A Method of Determination of an Acquisition Program of N Goods in Order to Maximize the Total Utility

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Abstract. This paper solves in a different way the problem of maximization of the total utility for n goods. The author uses the diophantic equations (equations in integer numbers) and after a decomposing in different cases, he obtains the maximal utility.

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A method of maximization the total utility for n goods

Let a consumer which has a budget of acquisition of r goods, $r\ge 2$, in value of $S\in\mathbf{N}$ u.m. The prices of the r goods x_i, $i=\overline{1,r}$ are $p_i\in\mathbf{N}$, $i=\overline{1,r}$. The marginal utilities corresponding to an arbitrary number of doses are in the following table:

No. of dose	U_{m1}	…	U_{mr}
1	u_{11}	…	u_{1r}
...	...	…	...
i	u_{i1}	…	u_{ir}
...	...	…	...
n	u_{n1}	…	u_{nr}

We want in what follows to determine the number of doses a_i for the good x_i, $i=\overline{1,r}$ such that the total utility: $U_t=\sum_{j=1}^{r}\sum_{i=1}^{a_j} u_{ij}$ to be maximal.

Let therefore $S_1 \le S$ and the equation:

$$(1)\quad \sum_{i=1}^{r} a_i p_i = S_1.$$

Let denote with $d=(p_1,\dots,p_r)$ the greatest common divisor of p_i, $\overline{1,r}$. We well know the fact that in order the equation has entire solutions we have necessarily $d \mid S_1$. Also, we shall consider:

$S_1 > S-\min\{p_1,\dots,p_r\}$ because if $S_1 \le S-\min\{p_1,\dots,p_r\}$ with a supplementary unit of o good i, where $1 \le i \le r$, the total utility will grow.

Dividing (1) at d, we have:

$$(2)\quad \sum_{i=1}^{r} a_i \frac{p_i}{d} = \frac{S_1}{d}$$

and with the notation: $q_i = \frac{p_i}{d}$ follows:

$$(3)\quad \sum_{i=1}^{r} a_i q_i = \frac{S_1}{d}.$$

where the greatest common divisor of q_i, $\overline{1,r}$ is $(q_1,\dots,q_r)=1$.

It is well known that for any relative prime numbers $A,B \in \mathbf{N}$ it exists α and $\beta \in \mathbf{Z}$ (determined eventually with the Euclid algorithm) such that: $\alpha A+\beta B=1$.

Let therefore $d_1=q_1$ and $d_i=(d_{i-1},q_i)$, $i=\overline{2,r}$. Because $(q_1,...,q_r)=1$ it follows obviously that $d_r=1$.

We have now $\exists \alpha_i,\beta_i \in \mathbf{N}$, $i=\overline{2,r}$, such that:

$$(4)\quad \alpha_i d_{i-1}+\beta_i q_i=d_i,\ i=\overline{2,r}$$

In what follows we shall denote: $d_0=0$, $\alpha_0=1$, $\beta_0=1$, $\alpha_1=1$, $\beta_1=1$ such that: $\alpha_1 d_0+\beta_1 q_1=d_1$.

Writing in detail the relation (4), we obtain:

$$(5)\quad \alpha_1 d_0+\beta_1 q_1=d_1$$

$$\alpha_2 d_1+\beta_2 q_2=d_2$$

$$\alpha_3 d_2+\beta_3 q_3=d_3$$

$$\dots$$

$$\alpha_r d_{r-1}+\beta_r q_r=d_r$$

Substituting the first of (5) in the second:

$$(6)\quad \alpha_2\alpha_1 d_0+\alpha_2\beta_1 q_1+\beta_2 q_2=d_2$$

after, the second in the third:

$$(7)\quad \alpha_3\alpha_2\alpha_1 d_0+\alpha_3\alpha_2\beta_1 q_1+\alpha_3\beta_2 q_2+\beta_3 q_3=d_3$$

we shall obtain, by induction:

$$(8)\quad \sum_{i=1}^{r-1}\prod_{j=i+1}^{r}\alpha_j\beta_i q_i+\beta_r q_r=d_r.$$

We have therefore:

$$(9)\quad \sum_{i=1}^{r}\sigma_i q_i=1$$

with the obvious notations: $\sigma_i=\beta_i\prod_{j=i+1}^{r}\alpha_j$, $i=\overline{1,r-1}$ and $\sigma_r=\beta_r$.

From (3), (9) we have now:

$$(10)\quad \sum_{i=1}^{r}a_i q_i=\frac{S_1}{d}\sum_{i=1}^{r}\sigma_i q_i$$

or, in other words:

$$(11)\quad \sum_{i=1}^{r}q_i\left(a_i-\frac{S_1}{d}\sigma_i\right)=0$$

For a fixed $k=\overline{1,r}$ we can write (11):

(12) $$q_k(a_k - \frac{S_1}{d}\sigma_k) + \sum_{\substack{i=1 \\ i\neq k}}^{r} q_i(a_i - \frac{S_1}{d}\sigma_i) = 0$$

Let now $\delta_k = (q_1, ..., \hat{q}_k, ..., q_r)$ where the sign $\wedge$ means that the term is missing. Because $(\delta_k, q_k) = 1$ follows:

(13) $$\delta_k \mid a_k - \frac{S_1}{d}\sigma_k$$

therefore:

(14) $$a_k - \frac{S_1}{d}\sigma_k = \zeta_k\delta_k,\ k = \overline{1,r}$$

From (11), (14) we have that:

(15) $$\sum_{i=1}^{r} \zeta_i\delta_i q_i = 0$$

We can write (14) also like:

(16) $$a_k = \frac{S_1}{d}\sigma_k + \zeta_k\delta_k,\ k = \overline{1,r}$$

Because $a_k \geq 0$, $k = \overline{1,r}$, we obtain that:

(17) $$S_1\sigma_k + \zeta_k\delta_k d \geq 0,\ k = \overline{1,r}$$

From (1) we can see easly that:

(18) $$a_k \leq \min\left(\frac{S_1}{p_k}, n\right),\ k = \overline{1,r}$$

From (16), (17) and (18) we find that:

(19) $$\begin{cases} \zeta_k \geq -\dfrac{S_1\sigma_k}{\delta_k d} \\ \zeta_k \leq \min\left(\dfrac{S_1(d - p_k\sigma_k)}{p_k\delta_k d}, \dfrac{nd - S_1\sigma_k}{\delta_k d}\right) \end{cases},\ k = \overline{1,r}$$

We have therefore:

(20) $$\zeta_k \in \left[-\frac{S_1\sigma_k}{\delta_k d}, \min\left(\frac{S_1(d - p_k\sigma_k)}{p_k\delta_k d}, \frac{nd - S_1\sigma_k}{\delta_k d} \right) \right] \cap \mathbf{N},\ k=\overline{1,r}$$

The length of the range is less than or equal with $\frac{S_1}{p_k\delta_k}$, therefore there exist at most $\left[\frac{S_1}{p_k\delta_k}\right]$+1 integer values of ζ_k (where [z] denotes the integer part of z) that verifies the acceptability conditions.

Example

No. of dose	U_{mx}	U_{my}	U_{mz}
1	10	20	15
2	8	16	12
3	7	15	10
4	6	14	7
5	5	13	5
6	4	10	3
7	3	8	2
8	2	7	1

$p_x=4$, $p_y=6$, $p_z=10$, S=50.

Solution

We have d=(4,6,10)=2 therefore $S_1>50-\min(4,6,10)=46$. Because $d \mid S_1$ we shall have $S_1 \in \{48,50\}$.

Dividing by 2 the reduced prices become: $q_1=2$, $q_2=3$, $q_3=5$.

Let now: $d_1=2$, $d_2=(d_1,q_2)=(2,3)=1$, $d_3=(d_2,q_3)=(1,5)=1$.

We have: $\alpha_1=1$, $\beta_1=1$ and from (4) the equation: $2\alpha_2+3\beta_2=1$ implies: $\alpha_2=-1$ and $\beta_2=1$. Also, the equation: $\alpha_3+5\beta_3=1$ implies: $\alpha_3=-4$ and $\beta_3=1$.

Let now: $\sigma_1=\beta_1\prod_{j=2}^{3}\alpha_j=\beta_1\alpha_2\alpha_3=4$, $\sigma_2=\beta_2\prod_{j=3}^{3}\alpha_j=\beta_2\alpha_3=-4$, $\sigma_3=\beta_3=1$

and: $\delta_1=(q_2,q_3)=(3,5)=1$, $\delta_2=(q_1,q_3)=(2,5)=1$, $\delta_3=(q_1,q_2)=(2,3)=1$.

The relation (15) becomes: $2\zeta_1+3\zeta_2+5\zeta_3=0$.

From (20):

$\zeta_k\in\left[-\frac{\sigma_k S_1}{2},\min\left(\frac{S_1(2-p_k\sigma_k)}{2p_k},8-\frac{S_1\sigma_k}{2}\right)\right]\cap\mathbf{N}$ therefore:

$$\zeta_1\in\left[-2S_1,\min\left(-\frac{7}{4}S_1,8-2S_1\right)\right]\cap\mathbf{N}$$

$$\zeta_2\in\left[2S_1,\min\left(\frac{13}{6}S_1,8+2S_1\right)\right]\cap\mathbf{N}$$

$$\zeta_3\in\left[-\frac{S_1}{2},\min\left(-\frac{2}{5}S_1,8-\frac{S_1}{2}\right)\right]\cap\mathbf{N}$$

and also from (16):

$a_1=\frac{S_1\sigma_1+\zeta_1\delta_1 d}{d}=\frac{4S_1+2\zeta_1}{2}=2S_1+\zeta_1$.

$a_2=\frac{S_1\sigma_2+\zeta_2\delta_2 d}{d}=\frac{-4S_1+2\zeta_2}{2}=-2S_1+\zeta_2$.

$a_3=\frac{S_1\sigma_3+\zeta_3\delta_3 d}{d}=\frac{S_1+2\zeta_3}{2}=\frac{S_1}{2}+\zeta_3$.

Finally we have the following cases:

$S_1=48\Rightarrow\zeta_1\in[-96,-88]\cap\mathbf{N}$

$\zeta_2\in[96,104]\cap\mathbf{N}$

$\zeta_3\in[-24,-20]\cap\mathbf{N}$

$S_1=50\Rightarrow\zeta_1\in[-100,-92]\cap\mathbf{N}$

$\zeta_2\in[100,108]\cap\mathbf{N}$

$\zeta_3 \in [-25,-20] \cap \mathbf{N}$

with $2\zeta_1+3\zeta_2+5\zeta_3=0$

For $S_1=48$ and certainly: $a_1=96+\zeta_1$, $a_2=-96+\zeta_2$, $a_3=24+\zeta_3$ we have:

$\zeta_1=-96$, $\zeta_2=99$, $\zeta_3=-21 \Rightarrow a_1=0$, $a_2=3$, $a_3=3$, $U_t=20+16+15+15+12+10=88$

$\zeta_1=-96$, $\zeta_2=104$, $\zeta_3=-24 \Rightarrow a_1=0$, $a_2=8$, $a_3=0$, $U_t=20+16+15+14+13+10+8+7=103$

$\zeta_1=-95$, $\zeta_2=100$, $\zeta_3=-22 \Rightarrow a_1=1$, $a_2=4$, $a_3=2$, $U_t=10+20+16+15+14+15+12=102$

$\zeta_1=-94$, $\zeta_2=96$, $\zeta_3=-20 \Rightarrow a_1=2$, $a_2=0$, $a_3=4$, $U_t=10+8+15+12+10+7=62$

$\zeta_1=-94$, $\zeta_2=101$, $\zeta_3=-23 \Rightarrow a_1=2$, $a_2=5$, $a_3=1$, $U_t=10+8+20+16+15+14+13+15=111$

$\zeta_1=-93$, $\zeta_2=97$, $\zeta_3=-21 \Rightarrow a_1=3$, $a_2=1$, $a_3=3$, $U_t=10+8+7+20+15+12+10=82$

$\zeta_1=-93$, $\zeta_2=102$, $\zeta_3=-24 \Rightarrow a_1=3$, $a_2=6$, $a_3=0$, $U_t=10+8+7+20+16+15+14+13+10=113$

$\zeta_1=-92$, $\zeta_2=98$, $\zeta_3=-22 \Rightarrow a_1=4$, $a_2=2$, $a_3=2$, $U_t=10+8+7+6+20+16+15+12=94$

$\zeta_1=-91$, $\zeta_2=99$, $\zeta_3=-23 \Rightarrow a_1=5$, $a_2=3$, $a_3=1$, $U_t=10+8+7+6+5+20+16+15+15=102$

$\zeta_1=-90$, $\zeta_2=100$, $\zeta_3=-24 \Rightarrow a_1=6$, $a_2=4$, $a_3=0$, $U_t=10+8+7+6+5+4+20+16+15+14=105$

$\zeta_1=-89$, $\zeta_2=96$, $\zeta_3=-22 \Rightarrow a_1=7$, $a_2=0$, $a_3=2$, $U_t=10+8+7+6+5+4+3+15+12=70$

$\zeta_1=-88$, $\zeta_2=97$, $\zeta_3=-23 \Rightarrow a_1=8$, $a_2=1$, $a_3=1$, $U_t=10+8+7+6+5+4+3+2+20+15=80$

For $S_1=50$ and: $a_1=100+\zeta_1$, $a_2=-100+\zeta_2$, $a_3=25+\zeta_3$ we have:

$\zeta_1=-100$, $\zeta_2=100$, $\zeta_3=-20 \Rightarrow a_1=0$, $a_2=0$, $a_3=5$, $U_t=15+12+10+7+5=49$

$\zeta_1=-100$, $\zeta_2=105$, $\zeta_3=-23 \Rightarrow a_1=0$, $a_2=5$, $a_3=2$, $U_t=20+16+15+14+13+15+12=105$

$\zeta_1=-99$, $\zeta_2=101$, $\zeta_3=-21 \Rightarrow a_1=1$, $a_2=1$, $a_3=4$, $U_t=10+20+15+12+10+7=74$

$\zeta_1=-99$, $\zeta_2=106$, $\zeta_3=-24 \Rightarrow a_1=1$, $a_2=6$, $a_3=1$, $U_t=10+20+16+15+14+13+10+15=113$

$\zeta_1=-98$, $\zeta_2=102$, $\zeta_3=-22 \Rightarrow a_1=2$, $a_2=2$, $a_3=3$, $U_t=10+8+20+16+15+12+10=91$

$\zeta_1=-98$, $\zeta_2=107$, $\zeta_3=-25 \Rightarrow a_1=2$, $a_2=7$, $a_3=0$, $U_t=10+20+16+15+14+13+10+8=106$

$\zeta_1=-97$, $\zeta_2=103$, $\zeta_3=-23 \Rightarrow a_1=3$, $a_2=3$, $a_3=2$, $U_t=10+8+7+20+16+15+15+12=103$

$\zeta_1=-96$, $\zeta_2=104$, $\zeta_3=-24 \Rightarrow a_1=4$, $a_2=4$, $a_3=1$, $U_t=10+8+7+6+20+16+15+14+15=111$

ζ_1=-95, ζ_2=100, ζ_3=-22 $\Rightarrow$ a_1=5, a_2=0, a_3=3, U_t=10+8+7+6+5+15+12+10=73

ζ_1=-95, ζ_2=105, ζ_3=-25 $\Rightarrow$ a_1=5, a_2=5, a_3=0,
U_t=10+8+7+6+5+20+16+15+14+13=**114**

ζ_1=-94, ζ_2=101, ζ_3=-23 $\Rightarrow$ a_1=6, a_2=1, a_3=2, U_t=10+8+7+6+5+4+20+15+12=87

ζ_1=-93, ζ_2=102, ζ_3=-24 $\Rightarrow$ a_1=7, a_2=2, a_3=1, U_t=10+8+7+6+5+4+3+20+16+15=94

ζ_1=-92, ζ_2=103, ζ_3=-25 $\Rightarrow$ a_1=8, a_2=3, a_3=0,
U_t=10+8+7+6+5+4+3+2+20+16+15=96

Finally, the maximal utility will be U_t=114 for 5 goods x and 5 goods y.