyzing different initial conditions. On the other hand, where possible, we have shown the concrete way of determining the parameters of the function.

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1 Introduction

Theory of production functions is vitally important in microeconomic analysis.

The need of economic phenomena mathematization, not only from a desire to give legitimacy to scientific economic theory but rather, to draw conclusions and prediction of enterprise activity required a careful analysis of them.

Well-thought literature profile, but especially practical applications encountered in all kinds of handouts, printed or online, we drew a number of issues that sometimes are neglected (probably considered insignificant) or omitted with true intent.

The first issue found by us is that of verification of sufficient conditions (not always necessary, but depending on the actual nature of the problem) as a function to be truly of production.

Another aspect which seems essential is the practical applicability. One question that could be asked of any student from any part of the Earth, is: "Departing from a

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series of discrete data, how you will generate the output and, especially, what kind of production function will choose?”

By own researches, I realized that maybe over 90% of production functions presented in teaching applications are of Cobb-Douglas type (requiring, however, the constancy of elasticity), the remainder being more or less created artificial (often they even unverified existing conditions).

It might object here that the learning exercises aims to increase math skills with these functions. The problem is not this, but what follow...

I rarely saw concrete applications, showing clearly how to practically apply these functions. Without this approach, the theory remains dry, with beautiful graphics (as an aside, all graphs of production looks pretty much the same, what will result in the following) and without practical application.

Following these minimum considerations, we will try in the following pages to generate major production functions based on practical conditions (the approach being not new, meeting in original papers), but systematized and then explaining in each case how can apply them practically.

2 General Notions

In what follows, we assume that resources are infinitely divisible, which implies the use of specific tools of mathematical analysis to analyze specific phenomena.

We thus define on $\mathbf{R}^n$ the space of production for n fixed resources as:

$$SP=\{(x_1,...,x_n) \mid x_i\geq 0, i=\overline{1,n}\}$$

where $x\in SP$, $x=(x_1,...,x_n)$ is an ordered set of resources (inputs).

Because within a production process, depending on the nature of applied technology, but also its specificity, not any amount of resources possible, we will restrict the production area to a subset $D_p\subset SP$ called production domain.

It is now called production function (output) an application:

$$Q:D_p\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D_p$$

For an effective and complex mathematical analysis of a production function we will require a number of axioms (not all essential) both its scope and its definition.

A1. The production domain D_p is convex i.e. $\forall x=(x_1,...,x_n)$, $y=(y_1,...,y_n)\in D_p$ $\forall\lambda\in[0,1]$ follows $(1-\lambda)x+\lambda y=((1-\lambda)x_1+\lambda y_1,...,(1-\lambda)x_n+\lambda y_n)\in D_p$.

Axiom A1 only mean that in the process of changing of the inputs from a level x to y, the linear shift is achieved through a series of successive steps which keeps them in the field of production, so by default the possibility of using the production function chosen. The condition could relax here, requiring domain to be, for example, connected by arches, that to be a continuous path between any two n-uple inputs.

A2. Q(0,0,...,0)=0

The axiom reflects a common sense assumption namely that in the absence of any input can not get any output.

A3. The production function is continuous.

Continuity, in purely mathematical sense, represents that for any fixed point $\left(\overline{x}_1,...,\overline{x}_n\right)$ of the production domain D_p and any string of inputs $(y_k)_{k\geq1}$, $y_k=\left(y_1^k,...,y_n^k\right)$ which converges to $\left(\overline{x}_1,...,\overline{x}_n\right)$ (or otherwise $y_i^k \to \overline{x}_i \quad \forall i=\overline{1,n}$) the production $Q\left(y_1^k,...,y_n^k\right)$ converges to $Q\left(\overline{x}_1,...,\overline{x}_n\right)$.

More simply, the continuity of the production function means that for two sets of resources $(x_1,...,x_n)$ and $(y_1,...,y_n)\in D_p$ close enough, result outputs $Q(x_1,...,x_n)$ and $Q(y_1,...,y_n)$ close enough. In other words, a very small change of inputs lead to a reasonable production obtained.

An axiom, not necessarily required, but particularly useful for obtaining significant results (using differential calculus) is:

A4. The production function is of class $C^2(D_p)$ i.e. admits 2nd order continous partial derivatives.

The condition of belonging to the class C^2 may seem, at first glance, restrictive, but is not really. All basic functions (constant, power, exponential, logarithmic, trigonometric functions as those obtained from them by arithmetic operations of addition, subtraction, multiplication, division, power lifting, composing or reversal) are of C^∞ class (implicitly of class C^2) on the definition domain i.e. have their partial derivatives of any order and these are continues. As a function of class C^k, $k\geq0$ is continuous implies that axiom A3, given that accept A4, is a simple consequence of the latter, so it can be removed.

What is actually at least C^1 class differentiability? If for a continuous function means, at an immediately approach (without much mathematical rigor) that its graph is not „broken" on the definition domain, the derivativability of class C^1 means that it does not have „corners" or „folds", the graph being smooth. In addition, for example in a corner point (for functions of one variable – different left and right derivatives) we can not make predictions, the behavior at left/right not anticipates the behavior at right/left.

A5. The production function is monotonically increasing in each variable.

A5 axiom states that in "ceteris paribus" hypotesis, $\forall i=\overline{1,n}$ if $x_i \geq y_i$ then> $Q(\bar{x}_1,...\bar{x}_{i-1},x_i,\bar{x}_{i+1},\bar{x}_n) \geq Q(\bar{x}_1,...\bar{x}_{i-1},y_i,\bar{x}_{i+1},\bar{x}_n)$ $\forall\, \bar{x}_k \geq 0$, $k=\overline{1,n}$, $k \neq i$ such that $(\bar{x}_1,...\bar{x}_{i-1},x_i,\bar{x}_{i+1},\bar{x}_n)$, $(\bar{x}_1,...\bar{x}_{i-1},y_i,\bar{x}_{i+1},\bar{x}_n) \in D_p$. If the function Q is at least $C^1(D_p)$ the character of monotonically increasing becomes $\frac{\partial Q}{\partial x_i} \geq 0$, $i=\overline{1,n}$. In terms of a "classic" production function with two variables: K – capital and L - labor, we have: $\frac{\partial Q}{\partial K} \geq 0$, $\frac{\partial Q}{\partial L} \geq 0$.

Also from the axiom A5 result, as an immediate consequence, that if $x_1 \geq y_1,...,x_n \geq y_n$ then: $Q(x_1,x_2,...,x_n) \geq Q(y_1,x_2,...,x_n) \geq Q(y_1,y_2,...,x_n) \geq ... \geq Q(y_1,y_2,...,y_n)$. It is obvious that the relationship occurs only if the nature of the inequalities between components is the same for all of them.

A condition often referred to in the definition of the production function is:â

A6. The production function is quasi-concave.

The quasi-concavity of a function means:

$$Q(\lambda x+(1-\lambda)y) \geq \min(Q(x),Q(y)) \ \forall \lambda \in [0,1] \ \forall x,y \in R_p$$

Geometrically speaking, a quasi-concave function has property to be above the lowest values recorded at the end of a certain segment. The property is equivalent to the convexity of the set $Q^{-1}[a,\infty)$ $\forall a \in \mathbf{R}$, where $Q^{-1}[a,\infty)=\{x \in R_p \mid Q(x) \geq a\}$.

What does the quasi-concavity so? Convexity of the set $Q^{-1}[a,\infty)$ lies in that if $Q(x) \geq a$, $Q(y) \geq a$ then Q $((1-\lambda)x+\lambda y) \geq a$. This specifies, in conjunction with the axiom A1, that the transition from one set of inputs x to y is at a production level equal to or greater than a specified lower limit. Neither this condition would not necessarily be required, existing situations (for example, the transition to a market economy of the former communist states) the refurbishment (thus changing the structure of inputs) was made with temporary dip in the level of production. But as economic analysis, most often refers (unfortunately) to the processes that are somewhat stabilized, we will retain this condition.

Considering so a production function $Q:D_p \to \mathbf{R}_+$, $(x_1,...,x_n) \to Q(x_1,...,x_n) \in \mathbf{R}_+$ $\forall (x_1,...,x_n) \in D_p$ let the bordered Hessian matrix:

$$H^B(f)=\begin{pmatrix} 0 & \frac{\partial Q}{\partial x_1} & \frac{\partial Q}{\partial x_2} & \cdots & \frac{\partial Q}{\partial x_n} \\ \frac{\partial Q}{\partial x_1} & \frac{\partial^2 Q}{\partial x_1^2} & \frac{\partial^2 Q}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 Q}{\partial x_1 \partial x_n} \\ \frac{\partial Q}{\partial x_2} & \frac{\partial^2 Q}{\partial x_2 \partial x_1} & \frac{\partial^2 Q}{\partial x_2^2} & \cdots & \frac{\partial^2 Q}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial Q}{\partial x_n} & \frac{\partial^2 Q}{\partial x_n \partial x_1} & \frac{\partial^2 Q}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 Q}{\partial x_n^2} \end{pmatrix}$$

and Δ_k^B - the boarded principal diagonal determinants formed with the first (k+1) rows and columns of the matrix $H^B(f)$. We have the following theorem:

Theorem If Q is a quasi-concave function then $(-1)^k \Delta_k^B \geq 0$, $k=\overline{1,n}$. If $(-1)^k \Delta_k^B > 0$ then Q is quasi-concave function.

Notes from the theorem that if at least one determinant is null we have not ensured the existence of quasi-concavity.

For classical production functions Q=Q(K,L) the sufficient condition for quasi-concavity becomes: $\begin{vmatrix} 0 & \frac{\partial Q}{\partial K} & \frac{\partial Q}{\partial L} \\ \frac{\partial Q}{\partial K} & \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\ \frac{\partial Q}{\partial L} & \frac{\partial^2 Q}{\partial K \partial L} & \frac{\partial^2 Q}{\partial L^2} \end{vmatrix} > 0$ therefore:

$$2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K \partial L} - \left(\frac{\partial Q}{\partial K}\right)^2 \frac{\partial^2 Q}{\partial L^2} - \left(\frac{\partial Q}{\partial L}\right)^2 \frac{\partial^2 Q}{\partial K^2} > 0.$$

Recall, near the end of this introduction, that a function is called homogeneous if $\exists r \in \mathbf{R}$ such that: $Q(\lambda x_1,...,\lambda x_n)=\lambda^r Q(x_1,...,x_n)$ $\forall \lambda \in \mathbf{R}^*$. r is called the degree of homogeneity of the function.

We say that a production function $Q:D_p \to \mathbf{R}_+$ is with constant return to scale if $Q(\lambda x_1,...,\lambda x_n)=\lambda Q(x_1,...,x_n)$ (so homogeneous of first degree), with increasing return to scale if $Q(\lambda x_1,...,\lambda x_n)>\lambda Q(x_1,...,x_n)$ and with decreasing return to scale if $Q(\lambda x_1,...,\lambda x_n)<\lambda Q(x_1,...,x_n)$ $\forall \lambda \in (1,\infty)$ $\forall (x_1,...,x_n) \in D_p$. The fact that a return to production is at constant scale means that the production has the same multiplication factor with those of the two factors. Similarly, the return of increasing (decreasing) scale production is multiplied by a factor higher (lower) than that of inputs.

We will note below for functions Q=Q(K,L): $\chi=\frac{K}{L}$.

In what follows we will analyze production functions of the form: Q=Q(K,L)

3 Main Indicators of a Production Function

Let a production function:

$$Q:D_p\to\mathbf{R}_+,\ (x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+\ \forall(x_1,...,x_n)\in D_p$$

We will call the marginal productivity relative to a production factor x_i: $\eta_{x_i}=\frac{\partial Q}{\partial x_i}$ and represents the trend of variation of production at the variation of the factor x_i. In particular, for a production function of the form: Q=Q(K,L) we have $\eta_K=\frac{\partial Q}{\partial K}$ - called the marginal productivity of capital and $\eta_L=\frac{\partial Q}{\partial L}$ - called the marginal productivity of labor.

If the output is given by discrete values, we define: $\eta_{x_i}=\frac{\Delta Q}{\Delta x_i}$ meaning the mean variation of the production on the interval of length Δx_i.

We call also the average productivity relative to a production factor x_i: $w_{x_i}=\frac{Q}{x_i}$ and represents the value of production at the consumption of a unit of factor x_i.

In particular, for a production function of the form: Q=Q(K,L) we have: $w_K=\frac{Q}{K}$ - called the productivity of capital, and $w_L=\frac{Q}{L}$ - the productivity of labor.

From [4], we have that in the general case of the variation of all inputs, for k_1 units of input 1,...,k_n units of input n, and Q(0,...,0)=0:

$$Q(k_1,...,k_n)=k_1\int_0^1\eta_{x_1}(k_1t,...,k_nt)dt+...+k_n\int_0^1\eta_{x_n}(k_1t,...,k_nt)dt$$

In particular, for Q=Q(K,L) we have: $Q(K,L)=K\int_0^1\eta_K(Kt,Lt)dt+L\int_0^1\eta_L(Kt,Lt)dt$.

Again, from [7], considering the factors i and j with $i\neq j$, we define the restriction of production area: $P_{ij}=\{(x_1,...,x_n)\mid x_k=a_k=\text{const}, k=\overline{1,n}, k\neq i,j, x_i,x_j\in D_p\}$ relative to the two factors when the others have fixed values and $D_{ij}=\{(x_i,x_j)\mid (x_1,...,x_n)\in P_{ij}\}$ - the domain of production relative to factors i and j.

Defining $Q_{ij}:D_{ij}\to\mathbf{R}_+$ - the restriction of the production function to the factors i and j, i.e.: $Q_{ij}(x_i,x_j)=Q(a_1,...,a_{i-1},x_i,a_{i+1},...,a_{j-1},x_j,a_{j+1},...,a_n)$ we obtain that Q_{ij} define a surface in $\mathbf{R}^3$ for every pair of factors (i,j).

We call partial marginal rate of technical substitution of the factors i and j, relative to D_{ij} (caeteris paribus), the opposite change in the amount of factor j to substitute a variation of the quantity of factor i in the situation of conservation production level and note: $RMS(i,j,\overline{x})=-\dfrac{dx_j}{dx_i}=\dfrac{\eta_{x_i}\big|_{D_{ij}}}{\eta_{x_j}\big|_{D_{ij}}}$ in an arbitrary point $\overline{x}=(\overline{x}_1,...,\overline{x}_n)$. We define also ([7]) the global marginal rate of substitution between the i-th factor and the others as: $RMS(i,\overline{x})=\dfrac{\eta_{x_i}(\overline{x})}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\eta_{x_j}^2(\overline{x})}}$. The global marginal rate of technical substitution is the minimum (in the meaning of norm) of changes in consumption of factors so that the total production remain unchanged.

In particular, for a production function of the form: Q=Q(K,L) we have:

$$RMS(K,L)=\frac{\eta_K}{\eta_L},\ RMS(L,K)=\frac{\eta_L}{\eta_K}$$

It is called elasticity of production in relation to a production factor x_i: $\varepsilon_{x_i}=\dfrac{\frac{\partial Q}{\partial x_i}}{\frac{Q}{x_i}}=\dfrac{\eta_{x_i}}{w_{x_i}}$ - the relative variation of production at the relative variation of factor x_i. In particular, for a production function of the form: Q=Q(K,L) we have $\varepsilon_K=\dfrac{\frac{\partial Q}{\partial K}}{\frac{Q}{K}}=$

$\frac{\eta_K}{w_K}$ - called the elasticity of production in relation to the capital and $\varepsilon_L = \frac{\frac{\partial Q}{\partial L}}{\frac{Q}{L}} =$

$\frac{\eta_L}{w_L}$ - the elasticity factor of production in relation to the labor.

If the production function is homogenous of degree r, after Euler's relation: $\sum_{i=1}^{n} x_i \frac{\partial Q}{\partial x_i} = rQ$ we obtain that $\sum_{i=1}^{n} \varepsilon_{x_i} = r$.

4 Conditions of Marginal Productivity

4.1. η_K =constant=α, η_L ≠constant

In this case, we have: $Q(K,L)= K\int_0^1 \eta_K(Kt, Lt)dt + L\int_0^1 \eta_L(Kt, Lt)dt =$ $K\int_0^1 \alpha dt + L\int_0^1 \eta_L(Kt, Lt)dt =$ $\alpha K+Lg(K,L)$. Because $\frac{\partial Q}{\partial K} = \alpha$ we have that $\frac{\partial g}{\partial K} = 0$ that is g=g(L). Therefore: $Q(K,L)=\alpha K+f(L)$. Now $\frac{\partial Q}{\partial L}$ $=f'(L)\neq 0 \Rightarrow f\neq$constant.

The conditions from the axioms become:

- $Q(0,0)=0 \Rightarrow f(0)=0$
- f – continuous
- $f\in C^2(D_p)$
- $\frac{\partial Q}{\partial K} = \alpha > 0$
- $\frac{\partial Q}{\partial L} = f'(L) > 0$
- $2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K \partial L} - \left(\frac{\partial Q}{\partial K}\right)^2 \frac{\partial^2 Q}{\partial L^2} - \left(\frac{\partial Q}{\partial L}\right)^2 \frac{\partial^2 Q}{\partial K^2} = -\alpha^2 f''(L) > 0 \Rightarrow f''(L) < 0$

After these considerations we obtain that $\alpha>0$ and f is a monotonically increasing, strictly concave differentiable function of class at least two and vanishing in 0.

If now Q is homogenous, we have: $\exists r\in\mathbf{R}$: $Q(\lambda K,\lambda L)=\lambda^r Q(K,L)$ that is: $\alpha\lambda K+f(\lambda L)=\lambda^r(\alpha K+f(L))$.

If $r\neq 1\Rightarrow \alpha\lambda K+f(\lambda L)=\alpha\lambda^r K+\lambda^r f(L)\Rightarrow K=\dfrac{\lambda^r f(L)-f(\lambda L)}{\alpha\lambda\left(1-\lambda^{r-1}\right)}$.Because K and L are independent variables follows K=constant therefore contradiction. We have r=1 that is: $f(\lambda L)=\lambda f(L)$, f being linear: $f(L)=\beta L$. We obtained: $Q=\alpha K+\beta L$ – the linear production function. Let note in this case that Q is quasi-concave even though $f''(L)=0$ for $f(L)=\beta L$.

For the linear production function, the determination of the parameters is very simple (using Least Square Method).

Let $(K_i,L_i,Q_i)_{i=1,\dots,n}$ values of the capital, labor and production at the moments 1 to n. The minimum condition of the expression: $E=\sum_{i=1}^{n}\left(\alpha K_i+\beta L_i-Q_i\right)^2$ (relative to α and β) becomes:

$$\begin{cases}\dfrac{1}{2}\dfrac{\partial E}{\partial\alpha}=\alpha\sum_{i=1}^{n}K_i^2+\beta\sum_{i=1}^{n}K_iL_i-\sum_{i=1}^{n}K_iQ_i=0\\ \dfrac{1}{2}\dfrac{\partial E}{\partial\beta}=\alpha\sum_{i=1}^{n}K_iL_i+\beta\sum_{i=1}^{n}L_i^2-\sum_{i=1}^{n}L_iQ_i=0\end{cases}$$

therefore:

$$\begin{cases}\alpha=\dfrac{\sum_{i=1}^{n}K_iQ_i\sum_{i=1}^{n}L_i^2-\sum_{i=1}^{n}L_iQ_i\sum_{i=1}^{n}K_iL_i}{\sum_{i=1}^{n}K_i^2\sum_{i=1}^{n}L_i^2-\left(\sum_{i=1}^{n}K_iL_i\right)^2}\\ \beta=\dfrac{\sum_{i=1}^{n}L_iQ_i\sum_{i=1}^{n}K_i^2-\sum_{i=1}^{n}K_iQ_i\sum_{i=1}^{n}K_iL_i}{\sum_{i=1}^{n}K_i^2\sum_{i=1}^{n}L_i^2-\left(\sum_{i=1}^{n}K_iL_i\right)^2}\end{cases}$$

4.2. η_L =constant=α, η_K ≠constant

Like previous, we obtain (permuting K with L): $Q(K,L)=\alpha L+f(K)$ with f satisfying the same conditions like above. The determination of the parameters is as above.

4.3. η_K **=constant=**α**,** η_L **=constant**$=\beta$

Q(K,L)= $K\int_0^1 \eta_K(Kt, Lt)dt + L\int_0^1 \eta_L(Kt, Lt)dt = K\int_0^1 \alpha dt + L\int_0^1 \beta dt = \alpha K + \beta L$ – the linear production function. The determination of the parameters is as above.

4.4. $\eta_K = \alpha\chi^\beta = \alpha\frac{K^\beta}{L^\beta}$

Q(K,L)= $K\int_0^1 \eta_K(Kt, Lt)dt + L\int_0^1 \eta_L(Kt, Lt)dt = K\int_0^1 \alpha\frac{K^\beta}{L^\beta}dt + L\int_0^1 \eta_L(Kt, Lt)dt =$

$\frac{\alpha K^{\beta+1}}{L^\beta} + g(K, L)$.

But $\frac{\partial Q}{\partial K} = \alpha\frac{K^\beta}{L^\beta} \Rightarrow \frac{\alpha(\beta+1)K^\beta}{L^\beta} + \frac{\partial g}{\partial K} = \alpha\frac{K^\beta}{L^\beta}$ from where: $\frac{\partial g}{\partial K} = -\alpha\beta\frac{K^\beta}{L^\beta} \Rightarrow$ $g = -\frac{\alpha\beta}{L^\beta}\int K^\beta dK = -\frac{\alpha\beta}{L^\beta}\left(\frac{K^{\beta+1}}{\beta+1} + f(L)\right)$ therefore: Q(K,L)= $\frac{\alpha K^{\beta+1}}{(\beta+1)L^\beta} - \frac{\alpha\beta}{L^\beta}f(L) =$

$\frac{\alpha}{\beta+1}K^{\beta+1}L^{-\beta} + h(L)$ where h(L)= $-\frac{\alpha\beta}{L^\beta}f(L)$.

The conditions from the axioms become:

- h – continuous
- $h \in C^2(D_p)$
- $\frac{\partial Q}{\partial K} = \alpha K^\beta L^{-\beta} > 0 \Leftrightarrow \alpha > 0$
- $\frac{\partial Q}{\partial L} = -\frac{\alpha\beta}{\beta+1}K^{\beta+1}L^{-\beta-1} + h'(L) > 0$
- $2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K\partial L} - \left(\frac{\partial Q}{\partial K}\right)^2\frac{\partial^2 Q}{\partial L^2} - \left(\frac{\partial Q}{\partial L}\right)^2\frac{\partial^2 Q}{\partial K^2} > 0 \Leftrightarrow$

$\beta\left(\frac{\alpha K^{\beta+1}L^{-\beta-1}}{\beta+1} + h'(L)\right)^2 + \alpha K^{\beta+1}L^{-\beta}h''(L) < 0$

After these considerations we obtain that $\alpha > 0$ and h has the properties:

- $h'(L) > \frac{\alpha\beta}{\beta+1}K^{\beta+1}L^{-\beta-1}$
- $\beta\left(\frac{\alpha K^{\beta+1}L^{-\beta-1}}{\beta+1} + h'(L)\right)^2 + \alpha K^{\beta+1}L^{-\beta}h''(L) < 0$

If now, we want that the function be homogenous, we have:

$Q(\lambda K,\lambda L) = \lambda\frac{\alpha}{\beta+1}K^{\beta+1}L^{-\beta} + h(\lambda L) = \lambda^r Q(K,L) = \lambda^r\frac{\alpha}{\beta+1}K^{\beta+1}L^{-\beta} + \lambda^r h(L)$ that is:

$$(\lambda^r - \lambda)\frac{\alpha}{\beta+1}K^{\beta+1}L^{-\beta} = h(\lambda L) - \lambda^r h(L)$$

If $r\neq 1$ we find that: $K^{\beta+1} = \frac{(\beta+1)}{\alpha(\lambda^r - \lambda)L^{\beta}}\left(h(\lambda L) - \lambda^r h(L)\right)$ that is K depends from L – contradiction.

We have therefore: r=1, that is: $h(\lambda L) = \lambda h(L)$, h being linear: $h(L)=\rho L$.

The production function becomes (after obvious notations):

$Q(K,L) = \frac{\alpha}{\beta+1}K^{\beta+1}L^{-\beta} + \rho L$ - Bruno production function with $\beta\in(-1,0)$ (after the above conditions), $\rho>0$, $\alpha>0$.

Let now $(K_i,L_i,Q_i)_{i=1,\ldots,n}$ values of the capital, labor and production at the moments 1 to n. The minimum conditions of the expression: $E = \sum_{i=1}^{n}\left(\frac{\alpha}{\beta+1}K_i^{\beta+1}L_i^{-\beta} + \rho L_i - Q_i\right)^2$ (relative to α, β and ρ) are very difficult to be solve (and is not relevant because the existence of this function requires the particular form of η_K), therefore we shall determine first, the discrete values of $\eta_K = \frac{\Delta Q}{\Delta K}$ that is: $\eta_{K,p} = \frac{Q_{p+1} - Q_p}{K_{p+1} - K_p}$, $p=\overline{1,n-1}$ and after, from the initial condition, that $\eta_K = \alpha\frac{K^{\beta}}{L^{\beta}}$ we have that: $\ln\eta_K = \ln\alpha + \beta\ln\frac{K}{L}$. Let now $E_1 = \sum_{p=1}^{n-1}\left(\Phi + \beta\ln\frac{K_p}{L_p} - \ln\eta_{K,p}\right)^2$ where $\Phi = \ln\alpha$.

The Least Square Method gives us:

$$\begin{cases} \frac{1}{2}\frac{\partial E_1}{\partial \Phi} = (n-1)\Phi + \beta\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p} - \sum_{p=1}^{n-1}\ln\eta_{K,p} = 0 \\ \frac{1}{2}\frac{\partial E_1}{\partial \beta} = \Phi\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p} + \beta\sum_{p=1}^{n-1}\left(\ln\frac{K_p}{L_p}\right)^2 - \sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}\ln\eta_{K,p} = 0 \end{cases}$$

therefore:

$$\begin{cases} \Phi^* = \dfrac{\sum_{p=1}^{n-1}\ln\eta_{K,p}\sum_{p=1}^{n-1}\left(\ln\frac{K_p}{L_p}\right)^2 - \sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}\ln\eta_{K,p}}{(n-1)\sum_{p=1}^{n-1}\left(\ln\frac{K_p}{L_p}\right)^2 - \left(\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}\right)^2} \\ \beta^* = \dfrac{(n-1)\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}\ln\eta_{K,p} - \sum_{p=1}^{n-1}\ln\eta_{K,p}\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}}{(n-1)\sum_{p=1}^{n-1}\left(\ln\frac{K_p}{L_p}\right)^2 - \left(\sum_{p=1}^{n-1}\ln\frac{K_p}{L_p}\right)^2} \end{cases}$$

and $\alpha^* = e^{\Phi^*}$.

After these: Q(K,L)= $\frac{\alpha^*}{\beta^*+1}K^{\beta^*+1}L^{-\beta^*} + \rho L$. The determination of ρ can be determined in the following way. Let note: $\Omega_i = Q_i - \frac{\alpha^*}{\beta^*+1}K_i^{\beta^*+1}L_i^{-\beta^*}$, i=$\overline{1,n}$ and the condition that the expression: E$_2$=$\sum_{i=1}^{n}(\rho L_i - \Omega_i)^2$ be minimum. We have therefore $\frac{1}{2}\frac{dE_2}{d\rho} = \sum_{i=1}^{n}L_i(\rho L_i - \Omega_i) = 0$ therefore: $\rho^* = \frac{\sum_{i=1}^{n}L_i\Omega_i}{\sum_{i=1}^{n}L_i^2}$ where at least one $L_i \neq 0$. Finally: Q(K,L)= $\frac{\alpha^*}{\beta^*+1}K^{\beta^*+1}L^{-\beta^*} + \rho^* L$. Let note here that because ρ^*=constant we must have that the values: $\frac{\Omega_i}{L_i}$, i=$\overline{1,n}$ must be approximately constant. To inquire this we can use the 3σ-rule that is in the interval (M-3σ,M+3σ)= $M\left(1-3\frac{\sigma}{M},1+3\frac{\sigma}{M}\right)$ lies over 89% values, where M is the average of these. Therefore, we shall compute for the values $\frac{\Omega_i}{L_i}$, i=$\overline{1,n}$ the average: M=$\frac{\sum_{i=1}^{n}\frac{\Omega_i}{L_i}}{n}$ and the standard deviation σ= $\frac{\sqrt{n\sum_{i=1}^{n}\left(\frac{\Omega_i}{L_i}\right)^2-\left(\sum_{i=1}^{n}\frac{\Omega_i}{L_i}\right)^2}}{n}$. If the value $\frac{\sigma}{M}$ is sufficiently small we can assume that ρ^* is almost a constant and the determination is as in the upper.

4.5. $\eta_L=\alpha\chi^{\beta}=\alpha\frac{K^{\beta}}{L^{\beta}}$

Because the relation can pe written as: $\eta_L=\alpha\frac{L^{-\beta}}{K^{-\beta}}$ we shall proceed as in 4.4. and we shall obtain (permuting K with L and replacing β with -β): Q(K,L)= $\frac{\alpha}{1-\beta}K^{\beta}L^{1-\beta}+h(K)$.

The conditions from the axioms become:

- h – continuous
- $h\in C^2(D_p)$
- $\frac{\partial Q}{\partial L}=\alpha K^{\beta}L^{-\beta}>0 \Leftrightarrow \alpha>0$
- $\frac{\partial Q}{\partial K}=\frac{\alpha\beta}{1-\beta}K^{\beta-1}L^{1-\beta}+h'(K)>0$
- $2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K\partial L}-\left(\frac{\partial Q}{\partial K}\right)^2\frac{\partial^2 Q}{\partial L^2}-\left(\frac{\partial Q}{\partial L}\right)^2\frac{\partial^2 Q}{\partial K^2}>0\Leftrightarrow$

$$-\beta\left(\frac{\alpha L^{1-\beta}K^{\beta-1}}{1-\beta}+h'(K)\right)^2+\alpha L^{1-\beta}K^{\beta}h''(K)<0$$

After these considerations we obtain that α>0 and h has the properties:

- $h'(K) > -\frac{\alpha\beta}{1-\beta}L^{1-\beta}K^{\beta-1}$
- $-\beta\left(\frac{\alpha L^{1-\beta}K^{\beta-1}}{1-\beta} + h'(K)\right)^2 + \alpha L^{1-\beta}K^{\beta}h''(K) < 0$

If now, we want that the function be homogenous, we have, as previous: r=1, that is: $h(\lambda L) = \lambda h(L)$, h being linear: h(L)=ρL, the production function becoming (after obvious notations):

Q(K,L)= $\frac{\alpha}{1-\beta}L^{1-\beta}K^{\beta} + \rho K$ - Bruno production type function with $\beta \in (0,1)$ (after the above conditions), ρ>0, α>0.

Let now $(K_i,L_i,Q_i)_{i=1,\ldots,n}$ values of the capital, labor and production at the moments 1 to n. We shall determine first, the discrete values of $\eta_L = \frac{\Delta Q}{\Delta L}$ that is: $\eta_{L,p} = \frac{Q_{p+1} - Q_p}{L_{p+1} - L_p}$, p=$\overline{1, n-1}$ and after, from the initial condition, that $\eta_L = \alpha\frac{L^{-\beta}}{K^{-\beta}}$ we have that: $\ln \eta_L = \ln \alpha - \beta \ln \frac{L}{K}$. Let now $E_1 = \sum_{p=1}^{n-1}\left(\Phi - \beta\ln\frac{L_p}{K_p} - \ln\eta_{L,p}\right)^2$ where Φ =ln α. The Least Square Method gives us (as upper):

$$\begin{cases} \Phi^* = \dfrac{\sum_{p=1}^{n-1}\ln\eta_{L,p}\sum_{p=1}^{n-1}\left(\ln\frac{L_p}{K_p}\right)^2 - \sum_{p=1}^{n-1}\ln\frac{L_p}{K_p}\sum_{p=1}^{n-1}\ln\frac{L_p}{K_p}\ln\eta_{L,p}}{(n-1)\sum_{p=1}^{n-1}\left(\ln\frac{L_p}{K_p}\right)^2 - \left(\sum_{p=1}^{n-1}\ln\frac{L_p}{K_p}\right)^2} \\ \beta^* = -\dfrac{(n-1)\sum_{p=1}^{n-1}\ln\frac{L_p}{K_p}\ln\eta_{L,p} - \sum_{p=1}^{n-1}\ln\eta_{L,p}\sum_{p=1}^{n-1}\ln\frac{L_p}{K_p}}{(n-1)\sum_{p=1}^{n-1}\left(\ln\frac{L_p}{K_p}\right)^2 - \left(\sum_{p=1}^{n-1}\ln\frac{L_p}{K_p}\right)^2} \end{cases}$$

and $\alpha^* = e^{\Phi^*}$.

After these: Q(K,L)= $\frac{\alpha^*}{1-\beta^*}L^{1-\beta^*}K^{\beta^*}+\rho K$. The determination of ρ can be determined in the following way. Let note: $\Omega_i=Q_i-\frac{\alpha^*}{1-\beta^*}L_i^{1-\beta^*}K_i^{\beta^*}$, i=$\overline{1,n}$ and the condition that the expression: E$_2$=$\sum_{i=1}^{n}(\rho K_i-\Omega_i)^2$ be minimum. We have therefore $\frac{1}{2}\frac{dE_2}{d\rho}=\sum_{i=1}^{n}K_i(\rho K_i-\Omega_i)=0$ therefore: $\rho^*=\frac{\sum_{i=1}^{n}K_i\Omega_i}{\sum_{i=1}^{n}K_i^2}$ where at least one $K_i\neq 0$. Finally: Q(K,L)= $\frac{\alpha^*}{1-\beta^*}L^{1-\beta^*}K^{\beta^*}+\rho^*K$. The demarche relative to the constancy of ρ^* is similarly to 4.4.

4.6. $\eta_K=\frac{aK^\alpha+bL^\beta}{K^\gamma}$, $\eta_L=\frac{cK^\alpha+dL^\beta}{L^\gamma}$

Q(K,L)= $K\int_0^1\eta_K(Kt,Lt)dt+L\int_0^1\eta_L(Kt,Lt)dt=$

$$K\int_0^1\frac{aK^\alpha t^\alpha+bL^\beta t^\beta}{K^\gamma t^\gamma}dt+L\int_0^1\frac{cK^\alpha t^\alpha+dL^\beta t^\beta}{L^\gamma t^\gamma}dt=$$

$$aK^{\alpha-\gamma+1}\int_0^1 t^{\alpha-\gamma}dt+bK^{1-\gamma}L^\beta\int_0^1 t^{\beta-\gamma}dt+cK^\alpha L^{1-\gamma}\int_0^1 t^{\alpha-\gamma}dt+dL^{1+\beta-\gamma}\int_0^1 t^{\beta-\gamma}dt=$$

$$K^\alpha\left(aK^{1-\gamma}+cL^{1-\gamma}\right)\int_0^1 t^{\alpha-\gamma}dt+L^\beta\left(bK^{1-\gamma}+dL^{1-\gamma}\right)\int_0^1 t^{\beta-\gamma}dt$$

<u>4.6.1.</u> If α-$\gamma\neq$-1, β-$\gamma\neq$-1 then: Q(K,L)= $\frac{1}{\alpha-\gamma+1}K^\alpha\left(aK^{1-\gamma}+cL^{1-\gamma}\right)+\frac{1}{\beta-\gamma+1}L^\beta\left(bK^{1-\gamma}+dL^{1-\gamma}\right)$.

The conditions from the axioms become:

- $\frac{\partial Q}{\partial K}=aK^{\alpha-\gamma}+\frac{c\alpha}{\alpha-\gamma+1}K^{\alpha-1}L^{1-\gamma}+\frac{b(1-\gamma)}{\beta-\gamma+1}K^{-\gamma}L^\beta>0$

- $\frac{\partial Q}{\partial L}=\frac{c(1-\gamma)}{\alpha-\gamma+1}K^{\alpha}L^{-\gamma}+\frac{b\beta}{\beta-\gamma+1}K^{1-\gamma}L^{\beta-1}+dL^{\beta-\gamma}>0$
- $2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K\partial L}-\left(\frac{\partial Q}{\partial K}\right)^2\frac{\partial^2 Q}{\partial L^2}-\left(\frac{\partial Q}{\partial L}\right)^2\frac{\partial^2 Q}{\partial K^2}>0$

If now, we want that the function be homogenous (of degree r), we have:

$$Q(\lambda K,\lambda L)=\lambda^{\alpha-\gamma+1}\frac{1}{\alpha-\gamma+1}K^{\alpha}\left(aK^{1-\gamma}+cL^{1-\gamma}\right)+\lambda^{\beta-\gamma+1}\frac{1}{\beta-\gamma+1}L^{\beta}\left(bK^{1-\gamma}+dL^{1-\gamma}\right)=$$

$$\lambda^{r}Q(K,L)=\lambda^{r}\frac{1}{\alpha-\gamma+1}K^{\alpha}\left(aK^{1-\gamma}+cL^{1-\gamma}\right)+\lambda^{r}\frac{1}{\beta-\gamma+1}L^{\beta}\left(bK^{1-\gamma}+dL^{1-\gamma}\right)$$ that is:

$$\left(\lambda^{\alpha-\gamma+1}-\lambda^{r}\right)\frac{1}{\alpha-\gamma+1}K^{\alpha}\left(aK^{1-\gamma}+cL^{1-\gamma}\right)+\left(\lambda^{\beta-\gamma+1}-\lambda^{r}\right)\frac{1}{\beta-\gamma+1}L^{\beta}\left(bK^{1-\gamma}+dL^{1-\gamma}\right)=0$$

4.6.1.a. If now $r\neq\beta-\gamma+1$ we have: $\frac{\lambda^{\alpha-\gamma+1}-\lambda^{r}}{\lambda^{\beta-\gamma+1}-\lambda^{r}}=-\frac{\alpha-\gamma+1}{\beta-\gamma+1}\frac{L^{\beta}\left(bK^{1-\gamma}+dL^{1-\gamma}\right)}{K^{\alpha}\left(aK^{1-\gamma}+cL^{1-\gamma}\right)}$.

4.6.1.a.i. If $r=\alpha-\gamma+1$ we obtain: $L^{\beta}\left(bK^{1-\gamma}+dL^{1-\gamma}\right)=0$ that is L= constant – contradiction or $K^{1-\gamma}=-\frac{d}{b}L^{1-\gamma}$ - contradiction with the independence of K and L (if $b\neq 0$) or $dL^{1-\gamma}=0$ (b=0) which is true only of d=0. But in this case, we have that: $\eta_K=aK^{\alpha-\gamma}$, $\eta_L=cK^{\alpha}L^{-\gamma}$ and $Q(K,L)=\frac{aK^{\alpha-\gamma+1}+cK^{\alpha}L^{1-\gamma}}{\alpha-\gamma+1}=$ $pK^{\alpha-\gamma+1}+qL^{\alpha-\gamma+1}\left(\frac{K}{L}\right)^{\alpha}$ with obvious notations.

Let now $(K_i,L_i,Q_i)_{i=1,...,n}$ values of the capital, labor and production at the moments 1 to n. We shall determine first, the discrete values of $\eta_K=\frac{\Delta Q}{\Delta K}$, $\eta_L=\frac{\Delta Q}{\Delta L}$ that is: $\eta_{K,p}=\frac{Q_{p+1}-Q_p}{K_{p+1}-K_p}$, $\eta_{L,p}=\frac{Q_{p+1}-Q_p}{L_{p+1}-L_p}$, $p=\overline{1,n-1}$ and after, from the initial condition, that $\eta_K=aK^{\alpha-\gamma}$, $\eta_L=cK^{\alpha}L^{-\gamma}$ we have that: $\ln\eta_K=\ln a+(\alpha-\gamma)\ln K$ and $\ln\eta_L=\ln c+\alpha\ln K-\gamma\ln L$. Let now first $E_1=$

$\sum_{p=1}^{n-1}\left(\Phi+\alpha\ln K_p-\gamma\ln L_p-\ln\eta_{L,p}\right)^2$ where Φ =ln c and The Least Square Method gives us (as upper):

$$\begin{cases}(n-1)\Phi+\alpha\sum_{p=1}^{n-1}\ln K_p-\gamma\sum_{p=1}^{n-1}\ln L_p=\sum_{p=1}^{n-1}\ln\eta_{L,p}\\ \Phi\sum_{p=1}^{n-1}\ln K_p+\alpha\sum_{p=1}^{n-1}\left(\ln K_p\right)^2-\gamma\sum_{p=1}^{n-1}\ln K_p\ln L_p=\sum_{p=1}^{n-1}\ln\eta_{L,p}\ln K_p\\ \Phi\sum_{p=1}^{n-1}\ln L_p+\alpha\sum_{p=1}^{n-1}\ln K_p\ln L_p-\gamma\sum_{p=1}^{n-1}\left(\ln L_p\right)^2=\sum_{p=1}^{n-1}\ln\eta_{L,p}\ln L_p\end{cases}$$

from where we shall find: $c^*=e^{\Phi^*}$, α^*, γ^*.

After these: $\ln\eta_K=\ln a+\left(\alpha^*-\gamma^*\right)\ln K$. For the determination of "a", let note here that because ln a is constant we must have that the values: $\ln\eta_{K,i}-\left(\alpha^*-\gamma^*\right)\ln K_i$, i=$\overline{1,n}$ must be approximately constant. To inquire this we can use the 3σ-rule that is in the interval (M-3σ,M+3σ)= $M\left(1-3\frac{\sigma}{M},1+3\frac{\sigma}{M}\right)$ lies over 89% values, where M is the average of these. Therefore, we shall compute for the values $\ln\eta_{K,i}-\left(\alpha^*-\gamma^*\right)\ln K_i$, i=$\overline{1,n}$ the average: M=$\frac{\ln\eta_{K,i}-\left(\alpha^*-\gamma^*\right)\ln K_i}{n}$ and the standard deviation σ= $\frac{\sqrt{n\sum_{i=1}^{n}\left(\ln\eta_{K,i}-\left(\alpha^*-\gamma^*\right)\ln K_i\right)^2-\left(\sum_{i=1}^{n}\ln\eta_{K,i}-\left(\alpha^*-\gamma^*\right)\ln K_i\right)^2}}{n}$. If the value $\frac{\sigma}{M}$ is sufficiently small we can assume that ln a is almost a constant and the determination is as in the upper. Let note also a^* this value.

Now we have Q(K,L)=$\frac{a^*K^{\alpha^*-\gamma^*+1}+c^*K^{\alpha^*}L^{1-\gamma^*}}{\alpha^*-\gamma^*+1}$ with obvious notations.

4.6.1.a.ii. If r≠α-γ+1 we have that $0\neq f(\lambda)=\frac{\lambda^{\alpha-\gamma+1}-\lambda^r}{\lambda^{\beta-\gamma+1}-\lambda^r}=-\frac{\alpha-\gamma+1}{\beta-\gamma+1}\frac{L^\beta\left(bK^{1-\gamma}+dL^{1-\gamma}\right)}{K^\alpha\left(aK^{1-\gamma}+cL^{1-\gamma}\right)}$ - contradiction with the fact that for constant K and L we shall have f=constant which is impossible.

4.6.1.b. Returning, now for r=β-γ+1 we have: $\left(\lambda^{\alpha-\gamma+1}-\lambda^{r}\right)\frac{1}{\alpha-\gamma+1}K^{\alpha}\left(aK^{1-\gamma}+cL^{1-\gamma}\right)=0$.

4.6.1.b.i. For α-γ+1≠r we shall obtain that the equality becomes true only if a=c=0 (as upper) that is: $\eta_K=bK^{-\gamma}L^{\beta}$, $\eta_L=dL^{\beta-\gamma}$, Q(K,L)=$\frac{bK^{1-\gamma}L^{\beta}+dL^{\beta-\gamma+1}}{\beta-\gamma+1}$ = $pL^{\beta-\gamma+1}+qK^{\beta-\gamma+1}\left(\frac{K}{L}\right)^{-\beta}$. The deterination of the parameters can be done like in 4.6.1.a.i replacing α with β, K with L, a with d and c with b.

4.6.1.b.ii. If α-γ+1=r we have an identity. In this case: α=β=r+γ-1 and: $\eta_K=aK^{\alpha-\gamma}+bK^{-\gamma}L^{\alpha}$, $\eta_L=cK^{\alpha}L^{-\gamma}+dL^{\alpha-\gamma}$, Q(K,L)= $\frac{aK^{\alpha-\gamma+1}+dL^{\alpha-\gamma+1}+cK^{\alpha}L^{1-\gamma}+bL^{\alpha}K^{1-\gamma}}{\alpha-\gamma+1}$. The determination of the parameters in this case is a little bit difficult because α and γ lies also at power of K and L and at the denominator of Q.

If, in particular, α=γ=$\frac{1}{2}$ we shall have: Q(K,L)=$aK+b\sqrt{K}\sqrt{L}+dL$ - Diewert production function (homogenous of degree 1).

4.6.2. If α-γ=-1 or β-γ=-1 then the integral becomes -∞ which is a contradiction which the nature of production.

4.7. $\eta_K=aK^{\alpha}L^{\beta}$, $\eta_L=bK^{\gamma}L^{\delta}$

In this case Q(K,L)=$K\int_0^1\eta_K(Kt,Lt)dt+L\int_0^1\eta_L(Kt,Lt)dt$ = $K\int_0^1 at^{\alpha+\beta}K^{\alpha}L^{\beta}dt+L\int_0^1 bt^{\gamma+\delta}K^{\gamma}L^{\delta}dt$

4.7.1. If α+β≠-1 and γ+δ≠-1 we have: Q(K,L)=$\frac{aK^{\alpha+1}L^{\beta}}{\alpha+\beta+1}+\frac{bK^{\gamma}L^{\delta+1}}{\gamma+\delta+1}$.

The conditions from the axioms become:

- $\frac{\partial Q}{\partial K}=aK^{\alpha}L^{\beta}>0$ that is a>0

- $\frac{\partial Q}{\partial L} = bK^{\gamma}L^{\delta}$ >0 that is b>0
- $2\frac{\partial Q}{\partial K}\frac{\partial Q}{\partial L}\frac{\partial^2 Q}{\partial K \partial L} - \left(\frac{\partial Q}{\partial K}\right)^2 \frac{\partial^2 Q}{\partial L^2} - \left(\frac{\partial Q}{\partial L}\right)^2 \frac{\partial^2 Q}{\partial K^2} =$

$a^2b(2\beta-\delta)K^{2\alpha+\gamma}L^{2\beta+\delta-1} - ab^2\alpha K^{\alpha+2\gamma-1}L^{\beta+2\delta}$ >0 that is: $a(2\beta-\delta)K^{\alpha}L^{\beta-1} - b\alpha K^{\gamma-1}L^{\delta}$ >0

If now, we want that the function be homogenous (of degree r), we have:

Q(λK,λL)= $\lambda^{\alpha+\beta+1}\frac{aK^{\alpha+1}L^{\beta}}{\alpha+\beta+1} + \lambda^{\gamma+\delta+1}\frac{bK^{\gamma}L^{\delta+1}}{\gamma+\delta+1} = \lambda^{r}\frac{aK^{\alpha+1}L^{\beta}}{\alpha+\beta+1} + \lambda^{r}\frac{bK^{\gamma}L^{\delta+1}}{\gamma+\delta+1}$ that is:

$$\left(\lambda^{\alpha+\beta+1} - \lambda^{r}\right)\frac{aK^{\alpha+1}L^{\beta}}{\alpha+\beta+1} + \left(\lambda^{\gamma+\delta+1} - \lambda^{r}\right)\frac{bK^{\gamma}L^{\delta+1}}{\gamma+\delta+1} = 0 .$$

If r≠γ+δ+1 we obtain: $\frac{\lambda^{\alpha+\beta+1} - \lambda^{r}}{\lambda^{\gamma+\delta+1} - \lambda^{r}} = -\frac{b(\alpha+\beta+1)K^{\gamma-\alpha-1}L^{\delta-\beta+1}}{a(\gamma+\delta+1)}$.

If α+β+1≠r and α+β≠γ+δ the expression from left depends from λ which is a contradiction with the right.

If α+β+1=r we shall find that: $b(\alpha+\beta+1)K^{\gamma-\alpha-1}L^{\delta-\beta+1}$=0 that is α+β+1=r=0 which is a contradiction with the hypothesis.

If α+β=γ+δ≠-1 we have: $bK^{\gamma-\alpha-1}L^{\delta-\beta+1} = -a$ that is a contradiction with the variability of K and L.

We have therefore: r=γ+δ+1 and with the same arguments r=α+β+1. In this case the production function is homogenous and has the expression: Q(K,L)= $\frac{aK^{\alpha+1}L^{\beta} + bK^{\alpha+\beta-\delta}L^{\delta+1}}{\alpha+\beta+1}$. With new notations: Q(K,L)= $AK^{\mu}L^{\beta} + BK^{\mu+\beta-\varepsilon}L^{\varepsilon}$. For A=0 or B=0 we obtain the classical Cobb-Douglas production function.

The determination of the parameters follows obviously (like upper) from the conditions: $\eta_K = aK^{\alpha}L^{\beta}$, $\eta_L = bK^{\gamma}L^{\delta}$.

4.7.2. If α+β=-1 or γ+δ=-1 then the integral becomes -∞ which is a contradiction which the nature of production.

5 Conditions of Marginal Rate of Substitution

5.1. RMS(K,L)= $\frac{a\chi+b}{c\chi+d}$ **where** $\chi=\frac{K}{L}$, Q being homogenous of degree 1.

Because Q(K,L)= $Q(\chi L, L)=LQ(\chi,1)$ we will note $q(\chi)=Q(\chi,1)$ and we have: Q(K,L)=L $q(\chi)$.

Now: $\eta_K=\frac{\partial Q}{\partial K}=Lq'(\chi)\frac{\partial\chi}{\partial K}=q'(\chi)$ and $\eta_L=\frac{\partial Q}{\partial L}=q(\chi)+Lq'(\chi)\frac{\partial\chi}{\partial L}=q(\chi)-\chi q'(\chi)$.

We have RMS(K,L)= $\frac{\eta_K}{\eta_L}=\frac{q'(\chi)}{q(\chi)-\chi q'(\chi)}=\frac{a\chi+b}{c\chi+d}$.

In this case: $\left[a\chi^2+(b+c)\chi+d\right]q'(\chi)=(a\chi+b)q(\chi)$ which it can be written as:

$\frac{q'(\chi)}{q(\chi)}=\frac{a\chi+b}{a\chi^2+(b+c)\chi+d}$ if $a\chi^2+(b+c)\chi+d\neq 0$.

Integrating: $\ln q(\chi)=\int\frac{a\chi+b}{a\chi^2+(b+c)\chi+d}d\chi$

<u>5.1.1.</u> If $(b+c)^2-4ad<0$ we have that:

$$\ln q(\chi)=\frac{1}{2}\ln\left|a\chi^2+(b+c)\chi+d\right|+\frac{b-c}{\sqrt{4ad-(b+c)^2}}\operatorname{arctg}\frac{2a\chi+b+c}{\sqrt{4ad-(b+c)^2}}+C,\ C\in\mathbf{R}$$

therefore: $q(\chi)=C\sqrt{\left|a\chi^2+(b+c)\chi+d\right|}e^{\frac{b-c}{\sqrt{4ad-(b+c)^2}}\operatorname{arctg}\frac{2a\chi+b+c}{\sqrt{4ad-(b+c)^2}}}$, $C\in\mathbf{R}_+^*$.

Expressing in function of K and L, we find that:

Q(K,L)= $C\sqrt{\left|aK^2+(b+c)KL+dL^2\right|}e^{\frac{b-c}{\sqrt{4ad-(b+c)^2}}\operatorname{arctg}\frac{2aK+(b+c)L}{L\sqrt{4ad-(b+c)^2}}}$, $C\in\mathbf{R}_+^*$.

In particular, for b=c we have: Q(K,L)= $C\sqrt{\left|aK^2+2bKL+dL^2\right|}$, $C\in\mathbf{R}_+^*$ with b^2-ad<0 – the Allen production function.

5.1.2. If $(b+c)^2-4ad=0$ we have that: $\ln q(\chi)=\ln\left|\chi+\frac{b+c}{2a}\right|-\frac{b-c}{2a\chi+b+c}+C$ therefore:

$$q(\chi)=C\left|\chi+\frac{b+c}{2a}\right|e^{-\frac{b-c}{2a\chi+b+c}},\ C\in\mathbf{R}_+^*.$$

Finally: Q(K,L)= $q(\chi)=C\left|K+\frac{b+c}{2a}L\right|e^{-\frac{(b-c)L}{2aK+(b+c)L}}$

5.1.3. $(b+c)^2-4ad>0$ we have that:

$$\ln q(\chi)=\frac{1}{2}\ln\left|a\chi^2+(b+c)\chi+d\right|+\frac{b-c}{2a}\int\frac{1}{(\chi-\chi_1)(\chi-\chi_2)}d\chi=$$

$\frac{1}{2}\ln\left|a\chi^2+(b+c)\chi+d\right|+\frac{b-c}{2a}\frac{1}{\chi_1-\chi_2}\ln\left|\frac{\chi-\chi_1}{\chi-\chi_2}\right|$ where χ_1 and χ_2 are the real roots of $a\chi^2+(b+c)\chi+d$ =0.

We have now: $q(\chi)=C\sqrt{\left|a\chi^2+(b+c)\chi+d\right|}\left|\frac{\chi-\chi_1}{\chi-\chi_2}\right|^{\frac{b-c}{2a(\chi_1-\chi_2)}}$, $C\in\mathbf{R}_+^*$. and finally:

$$Q(K,L)=C\sqrt{\left|aK^2+(b+c)KL+dL^2\right|}\left|\frac{K-\chi_1L}{K-\chi_2L}\right|^{\frac{b-c}{2a(\chi_1-\chi_2)}},\ C\in\mathbf{R}_+^*.$$

In particular, for b=c we have: Q(K,L)= $C\sqrt{\left|aK^2+2bKL+dL^2\right|}$, $C\in\mathbf{R}_+^*$ with b^2-ad>0 – the Allen production function.

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