Axiomatic Analysis of the Semi-Fuzzy Poverty Indices MI$_f$ and PG$_f$

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Abstract: Every poverty index can be classified into one of the two major classes; classical indices and fuzzy indices; except for the semi-fuzzy poverty indices such as PGf and MIf which hybridize between the theory of classical sets and that of fuzzy sets, which makes their axiomatic analysis very special since it uses both classical and fuzzy mathematical tools. In order to better exploit and characterize the PGf and MIf indices, we propose in this paper an axiomatic analysis by mathematically demonstrating, on the one hand, the satisfaction of these two indices of a set of axioms most desirable by economists, which shows their performance in describing poverty. On the other hand, we discuss their limits according to three axioms that we demonstrate in order to improve the formula of these semi-fuzzy indices of poverty.

Keywords: Poverty measure; fuzzy set theory; confidence intervals; semi fuzzy poverty indices PG_f and MI_f ; axiomatic analysis

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1. The Poverty Measure: A Scientific Challenge to the Development of More Effective Measures

Poverty is a socio-economic phenomenon faced by all nations of the world, starting from the marginalization and social exclusion in developed countries and arriving at hunger and death in very poor countries. It is a plague causing itself other terrible problems such as crime, prostitution, selling drugs, migration, terrorism, which aggravate increasingly health status, levels of economy, social, education and therefore deepen the poverty of these populations. It is a circle continuously extended to include more and more poor and worsens worse and worst living conditions.

Thus, the fight against poverty is a priority for all countries of the world, seen that the poverty of undeveloped countries has consequences that reach even indirectly developed countries, migration, the spread of disease and deadly viruses, terrorism,

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Or fight against poverty requires the implementation of a set of policies to improve the living standards of the poor, what remains a difficult task if we do not determine up front the real need of this target population.

For this, the researchers company also has contributed to the fight against poverty for many years, by developing several poverty indexes as quantitative analytical instruments that reflect the reality accordance with conditions of the poor, to optimize time and resources invested and establish best results.

2. Evolution of Poverty Indexes: From Basic Indices to Multidimensional and Complex Indices

The first of poverty indices that have been proposed is the Headcount ratio, denoted H, which represents the proportion of poor compared to the total population (Notes techniques, 2002), then the index Income gap ratio, denoted I, which is defined as the mean distance separating the poor from the poverty line (Notes techniques, 2002). These two indexes are the simplest and easiest to evaluate, and also remain the most used by several governments and international organizations as first poverty assessment tools of a given population. But after formalizing the study of aggregation of poverty by economists, several criticisms of both indexes were evaluated (Sen, 1976). By following, several indices and poverty measures have been proposed that we can assign them into two classes, the first is classic and the second is fuzzy.

- Class of classical approaches:

These are all based on the following hypothesis:

"it is possible to delimit poverty and thus to identify the poor by determining a poverty line" (Deaton, 2005; Hagenaars, 1986; Meyer & Sullivan, 2003; McKinnish, 2005).

Using the classical mathematics logic, the concept of these approaches is to declare that a person is poor compared to an attribute if the realization of this attribute is below a fixed threshold, said *line* or *poverty threshold*. Mathematically this is reflected by the definition of a deprivation function $\varphi\left(x_{ij}, z_j\right)$, (Delhausse, 2002, p. 55) (Bertin, 2007) such as:

$$\varphi\left(x_{ij}, z_j\right) = \begin{cases} 1 \ \ si\ x_{ij} \geq z_j \ \rightarrow \ non\ privation \\ 0 \ \ si\ x_{ij} < z_j \ \rightarrow \ \ \ privation \end{cases}$$

Where x_{ij} is the level of functioning carried out by the individual *i* for the attribute *j*, and z_j is the deprivation threshold for the attribute *j*.

As an example, there are several indices such as index H and I cited above, as well as Sen index, Thon index, FGT index, Clark, Hemming et Ulph index, Kakwani index which is among the generalized poverty indicators, since it is a generalization of the FGT, Sen, Tsui indexes, the human poverty index IPH… and the list is still open to new indexes more performing.

- Class of fuzzy approaches:

This class of measures refuses hypothesis seen above that there is not a threshold or line of poverty unanimously adapted by the various classical approaches, also it is difficult to accept that the passage of a state of poor to non-poor is brutal, because of some differences milimes in income for example. Thus, a fuzzy approach models poverty as a state of an individual who has a depth (level of poverty) and not a characteristic that an individual has.

Fuzzy approaches include fuzzy mathematical logic, or the fuzzy sets theory, to address these deficiencies cited in the first class of approach. Indeed, it consists in the adaptation of a membership function μ such that:

$$\mu_B\big(X_j(a_i)\big) = \begin{cases} 1 \quad \textit{does not possess the attribute } j \ (a_i \in B \textit{ certainly}) \\ x_{ij} \,; 0 < x_{ij} < 1 \ \textit{possesses partially the attribute } j \ (a_i \in B \textit{ partially}) \\ 0 \ \textit{possesses the attribute } j \ (a_i \textit{ non poor} : \ a_i \notin B \textit{ certainly}) \end{cases}$$

- $X_j(a_i)$ represents the realization of a poor individual a_i in terms of the attribute j (or also the indicator j).

In other words, the value of the membership function μ to the fuzzy subset B of the i^th individual (i = 1,2, ... n) relative to the j^th attribute (j = 1,2, ... m) is defined as next:

$$x_{ij} = \mu\left(X_j(a_i)\right) \ ; \ 0 \le x_{ij} \le 1$$

Where:

- $x_{ij} = 1$ if the i^{th} individual does not possess the j^{th} attribute;
- $x_{ij} = 0$ if the i^{th} individual possess the j^{th} attribute;
- $0 < x_{ij} < 1$ if the i^{th} individual possess the j^{th} attribute with an intensity between 0 and 1.

In this context, several indices have been developed such that the index of Cerioli and Zani 90 followed by Cheli and Lemmi 95, Belhadj B. in 2005 and the list of these indices is still more enriched by new ones.

As part of the two approaches of poverty, the indexes have evolved in the growing sense of performance and credibility of indexes. In fact, the construction of these indexes has passed through two main phases that have contributed to this development:

- The first phase: it was designed to provide a picture of the proportion or distribution of the poor compared to the overall studied population through global indexes (indices H and I).

- The second phase: through reproaches and critical analysis of the imperfections of the existing indexes, we could make improvements and modifications to some of these indexes to exceed their deficiencies. This prompted the researchers to establish axiomatic approaches, each of which rests on one or more axioms that we find essential in a poverty index. These axioms will be subsequently as standards for the qualification or not of a poverty index. Thereby we continue to construct a general axiomatic framework of poverty indexes that does not cease to include new axioms until now.

3. Axioms: A Means of Characterizing Poverty Indexes

The axiomatic approach was first founded by Sen. Indeed, to construct his measure, Sen proposes to satisfy a set of ethical and moral principles characterizing the population of the poor, that he translated into axioms that a good index must satisfy (Sen, 1976). Then, several researchers have adapted the same principle to construct more efficient indices, introducing new axioms, thus good indicators satisfy most of axioms and especially those most desirable by economists.

Among all the axioms that a poverty index must satisfy, we find the following list, the two first are those proposed by Sen:

- ***Monotony axiom:*** All things being equal, a reduction in the income of a person who is below the poverty line should increase the poverty measure.

This axiom has been created on the basis of a critique of the H index that does not satisfy this axiom despite its obviousness.

- ***Transfer axiom:*** All things being equal, a transfer of income between a person who is below poverty line and someone who is richer must increase the poverty measure.

- ***Axiom of continuity***: the poverty measure should not be very sensitive to a marginal variation of the quantity of an attribute.

- ***Symmetry axiom or anonymity:*** it characteristics other than the attributes used to define poverty does not affect the measurement of poverty.

- ***Transfer sensitivity axiom:*** All things being equal, a regressive transfer of an amount *w* of the i^{th} to the j^{th} poor cause a greater increase in the poverty measure than a regressive transfer of the same amount from the k^{th} to the l^{th} poor if:

$$y_j - y_i = y_l - y_k > 0 \quad and \quad y_k > y_l$$

Such as y_i is the income of individual i.

This axiom established that aggregate poverty increases with a regressive transfer, and that more people involved in this transfer are poorer, more increasing the poverty level will be high. It therefore gives greater importance to transfers made between the poorest people.

- ***Decomposition axiom:*** Let be a population consisting of m groups, each group containing n_j individuals $\left(\mathrm{j} = 1,2.., \mathrm{m} \text{ and} \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{n_j} = \mathrm{n} \right)$

If we note *P* aggregate measure poverty calculated on the entire population and P_j which is calculated on the j^{th} group, then:

$$P = \sum_{j=1}^{m} \frac{n_j}{n} P_j$$

In other words, the aggregate poverty of the entire population is a sum of the aggregate poverty for all groups weighted by the share of each group $\left(\frac{n_j}{n}\right)$ in the total population.

The impact of poverty's variation of a group on total poverty increases with the number of persons forming this group.

- ***Axiom of the population's principle:*** If an attribute matrix is replicated several times, then overall poverty remains unchanged.

- ***Axiom of the invariance to the scale's variations***: The poverty measure is homogeneous with a degree 0 with respect to X and Z, where Z is the threshold vector.

- ***Axiom of concentration:*** The poverty measure is unchanged if an attribute j increases for an individual i characterized by $x_{i,j} \geq Z_j$. ($x_{i,j}$ is the value of attribute j for individual i).

- ***Axiom of monotonicity:*** The measure of poverty decreases, or does not increase following an improvement in one of the attributes of a poor.

In the following of this work, we recall first the semi fuzzy index and semi fuzzy vector of poverty, and then we present an axiomatic analysis showing the advantages and limitations of these indices.

4. Semi Fuzzy Index PG_f and Semi Fuzzy Vector MI_f of Poverty

We recall in this section the construction and the general formula of PG_f and MI_f semi fuzzy indices. To do this:

Let μ_B be a membership function chosen by the decision maker to integrate different criteria, that he finds necessary to measure poverty in a given population Ω.

Let Y_{qf} the total income of all the poor in population determined by the membership function μ_B, where:

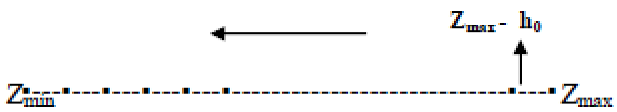
$$Y_{qf} = \sum_{i=1}^{qf} y_i \quad : \quad y_i \textit{ is the income of the individual } i$$

With:

$$q_f = Cardinal(B) \quad where \quad B = \{i \in \Omega : \mu_B(i) > 0\}$$

Let [Zmin, Zmax] a confidence interval (Belhadj & Matoussi, 2007), as Z_{min} is the minimum value that is desired to take the poverty line, and Z_{max} is its maximum value (Ravallion, 1994; Ravallion, 2003).

Consider $n \in \mathbb{N}^*$ the order of the discretization of the confidence interval [Zmin, Zmax], and $(h_1, h_2, h_3, h_4, \ldots, h_n) \in IR_+^{n*}$ steps of this discretization. These steps h_i express the differences that the expert considers reasonable, to measure income degradation, as is known to the evaluation and devaluation of wages.



A first step in our index construction process consists in a Euclidean division of Yqf by Zmax, which gives us:

$$Y_{qf} = a_0(Z_{max} - h_0) + r_0 \quad where \ 0 \le r_0 < Z_{max}$$

If $Z_{max} - h_0 < r_0$, we still perform the following division:

$$r_0 = a_1(Z_{max} - h_0) + r_1 \quad where \quad 0 \le r_1 < Z_{max} - h_0$$

Furthermore, if $Z_{max} - h_1 < r_1$, we can write:

$$r_1 = a_2(Z_{max} - h_1) + r_2 \quad where \quad 0 \le r_2 < Z_{max} - h_1$$

If $Z_{max} - h_m < r_m$, we can write:

$r_{m-2} = a_{m-1}(Z_{max} - h_{m-2}) + r_{m-1}$

Until last division we can perform if $Z_{min} < r_{m-1}$

$r_{m-1} = a_m Z_{min} + r_m$

From the first division, we have a_0 persons supposed to live with an income Zmax. Similarly, according to the second Euclidean division, there is a_1 persons assumed to have an income ($Z\max - h0$), so on until the last equality that explains the existence of a_m 9 persons supposed to live on an income Zmin, the rest of the population of q_f poor is $\left(q_f - (a_0 + a_1 + .. + a_m)\right)$ persons supposed to live with an income near to zero, and we note that the set B^*. So we get the construction of $m + 1$ subpopulations of poor forming a disjointed recovery of the poor population B, where each requires special treatment. Consequently, the class B of the poor is decomposed into disjoint union of the following sets:

$$B = \dot{\bigcup_{i=0}^{m}} B_{ai} \dot{\bigcup} B^*$$

The choice of steps and the order of the discretization depends on the extent of the interval $[Zmin, Zmax]$ selected at the beginning, as it also depends on the description and the meaning associated with each terminal Zi such that:

$$Z_i = Z_{max} - h_i \quad : \quad i \in \{1,2,3, \dots m\}$$

If we choose a fixed discretization's step:

$$h_i = i.h \quad where \quad i \in \{1,2,3, \dots m\}$$

Classes will be equidistant, but with different cardinals according to data from the studied population. Therefore, we obtain a vector MI_f defined by:

$$MI_f = \begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_m \end{pmatrix}$$

Where each component Ij ($j = 1,2 \dots m$) is determined by:

$$I_j = \frac{q_f - \sum_{k=1}^{j} a_k}{n}$$

With a_k ($\mathrm{k} = 1, \dots \mathrm{j}$) the values obtained by the above process.

Note that $PG_f = I_m$ is the last component of the vector MI_f.

By construction, indices Ij $(j = 1,2 \dots m)$ are decreasing in the sense that we pass from the calculation of Ij to I_{j+1} by:

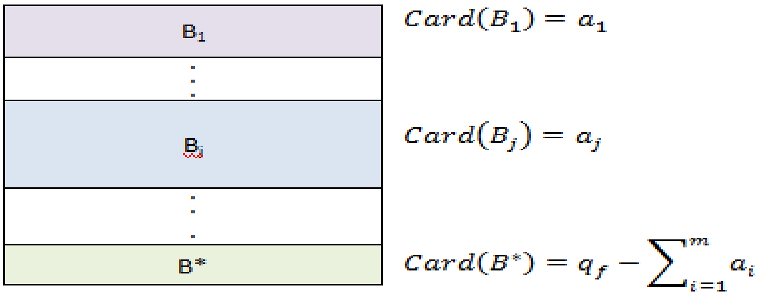
$$I_j - I_{j+1} = \frac{a_{j+1}}{n} \geq 0$$

which represents the weight of the $(j+1)^{th}$ set B_{j+1}relative to the entire population, thus, we have built a system of weights giving the thickness of each subset of poor.

The last class B^* is a particular class as it represents the misery in the studied society, characterized by:

$$PG_f = \frac{q_f - \sum_{i=1}^{m} a_i}{n}$$

This index reflects the weight of people living misery in the studied population.



Poverty classes of a population Ω

An example of a case of four classes is detailed in (Fikri, El Hilali Alaoui & El Khomssi, 2012).

5. Axiomatic Analysis of Semi Fuzzy Index and Semi Fuzzy Vector of Poverty

The introduction of axioms allowed to characterize poverty indicators through the validation of properties clearly explained. Indeed, this approach represents an indicator verification tool for a number of social and economic properties of the poor. Thus, the more an indicator verifies more axioms, the more this indicator is reliable. Consequently, researchers tend to build new indicators based on the maximum satisfaction of regarded axioms.

In this section, we will demonstrate the validation of a set of axioms by the semi fuzzy vector MI_f, by restricting demonstrations to four classes, because the general case is a simple extension of the case of four classes.

In a first axiomatic analysis of the indices “PG_f” and vector “MI_f” semi fuzzy of poverty has allowed us to confirm the satisfaction of the following axioms:

1) **Focus axiom**; (Fikri, El Khomssi & Saoud, 2011)

2) **Axiom of monotony**; (Fikri, El Khomssi & Saoud, 2011)

3) **Transfer axiom**; (Fikri, El Khomssi & Saoud, 2011)

In our following axiomatic analysis, we consider the following data:

Ω is a study population, containing n individuals.

We consider that an individual *i* has an income noted $xi \in D$ such as $1 \leq x2 \leq \cdots \leq xn$; and D the set of values that can take the income, with $D \subset IR^+$.

Income distribution of all individuals is denoted $x = (x1, x2, \ldots, xn)$.

In the following, we consider μ_p the membership function selected[1] and defined on Ω, and **B** the subset of poor defined by:

$B = \{ i \in \Omega : \mu_p(i) > 0 \}$

B is also said support of the membership function μ_p.

We note $\boldsymbol{q_f}$ the number of poor in the distribution x (also of **Ω**) such as:

q_f = Cardinal(B)

Let $Y_{qf} = \sum_{i \in B} x_i$ total of poor incomes **Ω**,

and $Z_B = (Z_{max} + Z_{min})/2$ with $[Z_{max}; Z_{min}]$ confidence interval considered.

We note [m] the whole part of positive real m.

Let a, b and c natural integers, and r1,r2 and r3 in $\mathbf{IR}^+$ such as:

$$Y_{qf} = a.Z_{max} + r1 \; with \;\; 0 \leq r1 < Z_{max} \; (1)$$
$$r1 = b.Z_B + r2 \; with \;\; 0 \leq r2 < Z_B \quad (2)$$
$$r2 = c.Z_{min} + r3 \; with \;\; 0 \leq r3 < Z_{min} \; (3)$$

4) **Axiom of symmetry**: permutation between the incomes of two individuals does not influence the measurement of poverty.

Indeed, given a distribution $x = (x1, x2, \ldots, xn)$ income of all individuals. Permutation between two elements of x does not impact the values of the

[1] For the choice of the membership function specialists can make their choice according to the dimensions they want to integrate (income, illiteracy, wellness ...).

membership function considered[1] , seeing that this one depends on the values *xi* and not their round.

Consequently, the value of $Y_{qf} = \sum_{i \in B} x_i$ it does not change, and also the values of the components of the vector MI_f and PG_f indices do not change.

5) **Axiom of homogeneity**: a multiplication by a positive constant for all incomes of x and for the poverty line z, does not impact $P(z; x)$.

To justify this property, we consider a distribution $x = (x_1, x_2, . ., , x_n)$ of the population Ω.

Let *x'* be the distribution obtained by multiplying the elements of *x* by a positive number *k* nonzero.

The same for the confidence interval $[Z_{min}, Z_{max}]$ substituted by the interval $[kZ_{min}, kZ_{max}]$.

Before verifying the sensitivity of our semi-fuzzy indexes to the multiplication, we note that the classification in poor and non-poor with the first distribution is the same for the second distribution. Indeed, we are left with two possibilities:

- **First case**, if the membership function is not based in its formula only on income, then the problem is simple because the degree of membership of x_i in the interval$[Z_{min}, Z_{max}]$, is the same as that of membership of kx_i to $[kZ_{min}, kZ_{max}]$ seeing that all the function values are included between 0 and 1. For example, Belhadj in (Belhadj, 2005) proposed the following membership function based on x_i the income or expenses of the i^{th} household as a dimension of poverty:

$$\mu_Q(i) = \begin{cases} 1 & if \quad 0 < x_i < Z_{imin} \\ \dfrac{-4}{2Z_{imax} - Z_{imin}} x_i + \dfrac{4Z_{imax}}{2Z_{imax} - Z_{imin}} & if \quad Z_{imin} \le x_i < Z_{imax} \\ 0 & if \quad x_i \ge Z_{imax} \end{cases}$$

By multiplying all the elements of the distribution *x* with a positive *k* and considering the interval $[kZ_{min}, kZ_{max}]$ we will have:

a) If $0 < kx_i < kZ_{imin}$ $then$ $0 < x_i < Z_{imin}$ $thus$ $\mu_Q(i) = 1$;

b) If $kZ_{imin} \le kx_i < kZ_{imax}$ then $Z_{imin} \le x_i \le Z_{imax}$ therefore;

[1] The choice is free for the membership function.

$$\mu_Q(i) = \frac{-4}{2kZ_{imax} - kZ_{imin}} kx_i + \frac{4kZ_{imax}}{2kZ_{imax} - kZ_{imin}}$$
$$= \frac{-4}{2Z_{imax} - Z_{imin}} x_i + \frac{4Z_{imax}}{2Z_{imax} - Z_{imin}}$$

c) If $kx_i \geq kZ_{imax}$ then, $x_i \geq Z_{imax}$ which implies $\mu_Q(i) = 0$.

Thus $\mu_Q(i)$ the degree of membership of an individual *i* in the sub-population of the poor remains unchanged if we multiply the income of all individuals and the thresholds of the confidence interval by the same positive number.

- Second case, when the selected membership function includes several attributes when calculating the degree of membership (Multidimensional Poverty), such as income, health, education. In this case if there is a scale that allows the homogenization of new incomes with other dimensions, then the fuzzy set of the poor does not change. If not, this set of poor can be changed according to the weight of each of the dimensions considered in the formula of the membership function.

In cases where the sub fuzzy B of the poor remains invariant with respect to the new distribution *x'*, the calculation of our semi fuzzy indices for this new distribution gives:

Total income of the poor is $Y'_{qf} = \sum_{i \in B} x'_i$

That is to say: $Y'_{qf} = \sum_{i \in B} kx_i = k.\sum_{i \in B} x_i = k.Y_{qf}$

Thus: $Y'_{qf} = k.Y_{qf}$

Subsequently equations (1), (2) and (3) obtained for the distribution *x* become for the new distribution *x'* as follows:

$$Y'_{qf} = k.Y_{qf}$$
$$Y'_{qf} = k.(a.Z_{max} + r_1)$$
$$Y'_{qf} = a.(kZ_{max}) + r'_1 \qquad (*)$$
$$with\, r'_1 = k.r_1 \;\; and \;\; 0 \leq r'_1 \leq kZ_{max}$$

Euclidean division of r'_1 by kZ_B gives:

$$r'_1 = k.r_1$$
$$r'_1 = k.(b.Z_B + r_2)$$
$$r'_1 = b.(k.Z_B) + r'_2 \quad (**)$$
$$with\, r'_2 = k.r_2 \;\; and \; 0 \leq r'_2 \leq kZ_B$$

A new euclidean division of r'_2 by kZ_{min} gives us:

$$r'_2 = k.r_2$$
$$r'_2 = k.(c.Z_{min} + r_3)$$
$$r'_2 = c.(k.Z_{min}) + r'_3 \quad (***)$$
$$with\ r'_3 = k.r_3 \quad and\ 0 \leq r'_3 \leq kZ_{min}$$

According to equations (*), (**) and (***), we remark that the results of the Euclidean divisions of the new values $Y'_{qf};\ r'_1\ and\ r'_2$ using the new values $kZ_{min}; kZ_B\ \ and\ kZ_{max}$ of the new corresponding confidence interval, are exactly « a », « b » and « c » the results of Euclidean division in equations (1) , (2) and (3) corresponding to the distribution *x*.

We therefore conclude that the components of the MI_f vectors and the PG_f index well respect the homogeneity property if the appropriate membership function considers income as a single attribute, where if the membership function measures multidimensional poverty with a formula invariant with respect to the multiplication of revenue by a positive non-zero.

6) **Axiom for Standardisation**: *A measure is "normalized" when it takes a special value to indicate that there is no poverty.*

Generally, it said that a measure is normalized when:

If no one live if no one lives below the poverty threshold for a given threshold z then the measure is null: P(x; z) = 0.

Indeed, in cases where all individuals in the population Ω are above Z_{max}, then the fuzzy set B is empty, as a result:

q_f=Card(B) = 0 and $Y_{qf} = \sum_{i=1}^{qf} x_i = 0$

Considering the equations (1), (2) and (3),

Since $Y_{qf} = 0$ then all the numbers a, b and c are zero.

Consequently: $I_1 = \frac{q-a}{n} = 0$ and similarly for I2, I3 and PG_f found that they are all null.

Thus all components of our vector MI_f and the semi-fuzzy index PG_f respect the normalization axiom.

Reciprocally: if the MI_f vector is null i.e. that I_1=I_2=I_3=0

From the expression of I_1 : If I_1=0 then $q = a$,

But « a » is defined as the number of poor people supposed to live with an income Z_{max}

In this case, we have « q » poor people supposed to live with an income Z_{max}

In other words, all the poor are supposed to live with an income Z_{max} ,

i.e $x_i \geq Z_{max} \; \forall \; i \; \in B$

Absurd. Hence, the set B of poor is empty.

Note: Since all the other I_2 , I_3 and PG_f indices are always lower than I_1 (Fikri, El Khomssi & Saoud, 2011), so just to have I_1=0 so that to such indices are zero.

6. Limits of the PG_f Index and the Semi-Fuzzy Vector MI_f

In the remainder of this section, we demonstrate a set of axioms not validate by the semi fuzzy indices PG_f and MI_f, this deficiency will be the first step towards improving the formulation of these two semi fuzzy indices.

6.1. Axiom of Independence

Consider two distributions *x and y* presenting the same poverty level in the sense of the indicator *P* for a given poverty line *z*. If the two distributions in question have a common part so the poverty level within the meaning of *P* for the threshold *z*, is equal to the distributions x and y without their common part.

The PG_f index and components of MI_f vector do not validate this axiom.

Indeed, let be:

Z_{max}=8 the poverty line

a distribution $x = (x_1, x_2, .\,.,\, x_{k-1}, x_k, x_{k+1}, \ldots. x_{k+l}, x_{k+l+1}, .\,.\,, x_n)$

a distribution $y = (y_1, y_2, .., y_{k-1}, x_k, x_{k+1}, \ldots. x_{k+l}, y_{k+l+1}, \ldots, y_n)$.

Such as :

$Y_x = \sum_{i \in B_x} x_i = 53 \quad and \;\; Y_y = \sum_{i \in B_y} y_i = 162$

With B_x (resp. B_y) is the fuzzy set of the poor of the distribution *x* (resp.*y*) whose Cardinal qx=10 (resp. qy=24).

Suppose E=(14 ; 6) is the common part between the two distributions x and y.

So :

$F = \sum_{i \in E} x_i = 20$ the total income of poor individuals belonging to the common part between the two distributions.

The total income of the poor distribution is: $Y_x = \sum_{i \in B_x} x_i = 53$

So we have the following calculation:

$$Y_x = 6 \times Z_{max} + 5$$
$$therefore \quad a_x = 6 \ \ et \, r_x^1 = 5$$

Hence

$$I_1^x = \frac{q_x - a_x}{n} = \frac{10 - 6}{n} = \frac{4}{n}$$

For distribution *y*, we proceed in the same way and we find:

$$I_1^y = \frac{q_y - a_y}{n} = \frac{24 - 20}{n} = \frac{4}{n}$$

Hence we have: $I_1^x = I_1^y$

However, let x*'* (resp. *y'*) the distribution obtained from *x* (resp. *y*) with extraction of the common part.

The new value of the total of incomes of the poor of the distribution *x'* is :

$$Y_{x'} = \sum_{i \in B_{x'}} x_i = 53 - 20 = 33$$

Euclidean division by $Z_{max} = 8$ give:

$$Y_{x'} = 4 \times Z_{max} + 1$$
$$hence \quad a_{x'} = 4 \ \ and \ \ r_{x'}^1 = 1$$

Hence,

$$I_1^{x'} = \frac{q_{x'} - a_{x'}}{n - 2} = \frac{(10 - 2) - 4}{n - 2} = \frac{4}{n - 2}$$

Similarly for distribution *y'*, we find:

$$I_1^{y'} = \frac{q_{y'} - a_{y'}}{n - 2} = \frac{(24 - 2) - 17}{n - 2} = \frac{5}{n - 2}$$

Therefore : $I_1^{x'} \neq I_1^{y'}$

Through this against-example, we can conclude that the component I_1 of the vector MI_f does not respect the independence axiom.

Similarly we can prove that the other components as well as the index PG_f does not meet this axiom.

6.2. Invariance Axiom by Replication

An index of poverty P respects this axiom if:

Given a distribution $x=(x_1,x_2,\ldots,x_n)$, For any replication y having an order k of x (i.e $y = \underbrace{(x, x, x, \ldots\ldots x)}_{\text{k times}}$ with $k \in IN^* - \{1\}$ and for a fixed threshold z we have : P(x,z)=P(y,z).

Considering the hypotheses of the axiom, and by noting:

$Y_y = \sum_{i\in B'} x_i$ the total of incomes of the poor of the distribution y.

Euclidean division of this number by the threshold $\mathrm{Z_{max}}$ gives:

$$Y_y = \sum_{i\in B'} x_i = \sum_{i\in B} x_i + \sum_{i\in B} x_i \ldots\ldots \sum_{i\in B} x_i \quad (\,k\ times)$$

Hence the following calculation :

$$Y_y = k.\sum_{i\in B} x_i = k.Y_x$$

$$Y_y = k.(a.Z_{max} + r_1) \quad with\, r_1 < Z_{max}$$

$$Y_y = (k.a).Z_{max} + k.r_1 \quad with\, r_1 < Z_{max}$$

Since $r_1 < Z_{max}$ and $k \in IN^* - \{1\}$ so there are two possible cases:

$k.r_1 < Z_{max} \quad else \quad kr_1 \geq Z_{max}$

In the case where $kr_1 \geq Z_{max}$ we can write:

$$kr_1 = \alpha.Z_{max} + \beta \quad with\, \alpha \in IN^*\, and\, 0 \leq \beta < Z_{max}$$

Therefore

$$Y_y = (k.a + \alpha).Z_{max} + \beta$$

Hence the expression of the first component of the vector MI_f is as following:

$$I_1^y = \frac{k.q - (k.a + \alpha)}{n.k} \quad : \quad \alpha \in IN^* \; and\, k \in IN^* - \{1\}$$

But $I_1^x = \frac{q-a}{n}$

So $I_1^x \neq I_1^y$, and as a result, the vector MI_f does not respect the property of invariance by replication.

The same reasoning for the other components of $\mathrm{MI_f}$ and the $\mathrm{PG_f}$ index.

6.3. Axiom of Decomposability

Let n(x) the number of individuals in the distribution x, and z a poverty line

Given a distribution $x=(x',x'')$ such as $n(x)=n(x')+n(x'')$.

A poverty measure *P* is called decomposable if and only if:

$$P(x,z) = \frac{n(x')}{n(x)} P(x',z) + \frac{n(x'')}{n(x)} P(x'',z)$$

In other words :

$$P(x,z) = \frac{1}{n(x)} \sum_{i=1}^{n} P(x_i, z) \; where \;\; P(x_i,z) = 0 \; for \; all \; x_i \; non \; poor$$

Proof:

Suppose that:

n is the number of individuals in the distribution *x*.

n' (resp.*n''*) is the number of individuals in the distribution *x'* (resp. *x''*).

q the number of poor in the distribution *x*,

q' (resp.*q''* the number of poor in the distribution *x'* (resp. *x''*).

so we have *n=n'+n''* and *q=q'+q''*

let's remember that [m] denotes the integer part of the real m.

Given $Y'_q = \sum_{\substack{i \in B \\ x_i \in x'}} x_i$ the total incomes of the poor in the sub-distribution x'.

$Y''_q = \sum_{\substack{i \in B \\ x_i \in x''}} x_i$ the total incomes of the poor in the sub-distribution x''

As a result, the total of incomes of the poor of the distribution *x* is given by

$$Y_q = \sum_{i \in B} x_i = Y'_q + Y''_q = \sum_{\substack{i \in B \\ x_i \in x'}} x_i + \sum_{\substack{i \in B \\ x_i \in x''}} x_i$$

By performing a Euclidean division of the previous totals Z_{max}, we find:

$$Y_q = a.Z_{max} + r$$

$$Y'_q = a'.Z_{max} + r'$$

$$Y''_q = a''.Z_{max} + r''$$

The first components of the vector MI_f corresponding to each of the distributions are given by:

$$I_1^x = \frac{1}{n}(q - a) = \frac{1}{n}\left(q - \left[\frac{Y_q}{Z_{max}}\right]\right)$$
$$I_1^{x'} = \frac{1}{n'}(q' - a') = \frac{1}{n'}\left(q' - \left[\frac{Y'_q}{Z_{max}}\right]\right)$$
$$I_1^{x''} = \frac{1}{n"}(q" - a'') = \frac{1}{n"}\left(q" - \left[\frac{Y''_q}{Z_{max}}\right]\right)$$

Hence:

$$\frac{n'}{n} I_1^{x'} + \frac{n''}{n} I_1^{x'} = \frac{1}{n}\left(q' + q" - \left(\left[\frac{Y'_q}{Z_{max}}\right] + \left[\frac{Y"_q}{Z_{max}}\right]\right)\right)$$
$$= \frac{1}{n}\left(q - \left(\left[\frac{Y'_q}{Z_{max}}\right] + \left[\frac{Y"_q}{Z_{max}}\right]\right)\right)$$

But for all positive real numbers α and β we have $[\alpha] + [\beta] \leq [\alpha + \beta]$ is a property of the integer part function. Therefore,

$$\frac{n'}{n} I_1^{x'} + \frac{n''}{n} I_1^{x'} \leq I_1^x$$

Hence the result.

7. Conclusion

The classification of the PG_f index and the MI_f vector as semi fuzzy poverty indices puts at the crossroads of traditional approaches and fuzzy approaches of poverty. In fact, they call on the one hand, tools of fuzzy logic (a membership function and a confidence interval...), and on the other hand, calculations from a classic cardinal of a set of poor. This positioning between classical and fuzzy made the axiomatic characterization and analysis of these two indices itself semi fuzzy, thus, the verification of a set of axioms is original in the sense that every axiomatic analyzes are either in the fuzzy frame, or in the classic but not in a frame combining the two.

In this article we demonstrated a set of axioms that the PG_f index and the MI_f vector semi fuzzy poverty validate, reflecting their relevance in describing poverty. We have also shown the limits of these two semi fuzzy measures through three axioms which do not satisfy in order to improve future writing these two measurements, or find conditions under which these semi fuzzy measures exceed their limits, and thus improving performance and relevance.

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