Some Aspects of Production Functions Differential Geometry

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Abstract: The article deals with some aspects of differential production functions with examples for Cobb-Douglas function in two or three variables. There are studied in each case, the conditions of the parameters in order that the sectional curvature be constant.

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1. Introduction

Let H be a hypersurface in $\mathbf{R}^{n+1}$ of equation:

$x^{n+1}=f(x_1,...,x_n)$, $(x_1,...,x_n)\in D\subset\mathbf{R}^n$, D – open

We will assume in what follows that $f\in C^2(D)$.

A parametric representation of hypersurfaces is:

$$H:\ \begin{cases} x_i = f_i(x_1,...,x_n) = x_i,\ i=\overline{1,n} \\ x_{n+1} = f_{n+1}(x_1,...,x_n) == f(x_1,...,x_n) \end{cases}$$

We will note sometimes: $x=(x_1,...,x_n)\in\mathbf{R}^n$.

Considering the Jacobian matrix $J_f=\left(\frac{\partial f_i}{\partial x^j}\right)$ we will assume that all hypersurfaces points are regular i.e. $\text{rank}\left(\frac{\partial f_i}{\partial x^j}\right)=n$.

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We define the tangent hyperplane to the hypersurface at a point x_0 and noted $T_{x_0}H$:

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_0)(x_i - x_{0i}) - x_{n+1} + f(x_0) = 0$$

which is the locus of the tangents to the curves on *H* passing through x_0. Any element of $T_{x_0}H$ is called the tangent vector to the hypersurface.

The normal at the hypersurface is the straight line orthogonal in x_0 on H and has equation:

$$\frac{x_1 - x_{01}}{\frac{\partial f}{\partial x_1}(x_0)} = \ldots = \frac{x_n - x_{0n}}{\frac{\partial f}{\partial x_n}(x_0)} = \frac{x_{n+1} - f(x_0)}{-1}$$

We define (Eisenhart, 1926) the metric tensor of the hypersurface as having components of matrix $g=(g_{ij})$ where:

$$g_{ij} = \sum_{k=1}^{n+1}\frac{\partial f_k}{\partial x_i}\frac{\partial f_k}{\partial x_j},\ i,j=\overline{1,n}$$

The tensor g, called the first fundamental form of the hypersurfaces, allows to define the length of a vector $v \in T_{x_0}H$ as $\sqrt{g(v,v)}$, the angle of two non-null vectors being:

$$\cos\angle(v,w) = \frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(w,w)}}$$

where $g(X,Y)=\sum_{i,j=1}^{n} g_{ij}X^iY^j$, X_i and Y^j being the components of the vectors $X,Y \in T_{x_0}H$.

Taking into account the parametric expression of hypersurfaces, we have:

$$g_{ij} = \sum_{k=1}^{n}\frac{\partial f_k}{\partial x_i}\frac{\partial f_k}{\partial x_j} + \frac{\partial f_{n+1}}{\partial x_i}\frac{\partial f_{n+1}}{\partial x_j} = \sum_{k=1}^{n}\delta_{ki}\delta_{kj} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j},\ i,j=\overline{1,n}$$

where $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ is the Kronecker's symbol.

We therefore fundamental matrix of the first forms of hypersurfaces:

$$g=\begin{pmatrix} 1+\left(\frac{\partial f}{\partial x_1}\right)^2 & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} & 1+\left(\frac{\partial f}{\partial x_2}\right)^2 & \dots & \frac{\partial f}{\partial x_2}\frac{\partial f}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_n} & \frac{\partial f}{\partial x_2}\frac{\partial f}{\partial x_n} & \dots & 1+\left(\frac{\partial f}{\partial x_n}\right)^2 \end{pmatrix}$$

Consider also the inverse of the metric tensor: $g^{-1}=(g^{ij})$ where g^{ij} are the inverse matrix elements g^{-1}: $g^{ij}=\frac{1}{|g|}\left(\delta_{ij}+\delta_{ij}\sum_{k=1}^{n}\left(\frac{\partial f}{\partial x_k}\right)^2-\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right)$ where $|g|=1+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2\neq 0$.

We define now Christoffel's symbols of the first kind:

$$|ij,k|=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_k}\right),\ i,j,k=\overline{1,n}$$

and of the second kind:

$$\begin{vmatrix} k \\ ij \end{vmatrix}=\sum_{p=1}^{n} g^{kp}|ij,p|,\ i,j,k=\overline{1,n}$$

For considered hypersurface we have so:

$$|ij,k|=\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial f}{\partial x_k},\ i,j,k=\overline{1,n}$$

$$\begin{vmatrix} k \\ ij \end{vmatrix}=\frac{1}{|g|}\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial f}{\partial x_k},\ i,j,k=\overline{1,n}$$

Let us define also for two vectors tangent to the hypersurface at a point $x_0=\left(x_{0,1},\dots,x_{0,n}\right)\in D$ the covariant derivative of $Y=\sum_{j=1}^{n}Y^j\partial_j$ relative to $X=\sum_{i=1}^{n}X^i\partial_i$ where $X^i,Y^i:D\to\mathbf{R}$, $\partial_i=\frac{\partial}{\partial x_i}$, $i=\overline{1,n}$:

$$\nabla_X Y = \sum_{i=1}^{n}\sum_{j=1}^{n} X^i \left(\frac{\partial Y^j}{\partial x_i}\partial_j + Y^j \sum_{k=1}^{n} \left| \begin{matrix} k \\ ij \end{matrix} \right| \partial_k \right)$$

So we have for the considered hypersurface:

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \left| \begin{matrix} k \\ ij \end{matrix} \right| \partial_k = \frac{1}{|g|}\frac{\partial^2 f}{\partial x_i \partial x_j}\sum_{k=1}^{n}\frac{\partial f}{\partial x_k}\partial_k$$

The covariant derivative is the generalization of the concept of directional derivative, in the sense that if X is, locally, tangent to a curve $x_i=x_i(t)$, $i=\overline{1,n+1}$ and Y is the restriction of a field of vectors along the curve, then $\nabla_X Y$ represents “the rate of change” of Y in the movement on the curve. In other words, in a flat space, the components of $\nabla_X Y$ are given by the derivatives in the direction of the tangent to the curve.

A curve $x_i=c_i(t)$, $i=\overline{1,n+1}$, $t\in(a,b)$, a<b, $a,b\in\mathbf{R}$ is called geodesic if, considering the vector $X=\sum_{i=1}^{n} c_i'(t)\partial_i$ occurs: $\nabla_X X=0$. The equation of a geodesic is: (Ianuș, 1983)

$$\frac{d^2 c_i}{dt^2} + \left| \begin{matrix} i \\ jk \end{matrix} \right| \frac{dc_j}{dt}\frac{dc_k}{dt} = 0, \; i=\overline{1,n+1}$$

Considering 2-space π generated by two vectors tangent to the hypersurface at the point x_0, then all geodesics passing through x_0 and have as tangent vectors to the curve vectors of π, they generate a surface S passing through x_0 and having π as tangent plane. Considering the normal to S in x_0, this, together with an arbitrary tangent vector from π generates a normal plane. It intersects the surface S after a curve called normal section. Considering their curvature, that is their “deviation” from a straight line, there are obtain several curvatures whose extreme (minimum and maximum) form the so-called principal curvatures. The product of the two principal curvatures is called Gaussian curvature of the surface S. In the case of hypersurfaces, corresponding Gaussian curvature to the plane π determined by two vectors $X,Y\in T_{x_0}H$ is called the sectional curvature corresponding to the vectors X,Y in x_0 noted with $k(x_0,\pi)$ where π=<X,Y> (the subspace generated by X and Y).

We define now the Riemann curvature tensor: $R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$ and the Riemann-Christoffel curvature tensor: $R(X,Y,Z,V)=g(R(X,Y)V,Z)$ $\forall X,Y,Z,V\in T_{x_0}H$.

We will note:

$$R(\partial_j,\partial_k)\partial_i=\sum_{p=1}^{n}R_{ijk}^{p}\partial_p\,,\ R_{ijk}^{p}=\frac{\partial\left|\begin{matrix}p\\ ik\end{matrix}\right|}{\partial x_j}-\frac{\partial\left|\begin{matrix}p\\ ij\end{matrix}\right|}{\partial x_k}+\sum_{q=1}^{n}\left|\begin{matrix}q\\ ik\end{matrix}\right|\left|\begin{matrix}p\\ qj\end{matrix}\right|-\sum_{q=1}^{n}\left|\begin{matrix}q\\ ij\end{matrix}\right|\left|\begin{matrix}p\\ qk\end{matrix}\right|,$$

$$R(\partial_i,\partial_j,\partial_k,\partial_p)=R_{ijkp}\,,\ R_{ijkp}=\sum_{q=1}^{n}g_{iq}R_{jkp}^{q}$$

Taking into account that: $\frac{\partial|g|}{\partial x_i}=2\sum_{s=1}^{n}\frac{\partial f}{\partial x_s}\frac{\partial^2 f}{\partial x_s\partial x_i}$ we have for the considered hypersurface:

$$R_{ijk}^{p}=\frac{\partial\left|\begin{matrix}p\\ ik\end{matrix}\right|}{\partial x_j}-\frac{\partial\left|\begin{matrix}p\\ ij\end{matrix}\right|}{\partial x_k}+\sum_{q=1}^{n}\left|\begin{matrix}q\\ ik\end{matrix}\right|\left|\begin{matrix}p\\ qj\end{matrix}\right|-\sum_{q=1}^{n}\left|\begin{matrix}q\\ ij\end{matrix}\right|\left|\begin{matrix}p\\ qk\end{matrix}\right|=$$

$$\frac{\sum_{s=1}^{n}\frac{\partial f}{\partial x_s}\frac{\partial^2 f}{\partial x_s\partial x_k}}{|g|^2}\frac{\partial^2 f}{\partial x_i\partial x_j}\frac{\partial f}{\partial x_p}-\frac{\sum_{s=1}^{n}\frac{\partial f}{\partial x_s}\frac{\partial^2 f}{\partial x_s\partial x_j}}{|g|^2}\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial f}{\partial x_p}+\frac{1}{|g|}\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial^2 f}{\partial x_j\partial x_p}-\frac{1}{|g|}\frac{\partial^2 f}{\partial x_i\partial x_j}\frac{\partial^2 f}{\partial x_k\partial x_p}$$

$$R_{sijk}=\sum_{p=1}^{n}g_{sp}R_{ijk}^{p}=\frac{1}{|g|}\left(\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial^2 f}{\partial x_j\partial x_s}-\frac{\partial^2 f}{\partial x_i\partial x_j}\frac{\partial^2 f}{\partial x_k\partial x_s}\right)$$

The significance of the Riemann curvature tensor is that $\forall X,Y\in T_{x_0}H$:

$$R(X,Y,X,Y)=k(x_0,\pi)\left(g(X,X)g(Y,Y)-g(X,Y)^2\right)$$

A hypersurface is said to have constant curvature if the sectional curvature is the same independent of the point x_0 and the 2-space π determined by two arbitrary vectors of $T_{x_0}H$. It shows that if hypersurface has constant curvature k then:

$$R(X,Y,Z,V)=k\left(g(X,Z)g(Y,V)-g(X,V)g(Y,Z)\right)\ \forall X,Y,Z,V\in T_{x_0}H$$

Schur's theorem states that if the hypersurfaces dimension is larger or equal to 3 and the sectional curvature depends only on the point and not tangent vectors (isotropy property) than the sectional curvature is constant.

If Riemann curvature tensor is null then it can build a coordinate system in which metric tensor components are constant. If the metric tensor is constant then the Riemann curvature tensor is trivial null.

Considering Riemann curvature tensor, it defines the Ricci tensor as: S(X,Y) with $S_{ij}=S(\partial_i,\partial_j)$ and $S_{ij}=\sum_{k,p=1}^{n} g^{kp}R_{kijp}$ and also the scalar curvature $S=\sum_{i,j=1}^{n} g^{ij}S_{ij}$.

For considered hypersurface:

$$S_{ij}=\sum_{k,p=1}^{n} g^{kp}R_{kijp}=\frac{1}{|g|^2}\sum_{k=1}^{n}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\frac{\partial^2 f}{\partial x_k^2}+\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial^2 f}{\partial x_j\partial x_k}\right)\left(1-\left(\frac{\partial f}{\partial x_k}\right)^2\right)$$

$$S=\sum_{i,j=1}^{n} g^{ij}S_{ij}=$$

$$\frac{1}{|g|^3}\sum_{i,j=1}^{n}\left(\delta_{ij}+\delta_{ij}\sum_{k=1}^{n}\left(\frac{\partial f}{\partial x_k}\right)^2-\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\sum_{k=1}^{n}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\frac{\partial^2 f}{\partial x_k^2}+\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial^2 f}{\partial x_j\partial x_k}\right)\left(1-\left(\frac{\partial f}{\partial x_k}\right)^2\right)$$

$$=$$

$$\frac{1}{|g|^2}\sum_{i,k=1}^{n}\left(\frac{\partial^2 f}{\partial x_i^2}\frac{\partial^2 f}{\partial x_k^2}+\left(\frac{\partial^2 f}{\partial x_i\partial x_k}\right)^2\right)\left(1-\left(\frac{\partial f}{\partial x_k}\right)^2\right)-\frac{1}{|g|^3}\sum_{k,i,j=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\frac{\partial^2 f}{\partial x_k^2}+\frac{\partial^2 f}{\partial x_i\partial x_k}\frac{\partial^2 f}{\partial x_j\partial x_k}\right)\left(1-\left(\frac{\partial f}{\partial x_k}\right)^2\right)$$

A hypersurface is called Einstein hypersurface if $S(X,Y)=\lambda g(X,Y)$ $\forall X,Y\in T_{x_0}H$ $\forall x_0\in H$.

If a hypersurface has constant curvature, then it is Einstein, and if it is Einstein and has dimension 3 then it has constant curvature.

We now define the curvature tensor of Weyl:

$C(X,Y)Z=R(X,Y)Z-X\cdot L(Y,Z)+Y\cdot L(X,Z)+g(X,Z)\cdot lY-g(Y,Z)\cdot lX$ $\forall X,Y,Z\in T_{x_0}H$

where $L(X,Y)=\frac{1}{n-2}\left(S(X,Y)-\frac{S}{2(n-1)}g(X,Y)\right)$ and $g(lX,Y)=L(X,Y)$

and also: $C(X,Y,Z,V)=g(C(X,Y)V,Z)$ from where:

$$C(X,Y,Z,V)=R(X,Y,Z,V)-\frac{1}{n-2}\left(g(X,Z)R(Y,V)+g(Y,V)R(X,Z)-\frac{S}{n-1}g(X,Z)g(Y,V)\right)+$$

$$\frac{1}{n-2}\left(g(X,V)R(Y,Z)+g(Y,Z)R(X,V)-\frac{S}{n-1}g(X,V)g(Y,Z)\right)$$

On a hypersurface of dimension 3 the Weyl tensor is null.

A hypersurface is said to be applied conformal to another space (e.g. the flat Euclidean space) if there is an application that preserves angles between any two tangent vectors. If hypersurface is conformal to an Euclidian space we will say that it is conformally flat.

Weyl theorem states that any hypersurface of dimension 2 (i.e. surface in the usual sense) is conformally flat, one of dimension 3 is conformally flat if and only if the tensor Riemann vanishes identically and if the dimension is greater than 3, then the necessary and sufficient condition to be conformally flat is that the Weyl tensor vanishes identically.

2. The Production Function

We define on $\mathbf{R}^n$ – the production space for n resources: $SP=\{(x_1,...,x_n) \mid x_1,...,x_n\geq 0\}$ where $x=(x_1,...,x_n)\in SP$ is an ordered set of resources.

Because in a production process, depending on the nature of applied technology, but also its specificity, not any amount of resources are possible, we will restrict the production area to a subset $DP\subset SP$ called domain of production.

It is called production function an application $Q:DP\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+$ $\forall(x_1,...,x_n)\in DP$.

The production function must satisfy a number of axioms:

- The domain of production is convex;
- $Q(0,...,0)=0$;
- The production function is of class C^2 on DP that is it admits partial derivatives of order 2 and they are continuous;
- The production function is monotonically increasing in each variable;
- The production function is quasi-concave that is: $Q(\lambda x+(1-\lambda)y)\geq\min(Q(x),Q(y))$ $\forall\lambda\in[0,1]$ $\forall x,y\in DP$.

One of the most used production function in microeconomics or macroeconomics analysis is the Cobb-Douglas function:

$Q:\mathbf{R}_+^n\to\mathbf{R}_+$, $Q(x_1,...,x_n)=Ax_1^{k_1}...x_n^{k_n}$, $A>0$, $k_1,...,k_n>0$

3. The Differential Geometry of Cobb-Douglas Function in 2 Variables

In what follows we will consider the Cobb-Douglas function: $Q:\mathbf{R}_+^2\to\mathbf{R}_+$, $Q(x,y)=x^\alpha y^\beta$ (where for simplification we took $A=1$, $\alpha,\beta>0$.

The equation of the surface is therefore: $u=x^\alpha y^\beta$.

The parametric representation of this surface is:

$$H:\ \begin{cases} x = x \\ y = y \\ u = f(x, y) = x^{\alpha} y^{\beta} \end{cases}$$

First, we have:

$$\frac{\partial f}{\partial x} = \alpha x^{\alpha-1} y^{\beta}, \quad \frac{\partial f}{\partial y} = \beta x^{\alpha} y^{\beta-1}, \quad \frac{\partial^2 f}{\partial x^2} = \alpha(\alpha-1) x^{\alpha-2} y^{\beta}, \quad \frac{\partial^2 f}{\partial x \partial y} = \alpha\beta x^{\alpha-1} y^{\beta-1},$$
$$\frac{\partial^2 f}{\partial y^2} = \beta(\beta-1) x^{\alpha} y^{\beta-2}$$

The tangent plane to the surface at a point (x_0, y_0) is:

$$T_{x_0}H:\ \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - u + f(x_0, y_0) = 0$$

that is:

$$T_{x_0}H:\ \alpha x_0^{\alpha-1} y_0^{\beta} x + \beta x_0^{\alpha} y_0^{\beta-1} y - u + (1 - \alpha - \beta) x_0^{\alpha} y_0^{\beta} = 0$$

We can see that the tangent plane pass through origin if and only if $\alpha+\beta=1$ that is the function is homogenous of degree 1.

The unitary normal at the surface is:

$$N\left(\frac{(x^{\alpha-1}y^{\beta})\alpha}{\sqrt{(x^{2\alpha-2}y^{2\beta-2})(y^2\alpha^2+x^2\beta^2)+1}}, \frac{\beta x^{\alpha} y^{\beta-1}}{\sqrt{x^{2\alpha-2}y^{2\beta-2}(\beta^2x^2+\alpha^2y^2)+1}}, -\frac{1}{\sqrt{x^{2\alpha-2}y^{2\beta-2}(\beta^2x^2+\alpha^2y^2)+1}}\right)$$

The metric tensor has the components:

$$g_{11} = \alpha^2 x^{2\alpha-2} y^{2\beta} + 1,\ g_{12} = g_{21} = \alpha\beta x^{2\alpha-1} y^{2\beta-1},\ g_{22} = \beta^2 x^{2\alpha} y^{2\beta-2} + 1$$

and the inverse:

$$g^{11} = \frac{\beta^2 x^{2\alpha} y^{2\beta-2} + 1}{x^{2\alpha-2}y^{2\beta-2}(y^2\alpha^2+x^2\beta^2)+1}, g^{12} = g^{21} = -\frac{\alpha\beta x^{2\alpha+1} y^{2\beta+1}}{x^2y^2 + x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2)}, g^{22} = \frac{\alpha^2 x^{2\alpha-2} y^{2\beta} + 1}{x^{2\alpha-2}y^{2\beta-2}(\beta^2x^2+\alpha^2y^2)+1}$$

Christoffel's symbols of the first kind are:

$$|11,1| = (\alpha-1)\alpha^2 x^{2\alpha-3} y^{2\beta},$$

$|11,2| = \frac{1}{2}(2\alpha(2\alpha-1)\beta x^{2\alpha-2}y^{2\beta-1} - 2\alpha^2\beta x^{2\alpha-2}y^{2\beta-1})$, $|12,1| = |21,1| = \alpha^2\beta x^{2\alpha-2}y^{2\beta-1}$, $|12,2| = |21,2| = \alpha\beta^2 x^{2\alpha-1}y^{2\beta-2}$,

$|22,1| = \frac{1}{2}\left(2\alpha\beta(2\beta-1)x^{2\alpha-1}y^{2\beta-2} - 2\alpha\beta^2 x^{2\alpha-1}y^{2\beta-2}\right)$, $|22,2| = (\beta-1)\beta^2 x^{2\alpha}y^{2\beta-3}$

Christoffel's symbols of the second kind are:

$\left|\begin{matrix}1\\11\end{matrix}\right| = \frac{(\alpha-1)\alpha^2 y^{2\beta+2}}{y^2x^{3-2\alpha}+xy^{2\beta}(\beta^2x^2+\alpha^2y^2)}$, $\left|\begin{matrix}2\\11\end{matrix}\right| = \frac{(\alpha-1)\alpha\beta x^{2\alpha}y^{2\beta+1}}{x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2)}$,

$\left|\begin{matrix}1\\12\end{matrix}\right| = \left|\begin{matrix}1\\21\end{matrix}\right| = \frac{\alpha^2\beta x^{2\alpha}y^{2\beta+1}}{x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2)}$, $\left|\begin{matrix}2\\12\end{matrix}\right| = \left|\begin{matrix}2\\21\end{matrix}\right| = \frac{\alpha\beta^2 x^{2\alpha+1}y^{2\beta}}{x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2)}$,

$\left|\begin{matrix}1\\22\end{matrix}\right| = \frac{\alpha(\beta-1)\beta x^{2\alpha+1}y^{2\beta}}{x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2)}$, $\left|\begin{matrix}2\\22\end{matrix}\right| = \frac{(\beta-1)\beta^2 x^{2\alpha+2}}{x^2y^{3-2\beta}+yx^{2\alpha}(\beta^2x^2+\alpha^2y^2)}$

Riemann-Christoffel curvature tensor is:

$$R_{1212} = \frac{\alpha\beta(1-\alpha-\beta)x^{2\alpha}y^{2\beta}}{x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2)}$$

The components of Ricci tensor are:

$S_{11} = -\frac{\alpha\beta(\alpha+\beta-1)x^{2\alpha}y^{2\beta+2}(\alpha^2x^{2\alpha}y^{2\beta}+x^2)}{(x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2))^2}$, $S_{12} = S_{21} = -\frac{\alpha^2\beta^2(\alpha+\beta-1)x^{4\alpha+1}y^{4\beta+1}}{(x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2))^2}$,

$$S_{22} = -\frac{\alpha\beta(\alpha+\beta-1)x^{2\alpha+2}y^{2\beta}(\beta^2x^{2\alpha}y^{2\beta}+y^2)}{(x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2))^2}$$

and the scalar curvature:

$S = -\frac{\alpha\beta(\alpha+\beta-1)x^{2\alpha+2}y^{2\beta+2}(\alpha^2\beta^2x^{4\alpha}y^{4\beta}+2x^2y^2+2x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2))}{(x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2))^3}$

Finally, the sectional curvature is:

$$K(1,2) = -\frac{\alpha\beta(\alpha+\beta-1)x^{2\alpha+2}y^{2\beta+2}}{(x^2y^2+x^{2\alpha}y^{2\beta}(\beta^2x^2+\alpha^2y^2))^2}$$

where K(1,2) corresponds to the plane determined by the vectors: $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$.

We can see that the sectional curvature vanishes if and only if $\alpha+\beta=1$ that is the function is homogenous of degree 1. In this case we have that Riemann-Christoffel curvature tensor vanishes and obviously Ricci tensor and the scalar of curvature.

4. The Differential Geometry of Cobb-Douglas Function in 3 Variables

In what follows we will consider the Cobb-Douglas function: $Q:\mathbf{R}_+^3 \to \mathbf{R}_+$, $Q(x,y,z) = x^\alpha y^\beta z^\gamma$ (where for simplification we took A=1, $\alpha,\beta,\gamma>0$.

The equation of the surface is therefore: $u = x^\alpha y^\beta z^\gamma$.

The parametric representation of this hypersurface is:

$$H: \begin{cases} x = x \\ y = y \\ z = z \\ u = f(x,y,z) = x^\alpha y^\beta z^\gamma \end{cases}$$

First, we have:

$$\frac{\partial f}{\partial x} = \alpha x^{\alpha-1} y^\beta z^\gamma,\ \frac{\partial f}{\partial y} = \beta x^\alpha y^{\beta-1} z^\gamma,\ \frac{\partial f}{\partial z} = \gamma x^\alpha y^\beta z^{\gamma-1},$$

$$\frac{\partial^2 f}{\partial x^2} = \alpha(\alpha-1)x^{\alpha-2}y^\beta,\ \frac{\partial^2 f}{\partial x \partial y} = \alpha\beta x^{\alpha-1} y^{\beta-1} z^\gamma,\ \frac{\partial^2 f}{\partial x \partial z} = \alpha\gamma x^{\alpha-1} y^\beta z^{\gamma-1},$$

$$\frac{\partial^2 f}{\partial y^2} = \beta(\beta-1)x^\alpha y^{\beta-2} z^\gamma,\ \frac{\partial^2 f}{\partial y \partial z} = \beta\gamma x^\alpha y^{\beta-1} z^{\gamma-1},\ \frac{\partial^2 f}{\partial z^2} = \gamma(\gamma-1)x^\alpha y^\beta z^{\gamma-2}$$

The tangent hyperplane to the hypersurface at a point (x_0, y_0, z_0) is:

$T_{x_0}H$:

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial f}{\partial z}(x_0,y_0,z_0)(z-z_0) - u + f(x_0,y_0,z_0) = 0$$

that is:

$$T_{x_0}H:\ \alpha x_0^{\alpha-1} y_0^\beta z_0^\gamma x + \beta x_0^\alpha y_0^{\beta-1} z_0^\gamma y + \gamma x_0^\alpha y_0^\beta z_0^{\gamma-1} z - u + (1-\alpha-\beta-\gamma)x_0^\alpha y_0^\beta z_0^\gamma = 0$$

We can see that the tangent hyperplane pass through origin if and only if $\alpha+\beta+\gamma=1$ that is the function is homogenous of degree 1.

The unitary normal at the hypersurface is:

$$N\left(\frac{\alpha x^{\alpha-1}y^{\beta}z^{\gamma}}{\sqrt{x^{2\alpha-2}y^{2\beta-2}z^{2\gamma-2}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2))+1}},\right.$$

$$\frac{\beta x^{\alpha}y^{\beta-1}z^{\gamma}}{\sqrt{x^{2\alpha-2}y^{2\beta-2}z^{2\gamma-2}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2))+1}},$$

$$\frac{\gamma x^{\alpha}y^{\beta}z^{\gamma-1}}{\sqrt{x^{2\alpha-2}y^{2\beta-2}z^{2\gamma-2}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2))+1}},$$

$$\left.-\frac{1}{\sqrt{x^{2\alpha-2}y^{2\beta-2}z^{2\gamma-2}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2))+1}}\right)$$

Computing as in the upper we find thate the sectional curvatures are:

$$K(1,2) = -\frac{\alpha\beta(\alpha+\beta-1)x^{2\alpha+2}y^{2\beta+2}z^{2\gamma+2}}{(x^2y^2+x^{2\alpha}y^{2\beta}z^{2\gamma}(\beta^2x^2+\alpha^2y^2))(x^2y^2z^2+x^{2\alpha}y^{2\beta}z^{2\gamma}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2)))}$$

$$K(1,3) = -\frac{\alpha\gamma(\alpha+\gamma-1)x^{2\alpha+2}y^{2\beta+2}z^{2\gamma+2}}{(x^2z^2+x^{2\alpha}y^{2\beta}z^{2\gamma}(\gamma^2x^2+\alpha^2z^2))(x^2y^2z^2+x^{2\alpha}y^{2\beta}z^{2\gamma}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2)))}$$

$$K(2,3) = -\frac{\beta\gamma(\beta+\gamma-1)x^{2\alpha+2}y^{2\beta+2}z^{2\gamma+2}}{(x^{2\alpha}y^{2\beta}z^{2\gamma}(\gamma^2y^2+\beta^2z^2)+y^2z^2)(x^2y^2z^2+x^{2\alpha}y^{2\beta}z^{2\gamma}(\gamma^2x^2y^2+z^2(\beta^2x^2+\alpha^2y^2)))}$$

where K(1,2) corresponds to the plane determined by the vectors: $\partial_1=\frac{\partial}{\partial x}$ and $\partial_2=\frac{\partial}{\partial y}$, K(1,3) corresponds to the plane determined by the vectors: $\partial_1=\frac{\partial}{\partial x}$ and $\partial_3=\frac{\partial}{\partial z}$ and K(2,3) corresponds to the plane determined by the vectors: $\partial_2=\frac{\partial}{\partial y}$ and $\partial_3=\frac{\partial}{\partial z}$.

We can see that the upper sectional curvatures vanishes if and only if $\alpha+\beta=1$, $\alpha+\gamma=1$ and $\beta+\gamma=1$ from where we find that $\alpha=\beta=\gamma=\frac{1}{2}$ that is the function becomes: $Q(x,y,z)=\sqrt{xyz}$.

5. References

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