Financial Institutions and Services

The Current Models of Credit Portfolio Management: A Comparative Theoretical Analysis

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Abstract: The main purpose of this paper is to examine theoretically the current models of credit portfolio management. There are currently three types of models to evaluate the risk of credit portfolio; the structural models (Moody's KMV model and CreditMetrics model) also defined as the models of the value of the firm, the actuarial models and the econometric models (the Macro-factors model). The development of these models is based on a theoretical analysis developed by several researchers. Then, the evaluation of the default frequencies and the size of the loan portfolio are defined by credit risk factors which are conditioned by macroeconomic and microeconomic circumstances. Also, we sundeexplain the different characteristics of these models. Additionally, the purpose of these models is to assess the default probability of credit portfolios.

Keywords: Risk management; Credit risk; Default probability; Structural models; KMV model; CreditRisk+; Credit Portfolio View

JEL Classification: G13; G21; G28

1. Introduction

The problem of evaluation of the failure probability of any borrower was the center of the bankers as soon as they began to lend some money. The quantitative modeling of the credit risk for a debtor is rather recent in fact. Besides, the modeling of the credit risk associated with instruments of a portfolio of credit such as, the loans, the pledges, the guarantees and the by-products (who constitute a recent concept).

Glasserman (2010) analyzes portfolio risk and volatility in the presence of constraints on portfolio rebalancing frequency. This investigation is motivated by the incremental risk charge (IRC) introduced by the Basel Committee on Banking Supervision. In contrast to the standard market risk measure based on a 10-day value-at-risk calculated at 99% confidence, the IRC considers more extreme losses and is measured over a 1-year horizon. More highly, whereas 10-day VaR is ordinarily calculated with a portfolio's holdings held fixed, the IRC assumes a portfolio is

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managed dynamically to a target level of risk, with constraints on rebalancing frequency. The IRC employs discrete rebalancing intervals (e.g., monthly or quarterly) as a rough measure of potential illiquidity in underlying assets. Glasserman (2010) analyzes the effect of these rebalancing intervals on the portfolio's profit and loss distribution over a risk-measurement horizon. Glasserman (2010) derives limiting results, as the rebalancing frequency increases, for the difference between discretely and continuously rebalanced portfolios; we employ these to approximate the loss distribution for the discretely rebalanced portfolio relative to the continuously rebalanced portfolio. This analysis leads to explicit measures of the impact of discrete rebalancing under a simple model of asset dynamics.

A certain number of models were developed, including at the same time the applications of property developed for the internal custom by the financial institutions, and the applications intended for the sale or for the distribution (Hickman & Koyluoglu, 1999).

The big financial institutions recognize his necessity, but there is a variety of approaches and rival methods. There are three types of models of credit portfolio in the course of use at present (Crouhy et al., 2000):

- The structural models: there are two models of management of credit portfolio who are supplied in the literature: Moody's KMV model (Portfolio Model) and CreditMetrics model by JPMorgan;
- The Macro-factors model (Econometric model): The Credit Portfolio View model introduces in 1998 by Mckinsey;
- The actuarial models CSFP (Credit Suisse First Boston): this model (CreditRisk+) is developed in 1997.

The main idea for this study is to answer the question follows: *Haw the default probability is defined by the credit portfolio models?*

Then, the organization of this paper is as follows. In section 2, we present the structural models and we define the forces and the weaknesses of each model. We provide the presentation of the econometric models in section 3. The section 4 is considered to present the development of the CreditRisk+ models and the final section is our conclusion.

2. The Structural Models

The structural models of management of credit portfolio were presented by Merton (1974) and then, developed by Leland (1994), Leland and Toft (1996), Anderson and

Sundaresan (1996) and Jarrow (2011). The characteristics to define a structural model are given by two conditions:

- The process of management of the assets of the company has to be known on the market in which this one operates;

- The structure of the liabilities of the company has to be known by all the actors operating on the market of this one.

In the practice, to examine the models of management of credit portfolio, it is necessary to use parameters estimated implicitly because the values of the assets of the company are not observable. Nevertheless, the majority of the empirical evidence does not retain the structural models. The implicit prices obtained from the structural models does not seem to match the structure of maturity of the efficiencies on the assets of the company (Eom et al., 2004; Ericsson & Reneby, 2005; Jarrow et al., 2003; Schaefer & Strebulaev, 2008; Li & Wong, 2008; Jarrow, 2011) and to allow the forecasts of defect of the borrowers (Patel & Pereira, 2007; Bharath & Shumway, 2008).

The analysis of the model of Merton (1974) shows that this one supposes that the value of the firm follows a process of distribution and the defect occurs when the value of the firm falls below the nominal value of the debt on the date of maturity of this one. In this respect, this model serves to determine a threshold of defect.

The development of Merton's model is made by adding the other variables such as; the interest rate (Longstaff & Schwartz, 1995), the optimal permanent capital (Leland & Toft, 1996), the variable time of the threshold of default (Collin-Dufresne & Goldstein, 2001), the unfinished accounting information (Duffie & Lando, 2001) and the risk of the events of defect (Driessen, 2005).

The structural models are based on the theory of the options and the structure of the capital of the company (Hamisultane, 2008). In this aligned, the bankruptcy of a company took place when the value of assets is situated below the value of its debt. The structural models or the models of the value of the firm are based on the approach of Merton (1974) which supposes that the failure of a company appears in case the market value of its assets is lower than a certain threshold of its debts.

Generally, the models of credit portfolio management resting on the approach of Merton are the model KMV (Kealhofer, McQuown & Vasicek) of Moody and the CreditMetrics model of JPMorgan (1997). The distinction between both structural models was described in the table below.

Table 1. The comparison between the KMV model and the CreditMetrics model

The KMV approach	The CreditMetrics approach
▪ The conduct of the value of the asset; ▪ Companies are decomposed into systematic components and that no-systematic; ▪ The systematic risk is based on the industry and the country of debtor; ▪ The correlation of defect ensues from the correlation of assets.	▪ The indication of own capital; ▪ Companies are decomposed systematic components and that no-systematic; ▪ The systematic risk is based on the industry and the country of debtor and can be sensitive to the size of the asset; ▪ The correlation of the defect ensues from the correlation of the efficiencies on own capital.

Source: Smithson (2003)

The structural models are also called models of the asset volatility. The Structural aspect of the models comes because there is a historical story behind by default that is something manages to start by default. The structural models are rooted in the knowledge of Merton. In Merton's model, the correlation of defect has to be a function of correlation of assets. The estimation of a structural model requires the implementation of the market value of the assets of the company and its volatility.

In the practice, the value of assets and their volatility are not observable for the most part of companies. The structural models lean strongly on the existence of assets quoted on the stock exchange so that we can estimate the necessary parameters.

2.1. The KMV Model

The KMV model of credit portfolio management was elaborated for the first time in 1993. This model allowed the development of several models of quantification of the credit risk: Credit Monitor, Credit Edge and Private Firm Model for the individual credit risk and Portfolio Manager for the credit risk of a portfolio.

The model KMV rests bases on the notion of default distance which is calculated by basing itself on the barrier which engages the defect. As soon as, the distance in the defect is calculated, it transformed into the probability of failure (Expected Default Frequency: EDF).

The KMV model which was developed by the Moody's-KMV company is based on the theory of the prices of Merton options. It is about an abstract frame used to estimate the default probability of a company. The KMV model supposes that the company is in situation of defect when the value of its asset is less than the value of its debts. The Figure 1 explains the relation between the estimated own capital and the value of the asset. According to Merton's basic idea, in the KMV model the value of the own capital of the company is considered as being an option to buy. So, the value of the asset is considered as being the underlying asset and the debt represents the price of exercise (Chen et al., 2010).

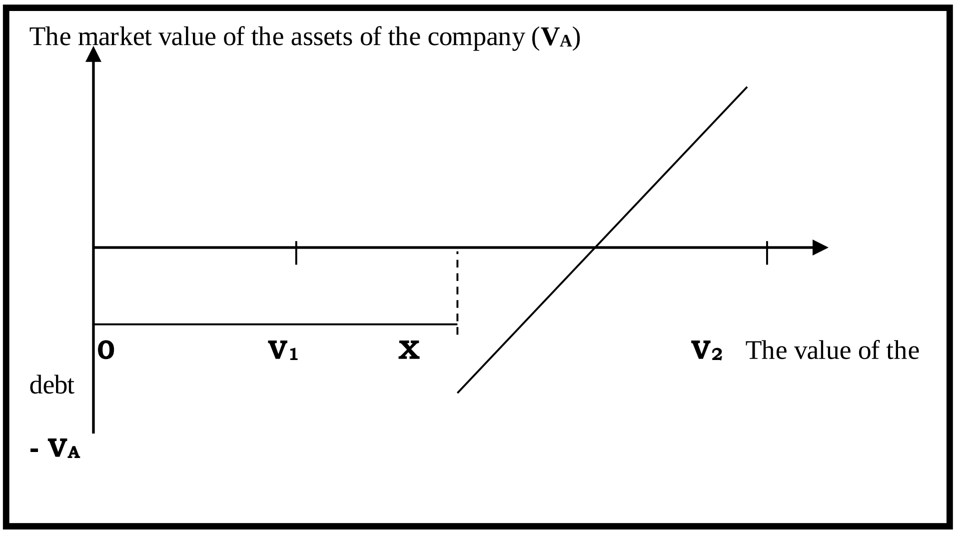

Figure 1. The relation between the market value of the assets of the company and the value of the debt (Merton, 1974)

In the Figure 1, $\mathbf{V_A}$ indicates the initial investment of the shareholders of the company; **X** indicates the point of default which corresponds to the sum of the long-term debt and half of the current liabilities. When the value of assets ($\mathbf{V_A}$) is superior to the debt (**X**), the shareholders will choose to gain profits staying after payment of the debts ($\mathbf{V_A - X}$) and these will be chosen by default, what is shaped with a net value raised in the Figure 1. In this case, the investor executes the option to buy.

So, if the value of assets is lower than the debt ($\mathbf{V_A < X}$), the shareholders will choose by default the transfer of the active total for the benefit of the creditors, what is coherent with a constant value of own capital indicated in the Figure 1, and it means that the option to buy is not executed (Caouandte et al., 1998; Kealhofer & Bohn, 2001; Saunders & Allen, 2002; Bohn & Crosbie, 2003).

Generally speaking, the shareholders receive Max ($\mathbf{V_A - X, 0}$) in the date of maturity **T**. According to Merton's model, the evolution of the market value of the assets of the company follows a process of geometrical distribution of the following shape:

$$\frac{dV_A}{V_A} = \mu\, dt + \sigma_A\, dW_t$$

Where, $W_t$the process of Wiener Standard is, μ is the average of the efficiency of assets and σ_A is the standard deviation of the efficiency on assets. The market value of the company is given by basing itself on the purchase price of a European option to buy supplies by Black and Scholes (1973).

$$V_E = V_A N(d_1) - e^{-rT} X N(d_2)$$

Where **N (.)** Indicate the function of distribution of the normal law with (Huang & Yu, 2010):

$$\begin{cases} d_1 = \dfrac{(ln\left(\dfrac{V_A}{X}\right) + \left(r + \dfrac{1}{2}\sigma_A^2\right)T)}{\sigma_A\sqrt{T}} = \dfrac{1}{\sigma_A\sqrt{T}}(ln\left(\dfrac{V_A}{X}\right) + \left(r + \dfrac{1}{2}\sigma_A^2\right)T) \\ d_2 = \dfrac{(ln\left(\dfrac{V_A}{X}\right) + \left(r - \dfrac{1}{2}\sigma_A^2\right)T)}{\sigma_A\sqrt{T}} = d_1 - \sigma_A\sqrt{T} \end{cases}$$

In the KMV model, there is a hypothesis which rests on the structure of the capital of the company. So, this capital has to consist only by actions, current liabilities and in the long term and convertible prices. Really, the value of the company V_A and the volatility of assets σ_A are not observable (Hull, 1997; Chen et al., 2010). We are going to deduct these two values by using the values of the options V_E.

So land us note that:

$$V_E = f(V_A, \sigma_A, X, c, r)$$

$$\sigma_E = g(V_A, \sigma_A, X, c, r)$$

Where, **c** is the coupon paid on the long-term debt, **r** is the interest rate without the risk and σ_E is the volatility of share prices.

By applying the Lemma of Itô to these two functions and by arranging the terms we obtain:

$$\sigma_E = (\frac{V_A}{V_E})\frac{\partial V_E}{\partial V_A}\sigma_A$$

With: $\dfrac{\partial V_E}{\partial V_A} = N(d_1)$ who is deducted from the equation which measures the value of the V_E which is defined by the following expression:

$$V_E = V_A N(d_1) - e^{-rT} XN(d_2)$$

Thus:

$$\sigma_E = \left(\frac{V_A N(d_1)}{V_E}\right)\sigma_A$$

Further to this transformation, we obtain a system of equation to two unknowns V_A andσ_A:

$$\begin{cases} V_A N(d_1) - e^{-rT} XN(d_2) - V_E = 0 \\ \sigma_E V_E - V_A N(d_1)\sigma_A = 0 \end{cases}$$

If the expressions of V_A and σ_A are determined, then we can arrive at the writing of the following formulation of the distance of defect (**DD**):

$$DD = \frac{\ln\left(\frac{V_A}{X}\right) + \left(\mu - \frac{1}{2}\sigma_A^2\right)}{\sigma_A\sqrt{T}}$$

According to the KMV model the distance of defect is defined in the following way (Crosbie & Bohn, 2003):

$$DD = \frac{V_A - X}{\sigma_A V_A}$$

From the distance of defect, we can deduct the value of the default probability as follows:

$$P_{KMV} = Prob\{V_A(T) < X\} = N\left(-\frac{ln\left(\frac{V_A}{X}\right) + \left(\mu - \frac{1}{2}\sigma_A^2\right)T}{\sigma_A\sqrt{T}}\right) = N(-DD)$$

Then we can obtain the frequency planned by default (Expected Default Frequency: EDF) such as:

$$EDF = N(-DD)$$

However, the default probability does not correspond to the normal law. KMV Company tries to obtain the empirical value of the EDF rather than the theoretical value of the models (Zheng, 2005). Fortunately, KMV Company possesses an enormous base of historical data concerning the default of the companies. By basing itself on these data KMV defined tables which associate with the various possible values of the distance of default (DD) on a temporal horizon considered a default probability definite and noticed empirically.

So, to protect itself against the risk which results from potential losses bound to the evolutions of the portfolio, Kealhofer, McQuown and Vasicek (1993) based on the determination of a random size **L** relative to the losses of the portfolio which is defined in a general way and on a horizon **H** as follows:

$$\boldsymbol{L} = \boldsymbol{V}_{\frac{H}{ND}} - \boldsymbol{V}_{\boldsymbol{H}}$$

Where $\boldsymbol{V}_{\frac{H}{ND}}$ indicates the value of the portfolio **H** in the absence of the losses and $\boldsymbol{V}_{\boldsymbol{H}}$ indicates the market value of the portfolio **H**. The development follows by KMV shows us that the distribution of **L** can be approached by an inverse normal distribution.

Table 2. The forces and the weaknesses relative to the KMV model

The forces	The weaknesses
▪ The default probability is connected with the information of the market. ▪ Contrary to CreditMetrics models and CreditRisk+ models the debtors are specific. We can distinguish them by basing itself on their default probability, on their own structure of capital and on their own assets. ▪ The threshold of defect is determined in an empirical way.	▪ A hypothesis which is not realistic because she supposes that the debt of the company consists by bonds with zero-coupon and shares. ▪ KMV supposes that the price of assets follows one moment Geometric Brownian. This modeling by a continuous process excludes all the early defaults. ▪ This method is difficult because it depend a several data which are in most of the time unobservable or with difficulty accessible. ▪ The interest rate is supposed constant.

Source: Hamisultane (2008)

2.2. The CreditMetrics Model

CreditMetrics was thrown for the first time in 1997 by JP Morgan's bank. CreditMetrics is considered as being an evaluation tool, for a portfolio, its variance of the values provoked by the changes of the quality of credit of the transmitter of the bonds (the credit migration) and leaves the defect of the counterpart. Unlike the approaches developed by the other models of management of a portfolio of credit, the probability of default in CreditMetrics is given by rating agencies for the big companies and by methods of scoring and mapping for small and medium-sized firms (Paleologo et al., 2010).

CreditMetrics belongs to the structural models since it rests on the model of Merton (1974) for the definition of the thresholds of the migration of credit. (Jarrow, 2011)

According to Hamisultane (2008), CreditMetrics makes it possible to calculate CreditVaR of a portfolio. The methodology of this model is based on the probability of moving of a quality of credit to the other in a given horizon of time (analysis of the migration of credit). The calculation of CreditVaR by CreditMetrics rests on the four stages following (Crouhy et al., 2000; Hamisultane, 2008):

- Determination of the risk isolated from each credit of the portfolio;
- The construction of the matrix of the probabilities of transition from a notation to another;
- The valuation of the assets of the portfolio according to the scenarios of transition from a notation to the other one;
- The calculation of CreditVaR.

The evaluation of a portfolio Value-at-Risk due to the credit (CreditVaR) by CreditMetrics is given the following Figure 2 (Crouhy et al., 2000):

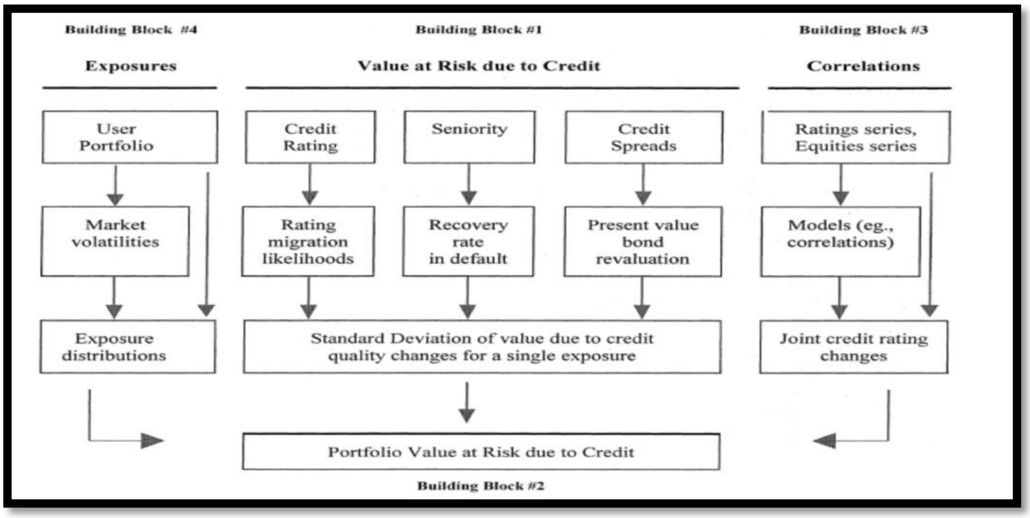

Figure 2. The evaluation of a portfolio

In the model CreditMetrics, there are three categories of estimation to be used according to the nature of the composition of the portfolio. We are going to try, in what follows, to present the various principles of the model according to the composition of the portfolio.

A. The portfolio in an Obligation

According to Hamisultane (2008), the system of rating used by CreditMetrics is the one rating agency. So, the broadcasting issuers of debt securities are noted according to a ladder of seven categories going from AAA to CCC according to the financial solidity of every company (Crouhy et al., 2000). The notation AAA is tuned to the healthy companies financially whereas those who are characterized by a bad financial situation are noted by CCC.

The notations offered by the agencies of rating are regularly published. These notations present information relative to the broadcasting issuers of debt securities. The agencies of rating include these notations in indicating tables, either the rate of historic default of broadcasting issuers according to their notation on a horizon of well determined time, or the evolutions of these notations in the time. These tables recapitulating the notations tuned to the broadcasting issuers of debt securities are defined by “the matrices of transition”.

The matrices of annual transition summarize all the changes of notation, on a horizon of time of one year, of a sand of broadcasting issuers is presented as follows:

Table 3. Transition matrix: Probabilities of credit rating migrating from one rating quality to another, within 1 year

Rating	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81%	8.33%	0.68%	0.06%	0.12%	0.00%	0.00%	0.00%
AA	0.70%	90.65%	7.79%	0.64%	0.06%	0.14%	0.02%	0.00%
A	0.09%	2.27%	91.05%	5.52%	0.74%	0.26%	0.01%	0.06%
BBB	0.02%	0.33%	5.95%	86.93%	5.30%	1.17%	0.12%	0.18%
BB	0.02%	0.14%	0.67%	7.73%	80.53%	8.84%	1.00%	1.06%
B	0.00%	0.11%	0.24%	0.43%	6.48%	83.46%	4.08%	5.20%
CCC	0.22%	0.00%	0.22%	1.30%	2.38%	5.00%	64.85%	19.79%
Default	0.00%	0.00%	0.00%	0.00%	0.00%	0.00	0.00%	100%

Source: Standard & Poor’s CreditWeek (1996)

According to Grundke (2009), this table must be carefully analyzed. So, by taking as an example the line corresponding to the BBB rating presented in the table above, we can deduct the probability of default as follows:

Table 4. The potential rating relative to the BBB rating

Initial rating	Potential rating in a one year	Probability
BBB	AAA	0.02%
	AA	0.33%
	A	5.95%
	BBB	86.93%
	BB	5.30%
	B	1.17%
	CCC	0.12%
	D	0.18%
	Total	100.00%

Source: Grundke (2009)

After a period of one year, and settling on the asset of initial notation BBB, we can deduct that the probability that this active rest BBB after a period of one year is 86,93 %, that to become AAA is 0,02 % and that to be lacking is 0,18 %.

The use of this model is based on three main hypotheses (Morgan & Co. Inc, 1997; Glasserman & Li, 2005; Hamisultane, 2008; Grundke, 2009; Figlewski et. al., 2012):

- The absence of multiple transitions: for a horizon of time given the number of transitions is in most of a single transition;
- The stability of the matrix of transition in time: for every class of notation, two companies in different sectors or in different countries have the same probability to migrate from a notation to the other one;
- The matrix of transition is of type Markov: for period given the probability to migrate of a class of notation in another class is independent from what took place for the last periods. These hypotheses are emitted for the simplification of the calculations of the matrix of transition for the posterior periods.

CreditMetrics determines the current value of the bond by using the curve of the rates with zero coupons to proceed with the calculations of CreditVaR. In that case, the transmitter of debt securities is not in situation of bankruptcy. By continuing in the same context of analysis, that is the use of the notation BBB as the example, we can use the table of the Forward rates following:

Table 5. One-year forward zero-curves for each credit rating (%)

Category	**Year 1**	**Year 2**	**Year 3**	**Year 4**
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.5	15.02	14.03	13.52

Source: CreditMetrics, JP Morgan

We suppose in our case which a noted transmitter BBB has emitted a Bond for 100 Euro over 4 years with a rate without annual risk of 6 %. The current value of the bond is given by the equation below:

$$\boldsymbol{V} = 6 + \frac{6}{(1+4.1\%)} + \frac{6}{(1+4.67\%)^2} + \frac{6}{(1+5.25\%)^3} + \frac{106}{(1+5.63\%)^4} = 107.55$$

By basing itself on the formula above, being able to us determine the various possible values of fire of type BBB according to his possible migrations towards other notations (Crouhy et al., 2000; Hamisultane, 2008). The possible values of a bond rated BBB according to the possible migrations are presented in the table 5.

In case the company had a bankruptcy, the value of the bond is determined by using the average recovery ratio calculated by CreditMetrics on historical data (Carty & Lieberman, 1996; Gordy, 1998).

Further to the representative table of the various values of BBB according to the possible migrations, we can subtract the distribution of the variations of the price of the obligation in the following table:

Table 6. Distribution of the bond values, and changes in value of a BBB bond, in 1 year

Rating	Probability: p (%)	Price of the obligation(bond) V ($)	Difference with regard to V: ΔV	Difference with regard to the average μ	$\mu^2 * p$ (%)
AAA	0.02	109.37	1.82	2.28	0.0010
AA	0.33	109.19	1.64	2.10	0.0146
A	5.95	108.66	1.11	1.57	0.1474
BBB	86.93	107.55	0	0.46	0.1853
BB	5.30	102.02	-5.53	-5.06	1.3592
B	1.17	98.10	-9.45	-8.99	0.9446
C	0.12	83.64	-23.91	-23.45	0.6598
Default	0.18	51.13	-56.42	-55.96	5.6358
	Average =	107.09 ($)		**Variance =** **Standard deviation =**	8.9477 2.99 ($)

Source: CreditMetrics, JP Morgan

The analysis of this table shows that CreditVaR in 1 % (at a level of 99 % confidence) is equal to the last value of the variation of the value of the bond which corresponds to the notation CCC. Thus, CreditVaR is equal to -23.91.

B. The Portfolio in Two Obligations

In the case of a portfolio consisted of two bands, the analysis is based on the level of correlation of the migrations. In fact, in a portfolio consisted of several assets the migrations of the various credits are correlated. CreditMetrics tries to estimate these correlations. As long, as there are no good data to be used. In that case, CreditMetrics used the correlations between the values of the assets of the broadcasting issuers of the credits which are approached by the correlations between the equity prices of these broadcasting issuers to calculate the correlations between the migrations of the credits (Treacy & Carey, 2000; Altman & Rijken, 2004; Gordy & Howells, 2006; Xing et. al., 2012).

According to Iscoe et al. (1999), to be able to divert the correlations of the migrations of the credits of the correlations of the values of assets, it is necessary to have a model linking the quality of a credit to the value of assets. The solution proposed by CreditMetrics is to use an extension of the model of Merton (1974) which incorporates the migrations of the credits. In this aligned, we suggest taking into account the probability of migration of a bond rated initially by BB. These probabilities are given by the following table:

Table 7. Transition matrix based on actual rating changes

Initial Rating	**Rating at year-end (%)**							
	AAA	AA	A	BBB	BB	B	CCC	Défaut
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

Source: Standard & Poor's CreditWeek (1996)

By basing itself on the model of Merton (1974), we can suppose that the efficiency on a bond modeled as follows:

$$r = \mu + \sigma\varepsilon$$

With: ε a term of error is such as $\varepsilon \sim N(0,1)$, μ is the average efficiency on the bond and σ is the standard deviation of the efficiencies of this bond. Then, the default probability of an issuer of the bond is given by the following expression:

$$Pr\{default\} = Pr\{r < Z_{Def}\} = Pr\{\mu + \sigma\varepsilon < Z_{Def}\}$$

Thus,

$$Pr\{default\} = Pr\{r < Z_{Def}\} = Pr\{\sigma\varepsilon < Z_{Def}\}$$

If $\mu = 0$

$$Pr\{default\} = \left\{\varepsilon < \frac{Z_{Def}}{\sigma}\right\} = \Phi\left(\frac{Z_{Def}}{\sigma}\right)$$

Where, Φ indicates the cumulative function of the normal law.

By using the table above, we can establish the table according to who summarizes the distribution of the probability of migration affected in conformance with BB rating:

Table 8. The distribution of the probability of migration of BB rating

Rating	Probability from the transition matrix (%)	Probability according to the asset value model
AAA	0.03	$\mathbf{1-\Phi(Z_{AA}/\sigma)}$
AA	0.14	$\mathbf{\Phi\,(Z_{AA}/\sigma)-\Phi(Z_{A}/\sigma)}$
A	0.67	$\mathbf{\Phi\,(Z_{A}/\sigma)-\Phi(Z_{BBB}/\sigma)}$
BBB	7.73	$\mathbf{\Phi\,(Z_{BBB}/\sigma)-\Phi(Z_{BB}/\sigma)}$
BB	80.53	$\mathbf{\Phi\,(Z_{BB}/\sigma)-\Phi(Z_{B}/\sigma)}$
B	8.84	$\mathbf{\Phi\,(Z_{B}/\sigma)-\Phi(Z_{CCC}/\sigma)}$
CCC	1.00	$\mathbf{\Phi\,(Z_{CCC}/\sigma)-\Phi(Z_{Def}/\sigma)}$
Default	1.06	$\mathbf{\Phi\,(Z_{Def}/\sigma)}$

Source: Crouhy and al. (2000)

With, $1-\mathbf{\Phi}\left(\frac{\boldsymbol{Z_{AA}}}{\boldsymbol{\sigma}}\right)$ represent the probability so that the bond of BB rating can pass in the notation AAA and $\mathbf{Z_{AA}}$ indicates the threshold from which BB passes to AAA. The transformation graphic of the data above is presented as follows:

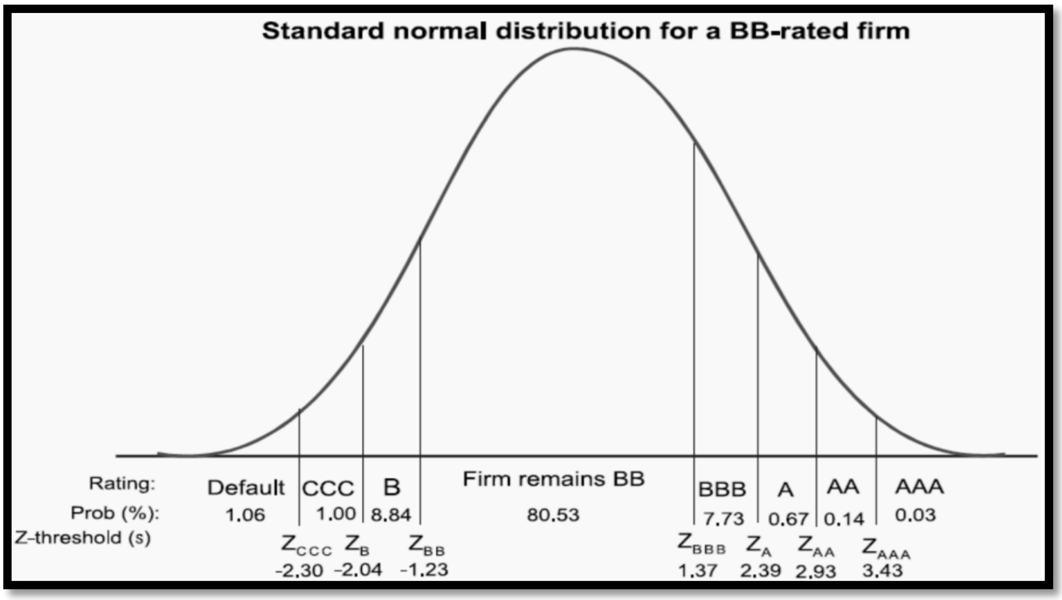

Figure 3. Generalization of the Merton model to include rating changes (Crouhy & al., 2000)

Thus:

$$\boldsymbol{Z_{def}} = \mathbf{\Phi}^{-1}(1.06\%).\boldsymbol{\sigma} = -2.30\ \boldsymbol{\sigma}$$

The values of the other thresholds are calculated according to whom corresponds itself aside type of the normal distribution of the random on the assets of the notation BB (Gupton et al., 1997; Crouhy et al., 2000; Nickell et al., 2000; Bangia et al., 2002; Albanese & Chen, 2003; Albanese et al., 2003; Rosch, 2005; Feng et al., 2008).

We suppose now, that a second issuer presents a rating A where the random on assets follow a normal distribution with a parameter $\boldsymbol{\sigma'}$. In that case, the values of thresholds relative for two bands who rated BB and A are presented as follows:

Table 9. Transition probabilities and credit quality thresholds for BB and A rated obligors

Rating in 1 year	Rated-A obligor		Rated-BB obligor	
	Probabilities (%)	**Thresholds: $Z_{(\sigma)}$**	**Probabilities (%)**	**Thresholds: $Z_{(\sigma)}$**
AAA	0.09	3.12	0.03	3.43
AA	2.27	1.98	0.14	2.93
A	91.05	-1.51	0.67	2.39
BBB	5.52	-2.30	7.73	1.37
BB	0.74	-2.72	80.53	-1.23
B	0.26	-3.19	8.84	-2.04
CCC	0.01	-3.24	1.00	-2.30
Default	0.06		1.06	

Source: Crouhy and al. (2000)

By taking into account the table above, we can calculate the probability of migration joined in the following way:

$$\boldsymbol{P}\left(\boldsymbol{Z_{BB}} < r < \boldsymbol{Z_{BBB}}, \boldsymbol{Z_A} < \boldsymbol{r}' < \boldsymbol{Z_{AA}}\right) = \int_{Z_{BB}}^{Z_{BBB}} \int_{Z_A}^{Z_{AA}} \boldsymbol{f}\left(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{\sigma}, \boldsymbol{\sigma'}\right) \boldsymbol{drdr'}$$

With **r** and **r’** indicate respectively the random on the assets who are rated by BB and A and $\boldsymbol{f}(\boldsymbol{r}, \boldsymbol{r'}, \boldsymbol{\sigma}, \boldsymbol{\sigma'})$ represent the joint density function by the Gaussian distribution which depends on the coefficient of correlation **ρ**.

The joint density function of the Gaussian distribution of two variables X and Y is presented by the form below:

$$\boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{y}\right) = \frac{1}{2\boldsymbol{\pi}\boldsymbol{\sigma}_x\boldsymbol{\sigma}_y\sqrt{1-\boldsymbol{\rho}^2}} \boldsymbol{exp}\left(-\frac{1}{2(1-\boldsymbol{\rho}^2)}\left(\frac{\boldsymbol{x}^2}{\boldsymbol{\sigma}_x^2} + \frac{\boldsymbol{y}^2}{\boldsymbol{\sigma}_y^2} - \frac{2\boldsymbol{\rho}\boldsymbol{xy}}{\boldsymbol{\sigma}_x\boldsymbol{\sigma}_y}\right)\right)$$

According to Hamisultane (2008), for **ρ** = 20% the matrix of joint transition which considers the correlation banding both entities BB and A is the following one:

Table 10. Joint rating probabilities (%) for BB and A rated obligors when correlation banding asset random is 20%

Rating of first company (BB)	Rating of second company (A)								
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**	**Default**	**Total**
AAA	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.03
AA	0.00	0.01	0.13	0.00	0.00	0.00	0.00	0.00	0.14
A	0.00	0.04	0.61	0.01	0.00	0.00	0.00	0.00	0.67
BBB	0.02	0.35	7.10	0.20	0.02	0.01	0.00	0.00	7.69
BB	0.07	1.79	73.65	4.24	0.56	0.18	0.01	0.04	80.53
B	0.00	0.08	7.80	0.79	0.13	0.05	0.00	0.01	8.87
CCC	0.00	0.01	0.85	0.11	0.02	0.01	0.00	0.00	1.00
Default	0.00	0.01	0.90	0.13	0.02	0.01	0.00	0.00	1.07
Total	0.09	2.29	91.06	5.48	0.75	0.26	0.01	0.06	100

Source: CreditMetrics, JP Morgan (Lucas, 1995)

The last column of the table and the last line of this one represent the marginal probability for the entities BB and A which are equal to the sum of the joint probability by line or by the column. According to Crouhy et al. (2000) these marginal probabilities correspond to the probability of migration of BB and of A taken individually. The variation of the portfolio of both bands is calculated for each of the joint probability. (Brady & Bos, 2002; Brady et al., 2003)

C. The Portfolio in Several Obligations

In case the portfolio consists further more than 2 bands calculates its joint probability will more be complicated. So, model CreditMetrics propose the use of the simulations of **Monte Carlo** and the decomposition of **Cholesky** to generate trajectories correlated to the bond and build the distribution of the values of the portfolio on certain horizon of time. (Gouriéroux **&** Monfort, 1995; Fishmen, 1997; Crouhy et al., 2000; Hamilton et al., 2002)

According to Hamisultane (2008) and Feng et al. (2008), to generate trajectories correlated to the variables which follow a normal distribution N (μ, ∑). The determination of these trajectories requires the respect for the following five stages:

Stage 1: The regression of the random $\boldsymbol{r_t}$ of the band on the sectorial indications. For example, in the case of three bands and two sectorial indications.

$$\boldsymbol{r}_{1,\,t} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_{1,\,X}\boldsymbol{I}_{X,\,t} + \boldsymbol{\alpha}_{1,\,Y}\boldsymbol{I}_{Y,\,t} + \boldsymbol{v}_{1,\,t}$$

$$\boldsymbol{r}_{2,\,t} = \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_{2,\,X}\boldsymbol{I}_{X,\,t} + \boldsymbol{\alpha}_{2,\,Y}\boldsymbol{I}_{Y,\,t} + \boldsymbol{v}_{2,\,t}$$

$$\boldsymbol{r}_{3,\,t} = \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_{3,\,X}\boldsymbol{I}_{X\,t} + \boldsymbol{\alpha}_{3,Y}\boldsymbol{I}_{Y,\,t} + \boldsymbol{v}_{3,\,t}$$

To pass in the second stage it is necessary to estimate the various parameters of three models.

Stage 2: The calculation of the variances and the covariance's banding 2 bands i and j:

$$\boldsymbol{cov}(\boldsymbol{r_i},\boldsymbol{r_j}) = \widehat{\boldsymbol{\alpha}}_{i,X}\widehat{\boldsymbol{\alpha}}_{j,X}\boldsymbol{V}(\boldsymbol{I_X}) + \widehat{\boldsymbol{\alpha}}_{i,Y}\,\widehat{\boldsymbol{\alpha}}_{j,Y}\boldsymbol{V}(\boldsymbol{I_Y}) + \left(\widehat{\boldsymbol{\alpha}}_{i,X}\,\widehat{\boldsymbol{\alpha}}_{j,Y} + \widehat{\boldsymbol{\alpha}}_{i,Y}\,\widehat{\boldsymbol{\alpha}}_{j,X}\right)\boldsymbol{cov}(\boldsymbol{I_X},\boldsymbol{I_Y})$$

And

$$\boldsymbol{V}(\boldsymbol{r_i}) = \widehat{\boldsymbol{\alpha}}^2_{i,X}\,\boldsymbol{V}(\boldsymbol{I_X}) + \widehat{\boldsymbol{\alpha}}^2_{i,Y}\boldsymbol{V}(\boldsymbol{I_Y}) + \boldsymbol{V}(\boldsymbol{v}_i^2)\boldsymbol{2}\left(\widehat{\boldsymbol{\alpha}}_{i,X}\,\widehat{\boldsymbol{\alpha}}_{i,Y}\right)\boldsymbol{cov}(\boldsymbol{I_X},\boldsymbol{I_Y})$$

By using these two formulae, we can obtain the matrix of the variances-covariance's $\sum$.

Stage 3: The decomposition of **Cholesky** of the matrix of the variances of the variances-covariance's $\sum$ in the following way (Hamisultane, 2008):

$$\sum = \boldsymbol{AA^T}$$

With $\mathbf{A}$ represent the lower triangular matrix and $\mathbf{A^T}$ transposed by the matrix A.

Stage 4: The simulation of variables $\boldsymbol{Z_{i,t}} \sim> N(0,1)$. In fact, the existence of the bond to be feigned allows the existence of $\boldsymbol{Z_{i,t}}$.

Stage 5: The simulation of the values of the correlated variables $\boldsymbol{V} \sim> \boldsymbol{N}\left(\boldsymbol{\mu},\sum\right)$ by basing itself on a process of geometrical distribution:

$$\frac{\boldsymbol{dV}}{\boldsymbol{V}} = \boldsymbol{\mu}\ \boldsymbol{dt} + \boldsymbol{A}\sqrt{\boldsymbol{dt}}\ \boldsymbol{Z}$$

Thus:

$$\frac{\boldsymbol{dV}}{\boldsymbol{V}} = \begin{pmatrix} \boldsymbol{dV_t^1/V_t^1} \\ \boldsymbol{dV_t^2/V_t^2} \\ \vdots \\ \boldsymbol{dV_t^i/V_t^i} \\ \vdots \\ \boldsymbol{dV_t^n/V_t^n} \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{lnV_t^1 - lnV_{t-1}^1} \\ \boldsymbol{lnV_t^2 - lnV_{t-1}^2} \\ \vdots \\ \boldsymbol{lnV_t^i - lnV_{t-1}^i} \\ \vdots \\ \boldsymbol{lnV_t^n - lnV_{t-1}^n} \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu_1} \\ \boldsymbol{\mu_2} \\ \vdots \\ \boldsymbol{\mu_i} \\ \vdots \\ \boldsymbol{\mu_n} \end{pmatrix}$$

$$\boldsymbol{dt} = \Delta\boldsymbol{t}$$

$$A = \begin{pmatrix} \beta_{11} & 0 & \cdots & & 0 \\ \beta_{21} & \ddots & & & 0 \\ \vdots & & \ddots & 0 & \vdots \\ \beta_{i1} & & \beta & \ddots & \vdots \\ \vdots & & & \beta & \ddots & 0 \\ \beta_{n1} & \beta_{n2} & \cdots & & \beta_{nn} \end{pmatrix}$$

$$\sqrt{dt}\, Z \approx \sqrt{\Delta t} \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \\ \vdots \\ Z_{i,t} \\ \vdots \\ Z_{n,t} \end{pmatrix}$$

According to Crouhy et al. (2000), Nickell et al. (2000) and Bangia et al. (2002), the forces and the weaknesses of this model are presented in the following table:

Table 11. The forces and the weaknesses relative to the CreditMetrics model

The forces	The weaknesses
▪ In the model CreditMetrics, both aspects of the credit risk are taken into account.	▪ The rating according to companies must be correct; ▪ The interest rates are supposed constant; ▪ The existence of a relation between the economic situation and the probability of defect. In that case, every economic cycle has to have matrices of transition appropriate for him; ▪ The variability of the actions of a company can be used to deduct the variability of the price of the assets of the company.

Source: Crouhy and al. (2000), Nickell al. (2000); Bangia and al. (2002)

3. The Econometric Models (Credit Portfolio View of Mackinsey)

Credit Portfolio View is a model with multiple factors which is used to feign the common conditional distribution of the default probability and migration for various groups of estimation and in different industries (Crouhy et al., 2000). This model was developed by Wilson (1997) within McKinsey. The approach developed by this author bases itself on the hypothesis that the probability of defect and migration are connected to macroeconomic factors such as the level of the long-term interest rate, the growth rate of the GDP, the global unemployment rate, the exchange rates, the public spending, the savings.

Credit Portfolio View is based on the occasional observation which supposes that the default probability, as well as the probability of migration, is connected to economic cycles. When the economy is in situation of recession, then the cycles of credit are also lesser. If it is the opposite case (the economy is in situation of expansion) then the cycles of credit become stronger. In other words the cycles of credit follow the tendency of economic cycles. Because the state of the economy is widely driven by macroeconomic factors, Credit Portfolio View proposes a methodology to connect these macroeconomic factors to the probability of default and migration.

Provided that the data are available, this methodology can be applied in every country, in the different sectors and in the diverse classes of borrowers of the obligors who react differently within the economic cycle.

The way that a model Credit Portfolio View works is as follows (Smithson, 2003):

- Simulate the state of the economy;
- Adjust the rate of default to the state of the simulation of the economy;
- Attribute a probability of default for every debtor on the basis of the simulations of the state of the economy;
- The value of the individual transactions attributed to the debtors according to the probability of defect is determined on the basis of the simulations of the state of the economy;
- Calculate the loss of the portfolio by adding the results for all the transactions;
- Repeat all the stages quoted above certain number of times to map finally the distribution of the losses;

In the model Credit Portfolio View of McKinsey, the historic rates of default for the various industries are described according to the macroeconomic variables specified by the user of the model:

$$(\boldsymbol{Probabillity\ of\ default} = \boldsymbol{f}\left(\boldsymbol{GDP}, \boldsymbol{Unemployment\ Rate}, \ldots, \boldsymbol{Exchange\ Rate}\right)$$

In the approach McKinsey, the rates of defect are commanded by a sensibility in a sand of the factors of the systematic risk, or the specific factors to the company. The table below summarizes the main characteristics of the model of McKinsey (Smithson, 2003):

Table 12. The main characteristics of the model Credit Portfolio View

Unit of analysis	Segmentation towards industries and on countries.
The data by default	Empirical estimation of the rates of default according to the macroeconomic variables. (For example: the GDP, the unemployment rate)
The structure of correlation	Obtained from the correlations banding the chosen macroeconomic variables and the estimated factors of sensibility.
The engine of the risk	The adjustment of the ARMA model (Autoregressive Moving Average model) with the evolution of the macroeconomic factors. The shocks undergone by the system determine the standard deviation of the average of the rates of defect concerning the level of the segment.
The distribution of the rates of defect	Logistic (Normal distribution).
The horizon	The maturity of the marginal default rate year by year.

Source: Smithson (2003)

3.1. The Forecast of the Default Rate

In the Credit Portfolio View model, the probabilities of default are modeled as being a Logit function. In this modeling the independent variable is a specific speculative index in every country and which depends on macroeconomic variables. The Logit function allows that the values of probability of default are included between 0 and 1 (Crouhy et al., 2000; Hamisultane, 2008).

$$\boldsymbol{P}_{j,t} = \frac{1}{1+\boldsymbol{e}^{-Y_{j,t}}}$$

$$\boldsymbol{Y}_{j,t} = \boldsymbol{\beta}_{j,0} + \boldsymbol{\beta}_{j,1}\boldsymbol{X}_{j,1,t} + \boldsymbol{\beta}_{j,2}\boldsymbol{X}_{j,2,t} + \ldots + \boldsymbol{\beta}_{j,m}\boldsymbol{X}_{j,m,t} + \boldsymbol{\varepsilon}_{j,t} \text{ And } \boldsymbol{\varepsilon}_{j,t} \sim> N(0, \boldsymbol{\sigma}^2_{\varepsilon,j})$$

Where, $\boldsymbol{P}_{j,t}$ indicate the conditional probability of default for period **t** for the debtors of the industry **j** and $\boldsymbol{Y}_{j,t}$ represent an indication stemming from a model in **m** factors. $\boldsymbol{\beta}_{j,0}$, $\boldsymbol{\beta}_{j,1}$, ..., $\boldsymbol{\beta}_{j,m}$ are coefficients to be estimated by the method the Ordinary Last Squares (**OLS**). $\boldsymbol{X}_{j,1,t}$ **,** $\boldsymbol{X}_{j,2,t}$, …, $\boldsymbol{X}_{j,m,t}$ are values of economic variables in the date **t** of the industry or the country **j**. $\boldsymbol{\varepsilon}_{j,t}$ represent a term of error which is normally distributed and independent of $\boldsymbol{Y}_{j,t}$.

The model of McKinsey so land us land us note, as it is a model of macro-factors $\boldsymbol{X}_{j,t}$ who are represented by variable macroeconomic who follow a Autoregressive model of order 2 (**AR2**):

$$\boldsymbol{X}_{j,i,t} = \boldsymbol{\gamma}_{j,i,0} + \boldsymbol{\gamma}_{j,i,1}\boldsymbol{X}_{j,i,t-1} + \boldsymbol{\gamma}_{j,i,2}\boldsymbol{X}_{j,i,t-1} + \boldsymbol{\omega}_{j,t} \text{ And } \boldsymbol{\omega}_{j,t} \sim> N(0, \boldsymbol{\sigma}^2_{\omega,j})$$

Where: $\gamma_{j,i,0}$, $\gamma_{j,i,1}$ and $\gamma_{j,i,2}$ are a coefficients to be estimated and $\omega_{j,t}$ is a term of error which is normally distributed and independent of $X_{j,i,t}$.

In this frame, our objective is to resolve the system below:

$$P_{j,t} = \frac{1}{1+e^{-Y_{j,t}}}$$

$$Y_{j,t} = \beta_{j,0} + \beta_{j,1}X_{j,1,t} + \beta_{j,2}X_{j,2,t} + \ldots + \beta_{j,m}X_{j,m,t} + \varepsilon_{j,t}$$

$$X_{j,i,t} = \gamma_{j,i,0} + \gamma_{j,i,1}X_{j,i,t-1} + \gamma_{j,i,2}X_{j,i,t-1} + \omega_{j,t}$$

Where E_t is the vector of the innovations such as:

$$E_t = \begin{bmatrix} \varepsilon_t \\ \omega_t \end{bmatrix} \sim N(0,\sum) \text{ And } \sum = \begin{bmatrix} \sum_{\varepsilon} & \sum_{\varepsilon,\omega} \\ \sum_{\omega,\ \varepsilon} & \sum_{\omega} \end{bmatrix}$$

With $\sum_{\varepsilon,\omega}$ and $\sum_{\omega,\varepsilon}$ Represent the matrices of correlation.

In case the parameters are estimated, then it is possible to feign the probability of default by basing itself on historical data. Credit Portfolio View uses tired matrices of transition of economic cycles.

3.2. The Conditional Matrices of Transition

By basing itself on the matrices of transition in the economic cycles which are proposed by the Credit Portfolio View, we can determine the situation of the economy (Crouhy et al., 2000). Noting in this respect that, the matrices of transition in the Credit Portfolio View are different to those of the matrices of migration in the CreditMetrics (Hamisultane, 2008).

Credit Portfolio View proposes a tool based on the following ratio: $\frac{P_{j,t}}{\varphi SDP}$

Where $P_{j,t}$ represent the probability of default feigned for date **t** and for the sector **j** and φSDP represent the historic default probability which is based on observed data.

If $\frac{P_{j,t}}{\varphi SDP} > 1$ then the economy is in period of recession and if $\frac{P_{j,t}}{\varphi SDP} < 1$ then the economy is in period of expansion.

Credit Portfolio View suggests employing this ratio to adjust the probability of migration. So, the matrix of transition multi-period is given by the following formula:

$$\boldsymbol{M} = \prod_{t=1}^{T} \boldsymbol{M}\left(\frac{\boldsymbol{P}_{j,t}}{\varphi \boldsymbol{SDP}}\right)$$

Where **M(.)** can take two different values. So, $\mathbf{M(.)} = \mathbf{M_L}$ if $\frac{P_{j,t}}{\varphi SDP} > 1$ and $\mathbf{M(.)} = \mathbf{M_H}$ if $\frac{P_{j,t}}{\varphi SDP} < 1$.

With $\mathbf{M_L}$ indicate the matrix of transition in the case of a period of recession and $\mathbf{M_H}$ indicates the matrix of transition in the case of a period of expansion.

We can simulate a lot of time the matrix of transition to determine the probability of default for any estimation and for any period. The methodology of Monte Carlo Simulation can be used to determine the distribution of the default probability for any period.

The forces and the weaknesses relative to the Credit Portfolio View model are presented in the table below:

Table 13. The forces and the weaknesses relative to the Credit Portfolio View model

The forces	The weaknesses
▪ Credit portfolio View connects the probability of default and the matrices of transition with economic indicators. In other words, the probability of default is stronger in period of recession than in period of expansion.	▪ In the Credit Portfolio View model, we use macroeconomic data which cannot be available for a country or a business sector. ▪ This model determines only the probability of default of a country or a business sector and not an issuer.

Source: Hamisultane (2008)

4. The Model CSFP: Credit Risk+ Market Risk

Since 1990s, Credit Suisse First Boston (CSFB) has developed new methods of risk management. In 1993, the credit Swiss Group launched, in parallel of an important project which aims at modernizing its credit risk management and by using the expertise of CSFB, new one management tool of the credit portfolio in the future. In December, 1996, Credit Suisse Group presented the CreditRisk+ model as being a model of the credit portfolio management.

The structural models present an inconvenience concerning the default. These models suppose that the default cannot have arisen by surprise because the market value of assets is supposed to follow a continuous process of distribution. In this aligned, a process of Fish was used in the actuarial models the purpose of which is to model the unpredictable character of the emergence of the default what is developed in the model CreditRisk+.

CreditRisk+ is a model with intensity is which presents no hypothesis on the causes of failure of a company. It is model statistical of the default of credit risk which makes no claim about the causes of the default. This approach is similar to that of the management of the market risk, in which no attempt is made model the causes of the movements of market prices. This model does not consider the consequences of a deterioration of the quality of the quality of the counterparty.

So, the number of failures in a credit portfolio during the given period justifies itself by a process of Fish. CreditRisk+ uses a methodology based on techniques and quantitative methods. The present model is based on an actuarial calculation to determine and present the distribution of the losses of a credit portfolio.

CreditRisk+ presents four hypotheses:

- Every individual credit presents only two possible states: failures or no failures;
- The default probability of an individual credit is low;
- The default probability for a big group of borrowers is very low;
- The number of default over a period is independent from that of any other period;

By basing itself on these hypotheses, the probability distribution of the number **X** of defaults over a given period (one month or one year for example) can be represented by using the law of Fish of average **μ** and of standard deviation $\sqrt{\boldsymbol{\mu}}$:

$$\boldsymbol{P}\left(\boldsymbol{X}=\boldsymbol{n}\right)=\frac{\boldsymbol{\mu}^{\boldsymbol{n}}\boldsymbol{e}^{-\boldsymbol{\mu}}}{\boldsymbol{n}!}$$

Where, **μ** is the average of the number of default a year.

$$\boldsymbol{\mu}=\sum \boldsymbol{P}_{\boldsymbol{A}}$$

With $\mathbf{P_A}$ indicate the default probability of the obligor A.

The annual number of the defaults, **n**, is a stochastic variable of average **μ** and a standard deviation $\sqrt{\boldsymbol{\mu}}$. According to CreditRisk+, the calculation of the distribution of the losses requires the use of an approach by bonds; that is issued in a portfolio are grouped and collected by edge of exposure.

The process of determination of the distribution of the losses of a portfolio is constituted by three stages:

- The determination of the generative function of probability for every bond;
- The diversion of the generative function of probability for the whole portfolio;
- The determination of the distribution of the losses for the whole portfolio.

The distribution of the losses of default for a portfolio is diverted in two stages as the watch represents it below:

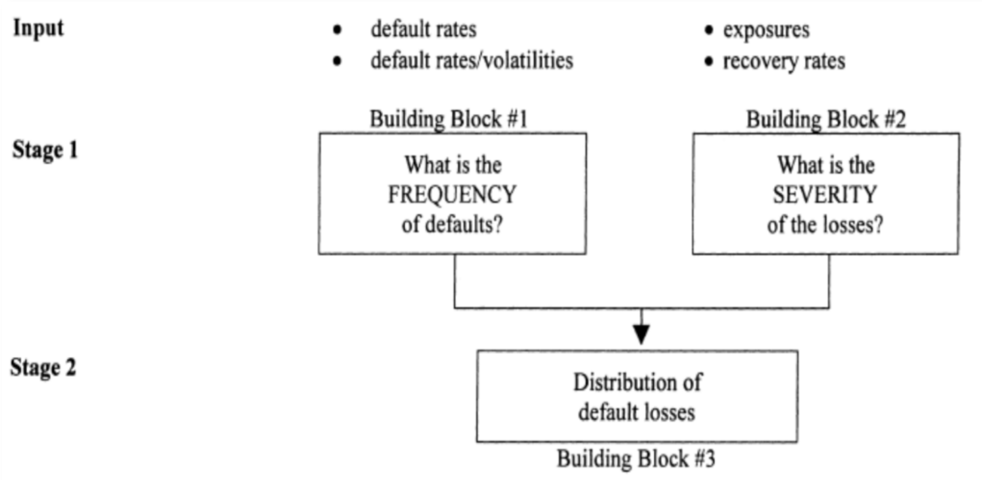

Figure 4. CreditRisk+ risk measurement framework (Crouhy et. al., 2000)

Until here, we supposed that the distribution of fish allows moving closer to the distribution of the number of the events of defect. Then we should expect that the standard deviation of the default rate is approximately equal to the square root of the average.

In case of defect of an obligor, the counterparty incurs a loss equal to the quantity possessed by the obligor less a quantity of restoring. In CreditRisk+ the exposure for every obligor is adjusted by the rate planned by restoring, to calculate the loss of default. These adjusted exposures are exogenous in the model, and are independent of the market risk and minimize the risk.

To divert the distribution of loss for a diversified portfolio, the losses are divided into bands with the level of the exposure in every band.

To analyze the distribution of the resultant losses of the whole portfolio, presenting us the default probability expressed by the function defined in terms of variables auxiliary **z** by respecting itself the following approach of the formulation of the generative function:

We considered **X** a whole and positive random variable. The generative function of **X** is the whole series:

$$G(z) = \sum_{k=0}^{\infty} P(X = k) z^k$$

Where **P(X=k)** is the probability that the random variable **X** takes the value **k**. to obtain **P(X=k)** from the generative function **G(z)**, we use the following formula:

$$P(X = k) = \frac{1}{k!} \frac{d^k G}{dz^k}(0)$$

In that case, the generative function associated among default **X** arisen among all the bonds of a portfolio is given by the expression below:

$$F(z) = \sum_{n=0}^{\infty} P(X = n).z^n = \sum_{n=0}^{\infty} \frac{\mu^n e^{-\mu}}{n!} z^n = \exp(\mu.(z-1))$$

This function can be written as follows:

$$F(z) = \prod_{A} F_A(z)$$

Where $F_A(z)$ indicate the generative function of a portfolio constituted by a single bond of the issuer **A**.

So, every portfolio consists of m identical bond of exposure of indications **j (j = 1, 2, m)**.

Every bond is characterized by: $\varepsilon_j = \mu_j * \vartheta_j$

Thus implies that: $\mu_j = \frac{\varepsilon_j}{\vartheta_j}$

With, ε_j indicate the expected average loss expressed in multiple of a standard exposure **L**, μ_j indicate the expected number of defaults which is a known value and ϑ_j indicate the exposure expressed in multiple of **L** in the band **j**.

In that case, the inputs of the model to be developed are: the individual exposure **L** and the probability of default P_A for the issuer (debtor) **A**. Then, the loss hoped for the debtor **A** is expressed as follows:

$$\lambda_A = L_A * P_A$$

$$\varepsilon_A = \frac{\lambda_A}{L}$$

The expression above is obtained when the expected loss is expressed in units of **L**. So, the expected loss $\boldsymbol{\varepsilon_j}$ for the bond **j** is given then as follows:

$$\varepsilon_j = \sum \varepsilon_A$$

In this perspective, the expected number of defects $\boldsymbol{\mu_j}$ for each of the indicated bond **j** is then given by:

$$\mu_j = \frac{\varepsilon_j}{\vartheta_j} = \sum \frac{\varepsilon_A}{\vartheta_j} = \sum_{\vartheta_j = \vartheta_j} \frac{\varepsilon_A}{\vartheta_A}$$

Thus, the number of waited defects total **μ** for them **m** bond is expressed as follows

$$\mu = \sum_{j=1}^{m} \mu_j = \sum_{j=1}^{m} \frac{\varepsilon_j}{\vartheta_j}$$

The expression of the generative function of the included losses is obtained by:

$$G(z) = \sum_{n=0}^{\infty} P\left(Agregate\ losses = n * L\right) z^n$$

$$G(z) = \prod_{j=1}^{m} G_j(z)$$

Thus:

$$G_j(z) = \sum_{n=0}^{\infty} P(V_j = k_j).\ z^{n\vartheta_j}$$

Where $\boldsymbol{V_j}$ represents the amount of the losses of the bond **j** and $\boldsymbol{P(V_j = k_j)}$ indicates the probability of the loss $\boldsymbol{k_j}$**.**

Furthermore, we have:

$$P\left(V_j = k_j\right) = P\left(X_j = n_j\right) = \frac{\mu_j^{n_j}\ e^{-\mu_j}}{n_j\,!}$$

Thus we obtain:

$$G_j(z) = \sum_{n=0}^{\infty} \frac{\mu_j^{n_j} e^{-\mu_j}}{n_j!} . z^{n_j \vartheta_j} = \mathbf{exp}(-\mu_j + \mu_j z^{\vartheta_j})$$

And

$$G(z) = exp(-\sum_{n=0}^{m} \mu_j + \sum_{j=1}^{m} \mu_j z^{\vartheta_j})$$

Then, if we put:

$$P(z) = \frac{1}{\mu} \sum_{j=1}^{m} \mu_j z^{\vartheta_j} = \frac{\sum_{j=1}^{m} \left(\frac{\varepsilon_j}{\vartheta_j}\right) z^{\vartheta_j}}{\sum_{j=1}^{m} \left(\frac{\varepsilon_j}{\vartheta_j}\right)}$$

Then, the generative function of the included losses can be written in the following way:

$$G(z) = \exp\left(\mu\left(P(z) - 1\right)\right) = F(P(z))$$

Where from, we can obtain the distribution of the losses of the total portfolio of an amount (**n*L**) as follows:

$$A_n = \frac{1}{n!} \frac{d^n G}{dz^n} (0)$$

Land us note in that case that, A_n can be calculated in continuous by basing itself on the following formula and under the hypothesis according to which **μ** is constant.

Where from we obtain:

$$\boxed{\begin{aligned} A_0 &= G(0) = \exp(-\mu) = exp\left(-\sum_{j=1}^{m} \frac{\varepsilon_j}{\vartheta_j}\right) \\ &= \sum_{j | \vartheta_j \le n} \left(\frac{\varepsilon_j}{n} A_{n-\vartheta_j}\right) \end{aligned}}$$

The CreditRisk+ model considers that every sector is driven by a simple fundamental factor. This factor explains the variability of the rate of average defect measured for this sector. The fundamental factor influences the rate of defects planned in the concerned sector which is modeled by a random variable of average **μ** and of standard deviation $\sqrt{\boldsymbol{\mu}}$ indicated for every sector.

The standard deviation reflects the degree to which, in all the probability of default, the obligors in the portfolio are exposed are more or less that their levels of the average. By continuing this analysis, the model CreditRisk+ bases on the hypothesis that **μ** is constant. So, by basing itself on the distribution of Fish of parameter **μ** the probability of failures are underestimated. In that case, it is necessary to take into account the existence of an average number of variable failures.

In this aligned, the parameter **μ** is considered as being a stochastic variable and depends on characteristics of the sector. In fact, and according to the CreditRisk+ model, a sector is considered as being a sand of credits the rates of failure of which are subjected to the same influences. In the CreditRisk+ model, every portfolio is divided into sectors indicated by **k** with **1 ≤ k ≤ K**.

In particular, for every sector **k**, we introduce one random variable $\boldsymbol{x_k}$ which represents the average number of defaults in this sector. The average number of the defects is equal in **μ**.

So, the hope of $\boldsymbol{x_k}$ for the sector **k** is noted **μ** and its standard deviation is equal in$\boldsymbol{\sigma_k}$. In this frame **μ** is calculated as follows:

$$\boldsymbol{\mu_k} = \sum_{j=1}^{m(k)} \frac{\boldsymbol{\varepsilon}_j^{(k)}}{\boldsymbol{\vartheta}_j^{(k)}}$$

In the case that **μ** is no constant; the generative function of the number of defaults is given by:

$$\boldsymbol{F}(\boldsymbol{z}) = \prod_{k=1}^{k} \boldsymbol{F}_k(\boldsymbol{z})$$

And

$$\boldsymbol{F}_k(\boldsymbol{z}) = \sum_{n=0}^{\infty} \boldsymbol{z}^n \int_{x=0}^{\infty} \boldsymbol{P}(\boldsymbol{n\ defaults})\, \boldsymbol{f}(x)\boldsymbol{dx} = \int_{x=0}^{\infty} \boldsymbol{e}^{x(z-1)} \boldsymbol{f}(x)\boldsymbol{dx}$$

Where $\boldsymbol{f(x)}$ indicates the density of the variable$\boldsymbol{x_k}$.

The continuation of the calculations is conditioned by the presence of a nature of distribution given in$\boldsymbol{x_k}$. In the CreditRisk+ model, the choice is fixed to a distribution Gamma **Γ** of average **μ** and of standard deviation$\boldsymbol{\sigma_k}$. Thus we obtain:

$$F_k(z) = \int_{x=0}^{\infty} e^{x(z-1)} \frac{e^{-\frac{x}{\beta_k}} x^{\alpha_k - 1}}{\beta_k^{\alpha_k} \Gamma(\alpha_k)}$$

Where the Gamma function written as follows:

$$\Gamma(\alpha) = \int_{x=0}^{\infty} e^{-x} X^{\alpha-1} dx$$

For every sector **k**, we have two parameters of Gamma function to be estimated $\boldsymbol{\alpha_k}$ and$\boldsymbol{\beta_k}$.

Thus:

$$\alpha_k = \frac{\mu_k^2}{\sigma_k^2}$$

$$\beta_k = \frac{\sigma_k^2}{\mu_k}$$

By substituting and by basing itself on the definition of the Gamma function, we obtain then:

$$F_k(z) = \int_{x=0}^{\infty} e^{x(z-1)} \frac{e^{-\frac{x}{\beta_k}} x^{\alpha_k - 1}}{\beta_k^{\alpha_k} \Gamma(\alpha_k)}$$

$$\Leftrightarrow \; F_k(z) = \frac{\Gamma}{\beta_k^{\alpha_k} \Gamma(\alpha) \left(1 + \beta_k^{-1} - z\right)^{\alpha_k}} = \frac{1}{\beta_k^{\alpha_k} \left(1 + \beta_k^{-1} - z\right)^{\alpha_k}}$$

After this simplification, the generating function of the distribution of the probabilities of default for the sector **K** is given by the following expression:

$$F_z(z) = \left(\frac{1 - p_k}{1 - p_k z} \right)^{\alpha_k}$$

Thus:

$$p_k = \frac{\beta_k}{1_+ \beta_k}$$

After the determination of the number of defaults in a portfolio, one goes in what follows to present the generating function of the losses incorporated in a portfolio functions written is the following;

$$G(z) = \sum_{n=0}^{\infty} p\left(Agregate\ losses = n * L\right) z^n$$

So:

$$G(z) = \prod_{k=1}^{k} G_k(z) = \prod_{k=1}^{k} F_k(P_k(z))$$

Where the polynomial function $P_k(z)$ is written as follows:

$$P_k(z) = \frac{\sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\vartheta_j^{(k)}} \right) z^{\vartheta_j^{(k)}}}{\sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\vartheta_j^{(k)}} \right)} = \frac{1}{\mu_k} \sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\vartheta_j^{(k)}} \right) z^{\vartheta_j^{(k)}}$$

One can deduce the expression from the generating function $\mathbf{G(z)}$ which is written in the following way:

$$G(z) = \prod_{k=1}^{k} \left(\frac{1 - p_k}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\vartheta_j^{(k)}} \right) z^{\vartheta_j^{(k)}}} \right)^{\alpha_k}$$

In this respect, we can deduct the distribution of the losses of portfolios from the $\mathbf{A_n}$ which is given by:

$$G(z) = \sum_{n=0}^{\infty} A_n Z^n$$

So, in case $\mathbf{G(z)}$ verify the following relation:

$$\frac{G'(z)}{G(z)} = \frac{A(z)}{B(z)}$$

Where $\mathbf{A(z)}$ and $\mathbf{B(z)}$ are two polynomials of the following shape:

$$A(z) = a_0 + \ldots + a_r z^r$$

$$B(z) = b_0 + \ldots + b_s z^s$$

Thus, the coefficients $\mathbf{A(z)}$ verify the relation of following recurrence:

$$A_{n+1} = \frac{1}{b_0(n+1)} \left(\sum_{i=0}^{\min(r,\, n)} a_i A_{n-i} - \sum_{l=0}^{\min(s-1,\, n-1)} b_l (n-l)\, A_{n-l} \right)$$

This relation is applied knowing that $\mathbf{G(z)}$ verify the following condition:

$$\frac{G'(z)}{G(z)} = \sum_{k=1}^{k} \left(\frac{\frac{p_k \alpha_k}{\mu_k} \sum_{j=0}^{m(k)} \varepsilon_j^{(k)} z^{\vartheta_j^{(k)}-1}}{1 - \frac{p_k}{\mu_k} \sum_{j=1}^{m(k)} \left(\frac{\varepsilon_j^{(k)}}{\vartheta_j^{(k)}} \right) z^{\vartheta_j^{(k)}}} \right)$$

Generally, the CreditRisk+ model is based on mathematical techniques in the modeling of the distribution of the losses in the field of the banking activities and of the insurance. The behavior of common default of the borrowers is incorporated by treating the rate of default as being a common random variable for multiple borrowers. So, the borrowers are assigned among the sectors among which each has a rate of average default and a volatility of rate of default. The volatility of rate of default is the standard deviation which would be observed on a portfolio of infinitely diversified homogeneous credit. The forces and the weaknesses relative to the CreditRisk+ model are presented in the table below (Hamisultane, 2008):

Table 14. The forces and the weaknesses relative to the CreditRisk+ model

The forces	The weaknesses
▪ The use of a minimum of data since the distribution of the losses depends only on one reduced number of parameters. This characteristic makes it possible the CreditRisk+ model to reduce and minimize the risk of errors due to the uncertainty of the parameters. ▪ The CreditRisk+ model uses models based on closed formulas what allows him a fast execution of calculations.	▪ The CreditRisk + model do not take into account the earnings or the loss of value of the portfolio provoked by changes of Rating. ▪ The interest rates are supposed constant. ▪ The used techniques of calculation are not simple and are not necessarily accessible to every user of the model.

Source: Hamisultane (2008)

5. Conclusion

In this paper we developed a comparative theoretical approach's concerning the model of management of credit portfolio. Then, we studied the four mains models of credit portfolio management. In the financial literature those models are grouped by three types of credit portfolio models (Crouhy et al., 2000).

The structural models: there are two models of management of credit portfolio who are supplied in the literature: Moody's KMV model (Portfolio Model) and CreditMetrics model by JPMorgan. The Macro-factors model. (Econometric model) The Credit Portfolio View model introduces in 1998 by Mckinsey. The actuarial models CSFP (Credit Suisse First Boston): this model (CreditRisk+) is developed in 1997.

The KMV model and Credit Portfolio View base their approach on the same empirical observation that default and migration probabilities vary over time. The KMV model adopts a microeconomic approach which relates the probability of default of any obligor, to the market value of its assets. The Credit Portfolio View model proposes a methodology which links macroeconomics factors to default and migration probabilities. The calibration of this model necessitates reliable default data for each country, and possibly for each industry sector within each country.

Structural models are based on option theory and capital structure the company. On econometric models, they link the probability fault of the company to the state of the economy. The probability of failure depends in these models of macroeconomic factors such as unemployment, the rate of increase GDP, the interest rate long-term. Moreover, in the CreditRisk+ models, the probability of default varies over time.

6. References

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