A Probability on Heuristic Subset of Integer Numbers that it is a Metric Space

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Abstract: In this work we will answer to the question, is there a probability space on a set such that it is a metric space? To answer this question, we prove that the probability that two random positive integers given from a heuristic set defined in this article are relatively prime, is a metric space. Hence, there is a probability that it is a metric. Also, we show that this probability space is a paracompact space.

Keywords: probability; metric space; paracompact space; random integers

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1 Introduction

Let Ω, $\mathcal{F}$ and P be the *sample space*, *collection of events* and *probability measure*, respectively. The triple $(\Omega, \mathcal{F}, P)$ is called the probability space and measure P satisfies the three *Kolmogorov axioms* (Gut, 2013); i.e.

I) For any $A \in \mathcal{F}$, there exist a number $P(A) \geq 0$, that this is the probability of A;

II) $P(\Omega) = 1$;

III) $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$, for every disjoint $\{A_n, n \geq 1\}$.

Now, we propose the question, is there a probability measure such that this measure be a *meteric* (the metric to be defined ahead)? In number theory, two integers x and y that they share no common positive factors except 1, are relatively prime. In 1970, S. W, Golomb (1970) studied a class of probability distributions on integer numbers , that under his work, in 1972, J. E. Nymann proved that if ζ be the Riemann ζ-function, then probability that k random positive integers, are relatively prime is equal to $\frac{1}{\zeta(k)}$ (Nymann, 1972). In this paper we will prove that

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the probability that two random positive integers are relatively prime, is a metric space. To prove the main theorem, the following notation will be used:

1- $gcd(x, y)$: greatest common factor of positive integers x and y.

2- $gcd(x, y) = 1$: two positive integers x and y , are relatively prime.

3- $Pr[gcd(x, y) = 1]$: probability that two random positive integers x and y, are relatively prime.

4- $\mathbb{R}$: real numbers.

5- $\mathbb{Z}$: integer numbers.

2. The Main Definitions and Lemmas

In mathematics a *metric space* is defined as follows (O'searcoid, 2006):

Definition 1 (metric space). *Let M be a set and $d: M \times M \to \mathbb{R}$ is a real function. Then (M, d) is a metric space for any $a, b, c \in M$, if the following conditions holds*

1- $d(a, b) \geq 0$;

2- $d(a, b) = 0 \Leftrightarrow a = b$;

3- $d(a, b) = d(b, a)$ (symmetry);

4- $d(a, c) \leq d(a, b) + d(b, c)$ (triangle inequality).

Hence, the function d is called a *metric* on the set M. In mathematics, there are many functions defined on their sets that with together constitute the metric spaces. For a nice study about metric space, see (Dress et al, 2001). Other related spaces are *paracompact spaces*. A topological space is called paracompact if it satisfies the condition that every *open cover* has a *locally finite open refinement* (Adhikari, 2016). For definitions of open cover, locally finite and open refinement, see (Adhikari, 2016). In 1968, M. E. Rudin presented a nice proof (Rudin, 1969) for the statement that every metric spaces are paracompact.

Note that, in this work, we will use the above definition for random variables x, y, z, which means that if M be a set and $x, y, z \in M$, then $d: M \times M \to \mathbb{R}$ is a probability function. Hence, in this article we assume that x, y, z are random integers greater than 1, and function d is the probability that two random positive integers, are relatively prime. Thus, for example if $x = y$ $(\forall x, y > 1)$, we know that $gcd(x, y) = gcd(x, x) = x \neq 1$ and $Pr[gcd(x, y) = 1] = 0$. Now, if $x \neq y$, then we assume that we don't know any details about common factors of x and y, since x and y are random integers, and hence by (Nymann, 1972), we have $Pr[gcd(x, y) = 1] = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$. The details mentioned are the foundation of our work and in the next definition we suggest a new heuristic subset of positive integer numbers that has the above properties.

Definition 2. *Let* $\mathcal{F}$ *be a subset of* $\mathbb{Z}$. *Then we say* $\mathcal{F}$ is a $\mathcal{F}-$set, if the following conditions holds

i) *If* $x \in \mathcal{F}$, *then* $x > 1$.

ii) *For every* $x, y \in \mathcal{F}$, *we have* $x = y$, *or* x *and* y *are relatively prime.*

Example 1. Let $M_1 = \{2, 3, 5, 2, 11\}$ and $M_2 = \{2, 3, 5, 2, 10, 11\}$. Then, M_1 is a $\mathcal{F}$-set and M_2 is not $\mathcal{F}$-set.

Example 2. Let P denotes the set of all prime numbers. We know that for every $p, q \in P$, we have $p, q > 1$ and p and q are relatively prime. Therefore, P is countably infinite $\mathcal{F}$- set.

Note that, if M be *an* $\mathcal{F}$-set, then for every $x, y \in M$ we have $x = y$ and $gcd(x, y) \neq 1$ or $x \neq y$ and $gcd(x, y) = 1$. Now, if $Pr[gcd(x, y) = 1]$ be the probability that two random positive integers x, y given from an arbitrary $\mathcal{F}$-set, are relatively prime, then we have

$$Pr[gcd(x, y) = 1] = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

To prove the main theorem we need the following lemmas:

Lemma 1. *Let* M *be an arbitrary* $\mathcal{F}$*-set. If* $x, y \in M$ *be two random positive integers, then* $Pr[gcd(x, y) = 1] = 0$ *if and only if* $x = y$.

Proof. We know that if $x = y$ then $gcd(x, y) = gcd(x, x) = x \neq 1$, for every positive integer $x > 1$, hence $Pr[gcd(x, x) = 1] = 0$. On the other hand, since M is an $\mathcal{F}$-set and $x, y \in M$, then if $Pr[gcd(x, y) = 1] = 0$, we have $x = y$.

Lemma 2. *Let* x, y *be two random positive integers given from an arbitrary* $\mathcal{F}$*-set. Then*

$$Pr[gcd(x, y) = 1] = Pr[gcd(y, x) = 1].$$

Proof. We know that $gcd(x, y) = gcd(y, x)$, the lemma is proved.

Lemma 3. *Let* x, y, z *be three random positive integers given from an arbitrary* $\mathcal{F}$*-set. Then*

$$Pr[gcd(x, z) = 1] \leq Pr[gcd(x, y) = 1] + Pr[gcd(y, z) = 1].$$

Proof. We know that x, y, z are three random positive integers given from an arbitrary $\mathcal{F}$-set. Hence, if $x \neq y$, then $Pr[gcd(x, y) = 1] = 1$, and also if $x = y$, then we have $Pr[gcd(x, y) = 1] = 0$. So, for all cases, we have

- Case 1:

$$\begin{cases} x \neq y \\ y \neq z \\ x \neq z \end{cases} \rightarrow \underbrace{Pr[gcd(x,z)=1]}_{1} < \underbrace{Pr[gcd(x,y)=1]}_{1} + \underbrace{Pr[gcd(y,z)=1]}_{1}$$

- Case 2:

$$\begin{cases} x \neq y \\ y = z \\ x \neq z \end{cases} \rightarrow \underbrace{Pr[gcd(x,z)=1]}_{1} = \underbrace{Pr[gcd(x,y)=1]}_{1} + \underbrace{Pr[gcd(y,z)=1]}_{0}$$

- Case 3:

$$\begin{cases} x = y \\ y \neq z \\ x \neq z \end{cases} \rightarrow \underbrace{Pr[gcd(x,z)=1]}_{1} = \underbrace{Pr[gcd(x,y)=1]}_{0} + \underbrace{Pr[gcd(y,z)=1]}_{1}$$

- Case 4:

$$\begin{cases} x = z \\ y \neq z \\ x \neq y \end{cases} \rightarrow \underbrace{Pr[gcd(x,z)=1]}_{0} < \underbrace{Pr[gcd(x,y)=1]}_{1} + \underbrace{Pr[gcd(y,z)=1]}_{1}$$

- Case 5:

$$\begin{cases} x = y \\ y = z \\ x = z \end{cases} \rightarrow \underbrace{Pr[gcd(x,z)=1]}_{0} = \underbrace{Pr[gcd(x,y)=1]}_{0} + \underbrace{Pr[gcd(y,z)=1]}_{0}$$

Hence, for every random positive integers x, y, z given from an arbitrary $\mathcal{F}$-set, we have

$$Pr[gcd(x,z)=1] \leq Pr[gcd(x,y)=1] + Pr[gcd(y,z)=1].$$

3. The Main Theorem

Theorem 1. *Let* $\mathbb{A}$ *be an* $\mathcal{F}$*-set and* $d: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{R}$. *If* $d = Pr[gcd(x,y)=1]$, *then* $(\mathbb{A}, d)$ *is a metric space.*

Proof. By the metric space definition, and using Lemma 1 (for condition 2) and Lemma 2 (for condition 3) and Lemma 3 (for condition 4), and since always $d = Pr[gcd(x,y)=1] \geq 0$ (for condition 1), the proof is complete.

Hence, we can answer to the question, is there a probability such that it is a metric?

Corollary 1. *There is at least a probability such that it is a metric.*

Since, every metric spaces are paracompact (Rudin, 1969), so we have the following corollary:

Corollary 2. *There is at least a probability space such that it is a paracompact space.*

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