Mathematical and Quantitative Methods

A Generalization of Some Production Functions

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Abstract: In this paper we shall give a generalization of Cobb-Douglas, CES, Lu-Fletcher, Liu-Hildebrand, VES, Kadiyala production functions. We compute the principal indicators like the marginal products, the marginal rate of substitution, the elasticities of factors and the elasticity of substitution. Finally we formulate two theorems of characterization for the functions with a proportional marginal rate of substitution and for those with constant elasticity+ of substitution (for n=1).

Keywords: production functions, Cobb-Douglas, CES, Lu-Fletcher, Liu-Hildebrand, VES, Kadiyala

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1. Introduction

In the economical analysis, the production functions had a long and interesting history.

A production function is defined like $P:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$, P=P(K,L) where P is the production, K - the capital and L – the labor such that:

(1) P(0,0)=0;

(2) P is differentiable of order 2 in any interior point of the production set;

(3) P is a homogenous function of degree 1, that is P(rK,rL)=rP(K,L) $\forall r\in\mathbf{R}$;

(4) $\frac{\partial P}{\partial K}\geq 0$, $\frac{\partial P}{\partial L}\geq 0$;

(5) $\frac{\partial^2 P}{\partial K^2} \leq 0,\ \frac{\partial^2 P}{\partial L^2} \leq 0.$

For any production function, we have a lot of indicators like:

(6) $\eta_K = \frac{\partial P}{\partial K}$ - the marginal product of K;

(7) $\eta_L = \frac{\partial P}{\partial L}$ - the marginal product of L;

(8) RMS= $\frac{\frac{\partial P}{\partial L}}{\frac{\partial P}{\partial K}}$ - the marginal rate of substitution;

(9) $E_K = \frac{\frac{\partial P}{\partial K}}{\frac{P}{K}}$ - the elasticity of K;

(10) $E_L = \frac{\frac{\partial P}{\partial L}}{\frac{P}{L}}$ - the elasticity of L;

(11) $\sigma = \frac{\frac{\partial P}{\partial L}\frac{\partial P}{\partial K}}{P\frac{\partial^2 P}{\partial K \partial L}}$ - the elasticity of substitution.

In [0] Charles Cobb and Paul Douglas formulate the well-known production function: $P(K,L)=\alpha K^p L^{1-p}$ where $p \in [0,1]$ which have many applications in various economical problems.

In [0] the authors generalize the preceding, obtained the CES production function: $P(K,L)=\alpha\left(\beta K^{\rho} + (1-\beta)L^{\rho}\right)^{\frac{1}{\rho}}$, which for $\rho=0$ becomes Cobb-Douglas function.

The Lu-Fletcher production function generalized also, the CES function into the form: $P(K,L)=\alpha\left(\delta K^{\beta}+(1-\delta)\eta\left(\frac{K}{L}\right)^{-c(1-\beta)}L^{\beta}\right)^{\frac{1}{\beta}}$, which for c=0, $\eta=1$ becomes CES function.

In [0] T.C. Liu and G.H. Hildebrand made a new generalization of CES function: $P(K,L)=\alpha\left((1-\delta)K^{\eta}+\delta K^{m\eta}L^{(1-m)\eta}\right)^{\frac{1}{\eta}}$ for m=0.

N.S. Revankar introduced the VES function: $P(K,L)=\alpha K^{\rho(1-\delta\mu)}[L+(\mu-1)K]^{\rho\delta\mu}$ which for $\mu=1$, $\rho=1$ is also a generalization of Cobb-Douglas production function.

In [0], K.R. Kadiyala made an important generalization with:

$P(K,L)=E(t)\left(c_{11}K^{\beta_1+\beta_2}+2c_{12}K^{\beta_1}L^{\beta_2}+c_{22}L^{\beta_1+\beta_2}\right)^{\frac{\rho}{\beta_1+\beta_2}}$ where $c_{11}+2c_{12}+c_{22}=1$, $c_{ij}\geq 0$, $\beta_1(\beta_1+\beta_2)>0$, $\beta_2(\beta_1+\beta_2)>0$.

For $c_{12}=0$, $\rho=1$ Kadiyala obtain the CES function, $c_{22}=0$ generates directly the Lu-Fletcher function, for $c_{11}=0$, $c_{22}=0$, $\rho=1$ – the Cobb-Douglas function and, finally, for $\beta_1=\frac{1}{\delta\mu}-1$, $\beta_2=1$, $c_{22}=0$ – the VES function.

In what follows, we shall make a new generalization, from another point of view, of these functions.

2. The sum production function

Let the production function:

(12) $P(K,L)=\sum_{i=1}^{n}\alpha_i\left(c_{i1}K^{p_{i1}+p_{i2}}+c_{i2}K^{p_{i1}}L^{p_{i2}}+c_{i3}L^{p_{i1}+p_{i2}}\right)^{p_{i3}}$, $n\geq 1$

where:

(13) $\alpha_i>0\ \forall i=\overline{1,n}$;

(14) $p_{i3}\in(-\infty,0)\cup[1,\infty)$, $p_{i1}p_{i2}>0$, $p_{i3}(p_{i1}+p_{i2})=1\ \forall i=\overline{1,n}$;

(15) $\sum_{i=1}^{n}\left(c_{i2}+c_{i1}c_{i3}\right)>0$, $c_{ij}\geq 0\ \forall i=\overline{1,n}\ \forall j=\overline{1,3}$.

From (14) follows that if $p_{i3}<0$ then p_{i1}, $p_{i2}<0$. If $p_{i3}\geq 1$ then p_{i1}, $p_{i2}>0$ and $p_{i1}+p_{i2}=\frac{1}{p_{i3}}\leq 1$ therefore: $1-p_{i1}\geq p_{i2}>0$, $1-p_{i2}\geq p_{i1}>0$.

We have then the following cases:

(16) a) $p_{i1}, p_{i2}, p_{i3}\in(-\infty,0)$ and $p_{i3}(p_{i1}+p_{i2})=1$;

b) $p_{i1}, p_{i2}\in(0,1)$, $p_{i3}\in[1,\infty)$ and $p_{i3}(p_{i1}+p_{i2})=1$.

From (15) we have that $\exists i=\overline{1,n}$ such that $c_{i2}+c_{i1}c_{i3}>0$ therefore, if for such an i, we have $c_{i2}=0$ follows that c_{i1}, $c_{i3}>0$ and if $c_{i2}>0$ follows that c_{i1}, c_{i3} are arbitrary (of course non-negative).

If we note:

(17) $\frac{K}{L}=\chi$

follows:

(18) $P=L\sum_{i=1}^{n}\alpha_i\left(c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}\chi^{p_{i1}}+c_{i3}\right)^{p_{i3}}$

Because $\chi\geq 0$ and for any $i=\overline{1,n}$ we have that $\alpha_i>0$ and at least one of c_{i1}, c_{i2} or c_{i3} is greater than 0 we obtain $P\geq 0$. Also from (12): $P(0,0)=0$ and P is differentiable of order 2 in any interior point of the production set.

We have now:

$P(rK,rL)=rL\sum_{i=1}^{n}\alpha_i\left(c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}\chi^{p_{i1}}+c_{i3}\right)^{p_{i3}}=r^1P(K,L)$ therefore P is homogenous of first degree.

Le note now:

(19) $A_i(\chi)=c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}\chi^{p_{i1}}+c_{i3}>0$, $i=\overline{1,n}$

From (18) and (19) we have that:

(20) $P=L\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}}=L\Phi(\chi)$

where:

(21) $\Phi(\chi)=\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}}$

From (19) we obtain easily:

(22) $A_i'(\chi)=\chi^{p_{i1}-1}\left[c_{i1}(p_{i1}+p_{i2})\chi^{p_{i2}}+c_{i2}p_{i1}\right]$, $i=\overline{1,n}$

(23) $A_i''(\chi)=\chi^{p_{i1}-2}\left[c_{i1}(p_{i1}+p_{i2})(p_{i1}+p_{i2}-1)\chi^{p_{i2}}+c_{i2}p_{i1}(p_{i1}-1)\right]$, $i=\overline{1,n}$

From (17) we obtain after partial derivation:

(24) $\frac{\partial\chi}{\partial K}=\frac{1}{L}$, $\frac{\partial\chi}{\partial L}=-\frac{K}{L^2}=-\frac{\chi}{L}$

From (20) we have:

(25) $\frac{\partial P}{\partial L}=\Phi(\chi)-\chi\Phi'(\chi)$, $\frac{\partial P}{\partial K}=\Phi'(\chi)$

therefore:

(26) $\frac{\partial P}{\partial L}=\frac{P}{L}-\chi\frac{\partial P}{\partial K}$

who can be derived also, from Euler's formula for homogenous functions.

By derivation with L and after with K in (26) we obtain:

$$\frac{\partial^2 P}{\partial L^2}=\frac{\frac{\partial P}{\partial L}L-P}{L^2}+\frac{\chi}{L}\frac{\partial P}{\partial K}-\chi\frac{\partial^2 P}{\partial L\partial K}=-\chi\frac{\partial^2 P}{\partial L\partial K}$$

$$\frac{\partial^2 P}{\partial L\partial K}=\frac{1}{L}\frac{\partial P}{\partial K}-\frac{1}{L}\frac{\partial P}{\partial K}-\chi\frac{\partial^2 P}{\partial K^2}=-\chi\frac{\partial^2 P}{\partial K^2}$$

therefore:

(27) $\frac{\partial^2 P}{\partial L^2}=-\chi\frac{\partial^2 P}{\partial L\partial K}$

(28) $\frac{\partial^2 P}{\partial K^2}=-\frac{1}{\chi}\frac{\partial^2 P}{\partial L\partial K}$

(29) $\frac{\partial^2 P}{\partial L^2}=\chi^2\frac{\partial^2 P}{\partial K^2}$

therefore $\frac{\partial^2 P}{\partial L^2}$ and $\frac{\partial^2 P}{\partial K^2}$ have the same sign.

We have now, from (20):

(30) $\eta_K = \frac{\partial P}{\partial K} = \sum_{i=1}^{n} \alpha_i A_i(\chi)^{p_{i3}-1} \chi^{p_{i1}-1}(c_{i1}\chi^{p_{i2}} + c_{i2}p_{i1}p_{i3})$

(31) $\eta_L = \frac{\partial P}{\partial L} = \sum_{i=1}^{n} \alpha_i A_i(\chi)^{p_{i3}-1}(A_i(\chi) - \chi p_{i3} A_i'(\chi))$

Because:

$A_i(\chi) - \chi p_{i3} A_i'(\chi) = (c_{i1}\chi^{p_{i1}+p_{i2}} + c_{i2}\chi^{p_{i1}} + c_{i3}) - \chi p_{i3}(c_{i1}(p_{i1}+p_{i2})\chi^{p_{i1}+p_{i2}-1} + c_{i2}p_{i1}\chi^{p_{i1}-1}) =$ $c_{i2}p_{i2}p_{i3}\chi^{p_{i1}} + c_{i3}$

we obtain from (31):

(32) $\eta_L = \frac{\partial P}{\partial L} = \sum_{i=1}^{n} \alpha_i A_i(\chi)^{p_{i3}-1}(c_{i2}p_{i2}p_{i3}\chi^{p_{i1}} + c_{i3})$

From (13)-(16) we can see easily that $\frac{\partial P}{\partial K} \geq 0$.

We have now the following lemma which will be useful in all what follows:

Lemma Let $q_i \in \mathbf{R}^*$, $i=\overline{1,m}$, $m \geq 1$, $q_i \neq q_j$ $\forall i,j=\overline{1,m}$, $i \neq j$. Therefore the functions χ^{q_i}, $i=\overline{1,m}$ and the constant function 1 are linear independent, that is from the equality: $\sum_{i=1}^{m} \beta_i \chi^{q_i} + \beta_{m+1} = 0$ follows $\beta_i = 0$, $i=\overline{1,m+1}$.

Proof. Differentiating the equality $\sum_{i=1}^{m} \beta_i \chi^{q_i} + \beta_{m+1} = 0$ m-times, we obtain:

$$\sum_{i=1}^{m} \beta_i \binom{q_i}{k} \chi^{q_i - k} = 0,\ k=\overline{1,m}$$

where $\binom{q_i}{k} = \frac{q_i(q_i-1)...(q_i-k+1)}{k!}$, $k=\overline{1,m}$.

Let compute now the determinant of the system. We have:

$$\begin{vmatrix} \binom{q_1}{1}\chi^{q_1-1} & \binom{q_2}{1}\chi^{q_2-1} & \dots & \binom{q_m}{1}\chi^{q_m-1} \\ \binom{q_1}{2}\chi^{q_1-2} & \binom{q_2}{2}\chi^{q_2-2} & \dots & \binom{q_m}{2}\chi^{q_m-2} \\ \dots & \dots & \dots & \dots \\ \binom{q_1}{m}\chi^{q_1-m} & \binom{q_2}{m}\chi^{q_2-m} & \dots & \binom{q_m}{m}\chi^{q_m-m} \end{vmatrix} =$$

$$\chi^{q_1-m}\dots\chi^{q_m-m} \begin{vmatrix} \binom{q_1}{1}\chi^{m-1} & \binom{q_2}{1}\chi^{m-1} & \dots & \binom{q_m}{1}\chi^{m-1} \\ \binom{q_1}{2}\chi^{m-2} & \binom{q_2}{2}\chi^{m-2} & \dots & \binom{q_m}{2}\chi^{m-2} \\ \dots & \dots & \dots & \dots \\ \binom{q_1}{m} & \binom{q_2}{m} & \dots & \binom{q_m}{m} \end{vmatrix} =$$

$$\chi^{q_1-m}\dots\chi^{q_m-m}\chi^{\frac{(m-1)m}{2}} \begin{vmatrix} \binom{q_1}{1} & \binom{q_2}{1} & \dots & \binom{q_m}{1} \\ \binom{q_1}{2} & \binom{q_2}{2} & \dots & \binom{q_m}{2} \\ \dots & \dots & \dots & \dots \\ \binom{q_1}{m} & \binom{q_2}{m} & \dots & \binom{q_m}{m} \end{vmatrix} = \chi^{q_1-m}\dots\chi^{q_m-m}\chi^{\frac{(m-1)m}{2}} D.$$

The degree of the determinant like function of $q_1, q_2, \dots, q_m$ is:

$$1+2+\dots+m=\frac{m(m+1)}{2}$$

If $q_i=q_j$, $i\neq j$ we have that columns i and j are equals then D=0. Also, if $q_i=0$ for an $i=\overline{1,m}$ follows that D=0. From this follows that: $D=\alpha\prod_{i=1}^{m} q_i \prod_{\substack{i,j=1 \\ i\neq j}}^{m} (q_i - q_j)$ with α a constant (because the degree of the right side is $m+\frac{m(m-1)}{2}=\frac{m(m+1)}{2}$. For m=2 we have that $D=q_1q_2(q_2-q_1)$ therefore $\alpha=1$.

We have that now the determinant of the system is:

$$\chi^{q_1-m}\ldots\chi^{q_m-m}\chi^{\frac{(m-1)m}{2}}\prod_{i=1}^{m}q_i\prod_{\substack{i,j=1\\ i\neq j}}^{m}(q_i-q_j)\neq 0$$

From the system we obtain that $\beta_i=0$, $i=\overline{1,m}$ and from the initial equality follows that $\beta_{m+1}=0$. **Q.E.D.**

Returning at the production functions we have from (30) that if $\frac{\partial P}{\partial K}=0$ follows that:

$c_{i1}\chi^{p_{i2}}+c_{i2}p_{i1}p_{i3}=0$ $\forall i=\overline{1,n}$ therefore from the lemma:

$c_{i1}=c_{i2}=0$ $\forall i=\overline{1,n}$ which is a contradiction with (15). We have finally that $\frac{\partial P}{\partial K}>0$.

From (32) we have that $\frac{\partial P}{\partial L}\geq 0$. If $\frac{\partial P}{\partial L}=0$ we have: $c_{i2}p_{i2}p_{i3}\chi^{p_{i1}}+c_{i3}=0$ therefore:

$c_{i2}=c_{i3}=0$ $\forall i=\overline{1,n}$ which is a contradiction with (15). We have finally that $\frac{\partial P}{\partial L}>0$.

Let compute now the second derivatives.

(33) $$L\frac{\partial^2 P}{\partial K^2}=\sum_{i=1}^{n}\alpha_i p_{i3}(p_{i3}-1)A_i(\chi)^{p_{i3}-2}A_i'^2(\chi)+\sum_{i=1}^{n}\alpha_i p_{i3}A_i(\chi)^{p_{i3}-1}A_i''(\chi)=$$

$$-\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-2}\chi^{p_{i1}-2}\left[c_{i2}^2p_{i1}p_{i2}p_{i3}^2\chi^{p_{i1}}+c_{i3}c_{i2}p_{i1}p_{i3}(1-p_{i1})+c_{i1}c_{i3}(1-p_{i1}-p_{i2})\chi^{p_{i2}}+\right.$$
$$\left.c_{i1}c_{i2}p_{i2}p_{i3}(1-p_{i2})\chi^{p_{i2}+p_{i1}}\right]$$

From (13)-(16) follows that $L\frac{\partial^2 P}{\partial K^2}\leq 0$. If $\frac{\partial^2 P}{\partial K^2}=0$ we have that:

$$c_{i2}^2p_{i1}p_{i2}p_{i3}^2\chi^{p_{i1}}+c_{i3}c_{i2}p_{i1}p_{i3}(1-p_{i1})+c_{i1}c_{i3}(1-p_{i1}-p_{i2})\chi^{p_{i2}}+c_{i1}c_{i2}p_{i2}p_{i3}(1-p_{i2})\chi^{p_{i2}+p_{i1}}=0$$

and from the lemma we have: $c_{i2}=0$, $c_{i1}c_{i3}=0$ which is a contradiction with (15). We have therefore $\frac{\partial^2 P}{\partial K^2}<0$. From (29) we obtain that $\frac{\partial^2 P}{\partial L^2}<0$ and from (28) that

$\frac{\partial^2 P}{\partial L\partial K}>0$.

The marginal rate of substitution is:

(34) $$\text{RMS}=\frac{\frac{\partial P}{\partial L}}{\frac{\partial P}{\partial K}}=\frac{\frac{P}{L}-\chi\frac{\partial P}{\partial K}}{\frac{\partial P}{\partial K}}=\frac{P}{L\frac{\partial P}{\partial K}}-\chi=\frac{\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\left(c_{i2}p_{i2}p_{i3}\chi^{p_{i1}}+c_{i3}\right)}{\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\chi^{p_{i1}-1}(c_{i1}\chi^{p_{i2}}+c_{i2}p_{i1}p_{i3})}.$$

The elasticities of L and K are:

(35) $$E_L=\frac{\frac{\partial P}{\partial L}}{\frac{P}{L}}=1-\chi\frac{\frac{\partial P}{\partial K}}{\chi\frac{P}{K}}=\frac{\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}(c_{i2}p_{i2}p_{i3}\chi^{p_{i1}}+c_{i3})}{\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}}}.$$

(36) $E_K=1-E_L$

The elasticity of substitution:

(37) $$\sigma=\frac{\frac{\partial P}{\partial L}\frac{\partial P}{\partial K}}{P\frac{\partial^2 P}{\partial K\partial L}}=$$

$$\frac{\sum_{i,j=1}^{n}\alpha_i\alpha_j A_i(\chi)^{p_{i3}-1}A_j(\chi)^{p_{j3}-1}\chi^{p_{j1}-1}(c_{i2}p_{i2}p_{i3}\chi^{p_{i1}}+c_{i3})(c_{j1}\chi^{p_{j2}}+c_{j2}p_{j1}p_{j3})}{\chi\sum_{i,j=1}^{n}\alpha_i\alpha_j A_i(\chi)^{p_{i3}}A_j(\chi)^{p_{j3}-2}\chi^{p_{j1}-2}\left[c_{j2}^2p_{j1}\chi^{p_{j1}}p_{j2}p_{j3}^2+c_{j3}c_{j2}p_{j1}p_{j3}(1-p_{j1})+c_{j1}c_{j3}(1-p_{j1}-p_{j2})\chi^{p_{j2}}+\right.}$$
$$\left.+c_{j1}c_{j2}\chi^{p_{j2}+p_{j1}}p_{j2}p_{j3}(1-p_{j2})\right]$$

For n=1 we have:

(38) $P(K,L)=\alpha\left(c_1K^{p_1+p_2}+c_2K^{p_1}L^{p_2}+c_3L^{p_1+p_2}\right)^{p_3}$

where the conditions (13)-(15) becomes:

(39) $\alpha>0$;

(40) $p_3\in(-\infty,0)\cup[1,\infty)$, $p_1p_2>0$, $p_3(p_1+p_2)=1$;

(41) $c_2+c_1c_3>0$, c_1, c_2, $c_3\geq 0$

From (30), (32)-(37) we obtain:

(42) $\eta_K=\frac{\partial P}{\partial K}=\alpha A(\chi)^{p_3-1}\chi^{p_1-1}(c_1\chi^{p_2}+c_2p_1p_3)$

(43) $\eta_L = \frac{\partial P}{\partial L} = \alpha A(\chi)^{p_3-1}(c_2 p_2 p_3 \chi^{p_1} + c_3)$

(44) $L\frac{\partial^2 P}{\partial K^2} = -\alpha A(\chi)^{p_3-2}\chi^{p_1-2}\left[c_2^2 p_1 \chi^{p_1} p_2 p_3^2 + c_3 c_2 p_1 p_3 (1-p_1) + c_1 c_3 (1-p_1-p_2)\chi^{p_2} + c_1 c_2 \chi^{p_2+p_1} p_2 p_3 (1-p_2)\right]$

(45) RMS$= \frac{\alpha A(\chi)^{p_3-1}\left(c_2 p_2 p_3 \chi^{p_1} + c_3\right)}{\alpha A(\chi)^{p_3-1}\chi^{p_1-1}(c_1 \chi^{p_2} + c_2 p_1 p_3)} = \frac{c_2 p_2 p_3 \chi^{p_1} + c_3}{\chi^{p_1-1}(c_1 \chi^{p_2} + c_2 p_1 p_3)}$

(46) $E_L = \frac{\alpha A(\chi)^{p_3-1}(c_2 p_2 p_3 \chi^{p_1} + c_3)}{\alpha A(\chi)^{p_3}} = \frac{c_2 p_2 p_3 \chi^{p_1} + c_3}{c_1 \chi^{p_1+p_2} + c_2 \chi^{p_1} + c_3}$

(47) $E_K = \frac{\chi^{p_1}(c_1 \chi^{p_2} + c_2 p_1 p_3)}{c_1 \chi^{p_1+p_2} + c_2 \chi^{p_1} + c_3}$

(48) $\sigma = \frac{(c_2 p_2 p_3 \chi^{p_1} + c_3)(c_1 \chi^{p_2} + c_2 p_1 p_3)}{c_2^2 p_1 \chi^{p_1} p_2 p_3^2 + c_3 c_2 p_1 p_3 (1-p_1) + c_1 c_3 (1-p_1-p_2)\chi^{p_2} + c_1 c_2 \chi^{p_2+p_1} p_2 p_3 (1-p_2)}$.

3. Particular cases

3.1. The Cobb-Douglas production function

For n=1, $p_1=1-\gamma$, $p_2=\gamma$, $\gamma \in (0,1)$, $c_1=0$, $c_2=1$, $c_3=0$ we have: $P(K,L)=\alpha K^{1-\gamma} L^{\gamma}$.

3.2. The CES production function

For n=1, $p_1=-\frac{\gamma}{2}$, $p_2=-\frac{\gamma}{2}$, $c_1=\delta$, $c_2=0$, $c_3=1-\delta$, $\delta \in (0,1)$ we have:

$P(K,L)=\alpha\left(\delta K^{-\gamma} + (1-\delta)L^{-\gamma}\right)^{-\frac{1}{\gamma}}$.

3.3. The Lu-Fletcher production function

For n=1, $p_1=-\gamma(1-\beta)$, $p_2=\gamma(1-\beta)+\beta$, $c_1=\delta$, $c_2=1-\delta$, $c_3=0$, $\delta\in(0,1)$ we obtain:

$P(K,L)=\alpha\left(\delta K^{\beta}+(1-\delta)K^{-\gamma(1-\beta)}L^{\gamma(1-\beta)+\beta}\right)^{\frac{1}{\beta}}$.

3.4. The Liu-Hildebrand production function

For n=1, $p_1=\delta\eta$, $p_2=(1-\delta)\eta$, $c_1=1-\beta$, $c_2=\beta$, $c_3=0$, $\delta\in(0,1)$ we have:

$P(K,L)=\alpha\left((1-\beta)K^{\eta}+\beta K^{\delta\eta}L^{(1-\delta)\eta}\right)^{\frac{1}{\eta}}$.

3.5. The VES production function

For n=1, $p_1=\frac{1}{\delta\mu}-1$, $p_2=1$, $c_1=\mu-1$, $c_2=1$, $c_3=0$, $\delta\mu\in(0,1)$, $\mu\geq 1$ we have: P(K,L)=

$\alpha\left((\mu-1)K^{\frac{1}{\delta\mu}}+K^{\frac{1}{\delta\mu}-1}L\right)^{\delta\mu}=\alpha K^{1-\delta\mu}\left((\mu-1)K+L\right)^{\delta\mu}$ a particular case of VES production function.

3.6. The Kadiyala production function

For $c_1+c_2+c_3=1$ and $c_2\neq 0$ or $c_2=0$, but c_1, $c_3>0$ we obtain a particular case of Kadiyala production function.

4. Theorems

Theorem 1 The only case when RMS=$k\frac{K}{L}$ where k is a positive constant is the Cobb-Douglas function with $\gamma=\frac{1}{k+1}$.

Proof. From (34) we have that:

$$\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\left(c_{i2}p_{i2}p_{i3}\chi^{p_{i1}}+c_{i3}\right)=k\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\chi^{p_{i1}}(c_{i1}\chi^{p_{i2}}+c_{i2}p_{i1}p_{i3})$$

therefore:

$$(49)\ \sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\left[c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}p_{i3}(p_{i1}-kp_{i2})\chi^{p_{i1}}+c_{i3}\right]=0$$

Let note I={i=$\overline{1,n}$ | p_{i1}, p_{i2}, p_{i3}<0} and J={j=$\overline{1,n}$ | p_{j1}, p_{j2}, p_{j3}>0}

Because (49) holds for every χ, we have with (16):

$$(50)\ \lim_{\chi\to\infty}\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\left[c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}p_{i3}(p_{i1}-kp_{i2})\chi^{p_{i1}}+c_{i3}\right]=$$

$$\lim_{\chi\to\infty}\sum_{i\in I}\alpha_i A_i(\chi)^{p_{i3}-1}\left[c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}p_{i3}(p_{i1}-kp_{i2})\chi^{p_{i1}}+c_{i3}\right]+$$

$$\lim_{\chi\to\infty}\sum_{i\in J}\alpha_j A_j(\chi)^{p_{j3}-1}\left[c_{j1}\chi^{p_{j1}+p_{j2}}+c_{j2}p_{j3}(p_{j1}-kp_{j2})\chi^{p_{j1}}+c_{j3}\right]=$$

$$\sum_{i\in I}\alpha_i c_{i3}^{p_{i3}}+\sum_{i\in J}\alpha_j c_{j1}^{p_{j3}}\infty .$$

$$(51)\ \lim_{\chi\to 0}\sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\left[c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}p_{i3}(p_{i1}-kp_{i2})\chi^{p_{i1}}+c_{i3}\right]=$$

$$\lim_{\chi\to 0}\sum_{i\in I}\alpha_i A_i(\chi)^{p_{i3}-1}\left[c_{i1}\chi^{p_{i1}+p_{i2}}+c_{i2}p_{i3}(p_{i1}-kp_{i2})\chi^{p_{i1}}+c_{i3}\right]+$$

$$\lim_{\chi\to 0}\sum_{i\in J}\alpha_j A_j(\chi)^{p_{j3}-1}\left[c_{j1}\chi^{p_{j1}+p_{j2}}+c_{j2}p_{j3}(p_{j1}-kp_{j2})\chi^{p_{j1}}+c_{j3}\right]=$$

$$\sum_{i\in I}\alpha_i c_{i1}^{p_{i3}}\infty+\sum_{i\in J}\alpha_j c_{j3}^{p_{j3}} .$$

From (50), (51) we have that: $c_{i1}=c_{i3}=0$ $\forall i\in I$ and $c_{j1}=c_{j3}=0$ $\forall j\in J$ therefore $c_{i1}=c_{i3}=0$ $\forall i=\overline{1,n}$.

From (49) we have now:

$$(52)\ \sum_{i=1}^{n}\alpha_i A_i(\chi)^{p_{i3}-1}\chi^{p_{i1}}c_{i2}p_{i3}(p_{i1}-kp_{i2})=0 \text{ where } A_i(\chi)=c_{i2}\chi^{p_{i1}}$$

that is:

$$(53)\ \sum_{i=1}^{n}\alpha_i c_{i2}^{p_{i3}}\chi^{p_{i1}p_{i3}}p_{i3}(p_{i1}-kp_{i2})=0.$$

From the lemma we have:

(54) p_{i1}-kp_{i2}=0 $\forall i=\overline{1,n}$

and with the notation p_{i2}=p we have that: p_{i1}=kp, p_{i2}=p, $p_{i3}=\frac{1}{(k+1)p}$, $p\le\frac{1}{k+1}$.

The production function becomes:

(55) $P(K,L)=\sum_{i=1}^{n}\alpha_i\left(c_{i2}K^{kp}L^{p}\right)^{\frac{1}{(k+1)p}} = \alpha K^{\frac{k}{(k+1)}}L^{\frac{1}{(k+1)}}$

after obvious notations. **Q.E.D.**

Theorem 2 The only cases when for n=1, σ=k where k is a positive constant are the Cobb-Douglas function and CES function with $\gamma=\frac{k}{1-k}$.

Proof. From (37) we have that:

$$(c_2p_2p_3\chi^{p_1}+c_3)(c_1\chi^{p_2}+c_2p_1p_3)=$$
$$k\left[c_2^2p_1\chi^{p_1}p_2p_3^2+c_3c_2p_1p_3(1-p_1)+c_1c_3(1-p_1-p_2)\chi^{p_2}+c_1c_2\chi^{p_2+p_1}p_2p_3(1-p_2)\right]$$

that is:

$$\left(kc_1c_2p_2p_3(1-p_2)-c_1c_2p_2p_3\right)\chi^{p_1+p_2}+\left(kc_2^2p_1p_2p_3^2-c_2^2p_1p_2p_3^2\right)\chi^{p_1}+$$
$$\left(kc_1c_3(1-p_1-p_2)-c_1c_3\right)\chi^{p_2}+kc_3c_2p_1p_3(1-p_1)-c_2c_3p_1p_3=0$$

From lemma, we obtain that:

(56) $c_1c_2p_2p_3\left(k(1-p_2)-1\right)=0$

(57) $c_2^2p_1p_2p_3^2(k-1)=0$

(58) $c_1c_3\left(k(1-p_1-p_2)-1\right)=0$

(59) $c_2c_3p_1p_3\left(k(1-p_1)-1\right)=0$

If $c_2\neq0$ follows from (57) that k=1 and from (56) we have that $c_1=0$ and $c_3=0$. The function is: $P(K,L)=\alpha\left(c_2K^{p_1}L^{p_2}\right)^{p_3}=\beta K^{p}L^{1-p}$ with obvious notations.

If c_2=0, from (58) we have that: k(1-p_1-p_2)-1=0 that is k=$\frac{p_3}{p_3-1}$. and the function is: P(K,L)=$\alpha\left(c_1K^p+c_3L^p\right)^{\frac{1}{p}}$ and k=$\frac{1}{1-p}$. **Q.E.D.**

5. References

Arrow, K.J.; Chenery, H.B.; Minhas, B.S. & Solow, R.M. (1961). *Capital Labour Substitution and Economic Efficiency*, *Review of Econ and Statistics* 63, pp. 225-250.

Cobb, C.W. & Douglas, P.H. (1928). A Theory of Production. *American Economic Review*. 18, pp. 139–165.

Ioan C. A. (2007). Applications of the space differential geometry at the study of production functions, *Euroeconomica*. 18, pp. 30-38.

Ioan C.A. (2004). Applications of geometry at the study of production functions. *The Annals of Danubius University, Fascicle I, Economics*, pp. 27-39.

Kadiyala, K. R. (1972). Production Functions and Elasticity of Substitution. *Southern Economic Journal*, 38(3). pp. 281-284.

Kmenta, J. (1967). On Estimation of the CES Production Function. *International Economic Review*. 8(2). pp. 180-189.

Liu, T.C. and Hildebrand, G.H. (1965). *Manufacturing Production Functions in the United States*. 1957. Ithaca: Cornell University Press.

Mishra, S.K. (2007). *A Brief History of Production Functions*. India: North-Eastern Hill University. Shillong.

Revankar, N.S. (1971). A Class of Variable Elasticity of Substitution Production Functions. *Econometrica*. 39(1), pp. 61-71.

Sato, R. (1974). On the Class of Seperable Non-Homothetic CES Functions. *Economic Studies Quarterly*, 15, pp. 42-55.