Again about Andrica's Conjecture...

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Abstract: The paper establishes an equivalence of the Andrica's conjecture in the direction of an increase of the difference of square root of primes by a combination of two consecutive primes.

Keywords: Andrica's conjecture; prime

JEL Classification: C002

1. Introduction

In a previous paper, entitled "About Andrica's conjecture" the authors have established an equivalence of conjecture Andrica by considering the ratio of two consecutive prime numbers. Because the average deviation calculated relative to the two terms, in this article will study another limit for the difference of square roots of two consecutive prime numbers.

A number $p \in \mathbf{N}$, $p \geq 2$ is called prime number if its only positive divisors are 1 and p.

Even if do not know much about prime numbers, there exist a lot of attempts to determine some of their properties, many results being at the stage of conjectures.

A famous conjecture relative to prime numbers is that of Dorin Andrica. Denoting by p_n - the n-th prime number ($p_1=2$, $p_2=3$, $p_3=5$ etc.), Andrica's conjecture ([1]) states that:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \quad \forall n \geq 1$$

In [3] we have found the following:

Theorem

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Let p_n the n-th prime number. The following statements are equivalent for n≥5:

1. $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$;
2. ∃α≥0 such that: $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{p_n}{p_{n+1}}\right)^{\alpha}$.

In the following, we shall prove a stronger theorem of equivalence of Andrica's conjecture.

2. Main Theorem

Theorem

Let p_n the n-th prime number. The following statements are equivalent for n≥5:

1. $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$;
2. ∃α≥0 such that: $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{{p_{n+1}}^{p_n}}{{p_n}^{p_{n+1}}}\right)^{\alpha}$.

Proof

First of all let the function f:[e,∞)→**R**, $f(x) = \frac{\ln x}{x}$. We have: $f'(x) = \frac{1-\ln x}{x^2} < 0$ therefore f is a strictly decreasing function. For n≥2 we have therefore: $f(p_n) > f(p_{n+1})$ that is: $\frac{\ln p_n}{p_n} > \frac{\ln p_{n+1}}{p_{n+1}} \Leftrightarrow p_{n+1}\ln p_n > p_n \ln p_{n+1} \Leftrightarrow {p_{n+1}}^{p_n} < {p_n}^{p_{n+1}}$

$\Leftrightarrow \frac{{p_{n+1}}^{p_n}}{{p_n}^{p_{n+1}}} < 1$.

<u>2⇒1</u> Because $\frac{{p_{n+1}}^{p_n}}{{p_n}^{p_{n+1}}} < 1$ follows that: α≥0 ⇒ $\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\frac{{p_{n+1}}^{p_n}}{{p_n}^{p_{n+1}}}\right)^{\alpha} < \left(\frac{{p_{n+1}}^{p_n}}{{p_n}^{p_{n+1}}}\right)^{0} = 1$.

<u>1⇒2</u> If we take the logarithm in the relationship, it becomes:

$$\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<\alpha\ln\left(\frac{p_{n+1}{}^{p_n}}{p_n{}^{p_{n+1}}}\right) \Leftrightarrow \ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<\alpha\left(\ln p_{n+1}{}^{p_n}-\ln p_n{}^{p_{n+1}}\right) \Leftrightarrow$$

$$\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)<\alpha\left(p_n\ln p_{n+1}-p_{n+1}\ln p_n\right) \Leftrightarrow \frac{\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)}{p_{n+1}\ln p_n-p_n\ln p_{n+1}}<-\alpha \Leftrightarrow$$

$$\frac{\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)}{\sqrt{p_{n+1}}^2\ln\sqrt{p_n}-\sqrt{p_n}^2\ln\sqrt{p_{n+1}}}<-2\alpha .$$

Let now the function:

$$g:(a,\infty)\to\mathbf{R},\ g(x)=\frac{\ln(x-a)}{x^2\ln a-a^2\ln x} \text{ with } a>2$$

We have now: $g'(x)=\dfrac{\frac{1}{x-a}\left(x^2\ln a-a^2\ln x\right)-\ln(x-a)\left(2x\ln a-\frac{a^2}{x}\right)}{\left(x^2\ln a-a^2\ln x\right)^2}=$

$$\frac{x^3\ln a-a^2x\ln x-2x^2(x-a)\ln a\ln(x-a)+a^2(x-a)\ln(x-a)}{x(x-a)\left(x^2\ln a-a^2\ln x\right)^2}$$

Because the denominator of g' is positive, we must inquire into the character of the function:

$h:(a,\infty)\to\mathbf{R},\ h(x)=x^3\ln a-a^2x\ln x-2x^2(x-a)\ln a\ln(x-a)+a^2(x-a)\ln(x-a)$.

Computing the derivative of h:

$$h'(x)=x^2\ln a-a^2\ln x-6x^2\ln a\ln(x-a)+4ax\ln a\ln(x-a)+a^2\ln(x-a)$$

Let now:

$y:(a,\infty)\to\mathbf{R}$,
$y(x)=x^2\ln a-a^2\ln x-6x^2\ln a\ln(x-a)+4ax\ln a\ln(x-a)+a^2\ln(x-a)$

and the derivative:

$$y'(x)=\frac{a^3-2x(2(x-a)(3x-a)\ln(x-a)+x(2x-a))\ln a}{x(x-a)}$$

Let now the function (the numerator of y):

$$z(x)=2x(2(x-a)(3x-a)\ln(x-a)+x(2x-a))\ln a$$

and, also, the derivative:

$$z'(x)=4\ln a\left(\left(9x^2-8ax+a^2\right)\ln(x-a)+2x(3x-a)\right)$$

Because x>a we have that $9x^2-8ax+a^2$>0 therefore z is a strictly increasing function.

But $\lim\limits_{x\to a} z(x)=2a^3\ln a$>0 therefore $z(x)$>0 ∀x>a.

In this case $y'(x)$>0 then y is also a strictly increasing function.

But $y(a+1)=(a+1)^2\ln a-a^2\ln(a+1)=a^2(a+1)^2\left(\frac{\ln a}{a^2}-\frac{\ln(a+1)}{(a+1)^2}\right)$.

The function $u(x)=\frac{\ln x}{x^2}$ has $u'(x)=\frac{x(1-2\ln x)}{x^4}$<0 for x>2 therefore u is decreasing and $\frac{\ln a}{a^2}-\frac{\ln(a+1)}{(a+1)^2}=u(a)-u(a+1)$>0.

We have now: $y(a+1)$>0 therefore $y(x)$>0 for x≥a+1.

Now $h'(x)$>0 which give us: h is increasing.

But $h(a+1)=a^2(a+1)^3\left(\frac{\ln a}{a^2}-\frac{\ln(a+1)}{(a+1)^2}\right)$>0 therefore $h(x)$>0 for x≥a+1.

Because now: $g'(x)$>0 implies that g is increasing and with $g(a+1)=0$ we find that g(x)>0 ∀x≥a+1.

From hypothesis 1 (Andrica's conjecture), we have: $\sqrt{p_{n+1}}-\sqrt{p_n}<1$ and noting: x=$\sqrt{p_{n+1}}$, a=$\sqrt{p_n}$ we have: x∈(a,a+1) therefore: $g(a)<g(x)<g(a+1)$ ⇔ $-\infty<\frac{\ln(x-a)}{x^2\ln a-a^2\ln x}<0$

Considering $\alpha=-\frac{1}{2}\sup\limits_{n}\frac{\ln\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)}{\sqrt{p_{n+1}}^2\ln\sqrt{p_n}-\sqrt{p_n}^2\ln\sqrt{p_{n+1}}}$≥0, the statement 2 is now obvious. **Q.E.D.**

3. Determination of the Constant α

Using the Wolfram Mathematica software, in order to determine the constant α (for the first 100000 prime numbers):

```
Clear["Global`*"];

numberiterations=100000;

minimum=1000;

k=0;

For[i=5,i<=numberiterations+4, i++,

difference=Sqrt[Prime[i+1]]-Sqrt[Prime[i]];

ratio=Log[Prime[i+1]]*Prime[i]-Log[Prime[i]]*Prime[i+1];

log=Log[difference]/ratio;

If[log<minimum,minimum=log]];

Print["Minimum=",N[minimum,1000]]
```

we found that the first 1000 decimals are:

α=0.001801787909180184090558881990879581852587815188626060829671181
9955181532280561858686616697228936379299051501383617413579875982175
2091249295800013427110224829129144010021192138295961103096235204621
3123107738700539021075748371514085755924571808071605072827284127643
7791095986635315223741002438617978237774820283643801709366814693751
8912461159503870105474089983531085085848126455516563425219916062338
0073272834451080219681979931918287609129486097360176969992548676629
7165720675277209011231194017976273680037341348819649636432410477964
8565485891418710372057051040019372330003785972735147995156530662746
8352075884099806617621474175589423220844469527382500914548671086635
2855099595409905960655726754630444411516619929414751645003809279755
6083075050236745852893416792192554737426491512157470711462277386533
5533852158934313781909407119398388818028233946073756228798804604974
6231538931008572428480523706827673078186615016687046047567231467115
6202235326608197057885854306504554998969783919670582435022650733176
2.

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