Miscellaneous

A Condition for a Sasakian Manifold to Be of Constant Curvature

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Abstract: In this paper we give a generalization of some results due to T. Takahashi [5] and M. Okumura [4]. Explicitly, we study an equation of the form R(X,Y)A=0 where X,Y are arbitrary vector fields on a Sasakian manifold and A a (1,3)-tensor field which generalizes the Riemann curvature tensor, Weyl conformal curvature tensor, Weyl projective curvature tensor and Yano concircular curvature tensor. The result which we obtain says that in complementary conditions the manifold is of constant curvature.

Keywords: Riemann curvature tensor, Yano concircular curvature tensor, constant curvature

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1. Introduction

In this paper we give a generalization of some results due to T. Takahashi [5] and M. Okumura [4]. Explicitly, we study an equation of the form R(X,Y)A=0 where X,Y are arbitrary vector fields on a Sasakian manifold and A a (1,3)-tensor field which generalizes the Riemann curvature tensor, Weyl conformal curvature tensor, Weyl projective curvature tensor and Yano concircular curvature tensor. The result which we obtain says that in complementary conditions the manifold is of constant curvature.

2. Preliminaries

Let $M=M^m$ a C^∞-differentiable connected manifold with dim $M=m=2n+1$.

Definition M is said to have an almost contact structure if it admits a field of endomorphisms, named φ, of the tangent spaces, a vector field ξ and a 1-form η satisfying:

$$\eta(\xi)=1$$

$$\eta(\varphi X)=0 \ \forall X\in \mathcal{X}(M)$$

$$\varphi^2X=-X+\eta(X)\xi \ \forall X\in \mathcal{X}(M)$$

$$\varphi\xi=0$$

Definition M has an almost contact metric structure if it admits an almost contact structure (φ,ξ,η) and a Riemannian metric g such that:

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y) \ \forall X,Y\in \mathcal{X}(M)$$

$$g(X,\xi)=\eta(X) \ \forall X\in \mathcal{X}(M)$$

where ∇ is the Levi-Civita connection corresponding to g.

Definition A manifold M with a normal contact metric structure is called Sasakian manifold.

On a Sasakian manifold we have:

(1) $R(X,Y)\xi=\eta(Y)X-\eta(X)Y \ \forall X,Y\in \mathcal{X}(M)$

(2) $R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X \ \forall X,Y\in \mathcal{X}(M)$

In what follows all the tensors are defined locally and we shall consider only the restrictions of tensor fields on neighborhoods of the manifold points.

Let $E_1,...,E_m$ an orthonormal basis for $\mathcal{X}(M)$ and let $\lambda^1,...,\lambda^m$ be the dual 1-forms for E_i, i=1,...,m. We have $\lambda^i(E_j)=\delta^i_j$, i,j=1,...,m. We define the Ricci tensor Ric: $\mathcal{X}(M)^2 \to \mathcal{F}(M)$:

$$Ric(X,Y)=\sum_{i=1}^{m}\lambda^i(R(E_i,X)Y) \quad \forall X,Y\in \mathcal{X}(M)$$

and the Ricci operator ric:$\mathcal{X}(M)\to\mathcal{X}(M)$: g(ric X,Y)=g(X,ric Y)=Ric(X,Y) $\forall X,Y\in \mathcal{X}(M)$.

The scalar of curvature is:

$$S=\sum_{i=1}^{m}Ric(E_i,E_i)=\sum_{i=1}^{m}\lambda^i(ricE_i)$$

From (2) we have on a Sasakian manifold:

(3) $Ric(\xi,X)=(m-1)\eta(X) \ \forall X\in \mathcal{X}(M)$

(4) $ric\ \xi=(m-1)\xi$

Let (M,g) a Riemannian manifold and $p\in M$. Let $X,Y\in T_pM$ independent vectors. We define the sectional curvature k(X,Y) by

$$k(X,Y)=\frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y)-g(X,Y)^2}$$

If k is independent of the choice of $p\in M$ and $X,Y\in T^pM$ we call M with constant curvature.

Theorem (F.SCHUR) A connected Riemannian manifold M with dim M≥3 for which the sectional curvature is constant at every point has constant curvature. In this case, we have:

(5) $R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y] \ \forall X,Y,Z\in \mathcal{X}(M)$,

k being the constant curvature.

On a Sasakian manifold we have ([1]) that $k(X,\xi)=1$ for $X\perp\xi$ therefore if a Sasakian manifold has constant curvature these must be 1.

3. Main theorem

Let $A:\mathcal{X}(M)^3\to\mathcal{X}(M)$,

(6) $A(X,Y)Z=R(X,Y)Z-a[Ric(Y,Z)X-Ric(X,Z)Y]-b[g(Y,Z)ricX-g(X,Z)ricY]+ c[g(Y,Z)X-g(X,Z)Y]\ \forall X,Y,Z\in \mathcal{X}(M),\ \forall a,b,c\in \mathcal{F}(M),\ a\neq -1.$

- For a=b=c=0 we obtain the Riemann curvature tensor
- For $a=\frac{1}{m-1}$, b=c=0 we have the Weyl projective tensor P
- For a=b=0, $c=-\frac{S}{m(m-1)}$ we have the Yano concircular curvature tensor
- For $a=b=\frac{1}{m-2}$, $c=\frac{S}{(m-1)(m-2)}$ we obtain the Weyl conformal curvature tensor.

If M is a Sasakian manifold, we have:

(7) $A(\xi,Y)Z=[1+c-b(m-1)]g(Y,Z)\xi+[a(m-1)-1-c]\eta(Z)Y-a\cdot Ric(Y,Z)\xi+b\cdot\eta(Z)\cdot ricY\ \forall Y,Z\in \mathcal{X}(M)$

(8) $A(\xi,Y)\xi=[1+c-(a+b)(m-1)]\eta(Y)\xi+[a(m-1)-1-c]Y+b\cdot ricY\ \forall Y\in \mathcal{X}(M)$

From [3] we know that we can consider R(X,Y) operating on the tensor algebra like a derivation. Therefore, let the equation:

(9) $R(X,Y)A=0\ \forall X,Y\in \mathcal{X}(M)$

From (6),(9) we have:

(10) $0=(R(X,\xi)A))(\xi,Y)Z=R(X,\xi)A(\xi,Y)Z-A(R(X,\xi)\xi,Y)Z-A(\xi,R(X,\xi)Y)Z- A(\xi,Y)R(X,\xi)Z\ \forall X,Y,Z\in \mathcal{X}(M)$

Using now (1),(2),(3),(4),(6),(7),(8) we have that (10) becomes:

(11) $R(X,Y)Z=-a(m-1)\eta(Y)g(X,Z)\xi-(a-b)(m-1)\eta(Z)g(X,Y)\xi+(1+b-bm)g(Y,Z)X- (1+a-am)g(X,Z)Y+b(m-1)\eta(Y)\eta(Z)X-b\eta(Y)\eta(Z)ricX+(a-b)\eta(Z)Ric(X,y)\xi+ a\eta(Y)Ric(X,Z)\xi-aRic(X,Z)Y+bg(Y,Z)ricX$

If in (11) we put $X=E_i$, apply λ^i in both sides of the equality and summing for i=1,...,m, we obtain:

(12) $(a+1)Ric(Y,Z)=b\eta(Y)\eta(Z)(m^2-m-S)+g(Y,Z)[(m-1)(a+1)-b(m^2-m-S)]\ \forall Y,Z\in \mathcal{X}(M)$

Introducing (12) in (11) we obtain:

(13) $R(X,Y)Z=\frac{2b(a-b)(m^2-m-S)}{a+1}\eta(X)\eta(Y)\eta(Z)\xi+[g(Y,Z)X-$

$g(X,Z)Y]+\frac{b(m^2-m-S)}{a+1}[b(\eta(Y)\eta(Z)X-g(Y,Z)X+\eta(X)g(Y,Z)\xi)-a(\eta(X)\eta(Z)Y-$

$g(X,Z)Y+\eta(Y)g(X,Z)\xi)-(a-b)\eta(Z)g(X,Y)\xi]$

If in (13) we replace b=0, we obtain:

(14) $R(X,Y)Z=g(Y,Z)X-g(X,Z)Y$

therefore M is of constant curvature.

If a=b we have from (13):

(15) $R(X,Y)Z=\left(1-\frac{a(m^2-m-S)}{a+1}\right)[g(Y,Z)X-g(X,Z)Y]-$

$a^2\frac{a(m^2-m-S)}{a+1}[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X-\eta(X)g(Y,Z)\xi+\eta(Y)g(X,Z)\xi]$

If in (15) we have S=m(m-1) we obtain (14) therefore M is of constant curvature 1.

If S=m(m-1) in (13) we obtain also (14).

Let suppose now that M is of constant curvature. From (13) and (14) we have:

(16) $\frac{b(m^2-m-S)}{a+1}[(a-b)\eta(Z)(2\eta(X)\eta(Y)-g(X,Y))\xi+b(\eta(Y)\eta(Z)X-g(Y,Z)X+$
$\eta(X)g(Y,Z)\xi)-a(\eta(X)\eta(Z)Y-g(X,Z)Y+\eta(Y)g(X,Z)\xi)]=0$

Suppose now that $b\neq 0$, $S\neq m^2-m$. If in (16) we take X=Y such that $\eta(X)=0$ and $Z=\xi$, $X\neq 0$ we have:

(16) $-(a-b)g(X,X)\xi=0$

therefore a=b. But from the preceding discussion we must have $S==m^2-m$. Contradiction. We can conclude:

Theorem 1 A Sasakian manifold with $R(X,Y)A=0$ $\forall X,Y\in \mathcal{X}(M)$ is of constant curvature if and only if b=0 or $b\neq 0$ but $S=m^2-m$ where m=dim M. The constant curvature is 1.

Corollary 1 ([5]) A Sasakian manifold satisfying R(X,Y)R=0 $\forall$X,Y$\in$ *X*(M) is of constant curvature.

Corollary 2 A Sasakian manifold with R(X,Y)P=0 where P is the Weyl projective curvature tensor is of constant curvature 1.

Corollary 3 A Sasakian manifold projectively flat is of constant curvature 1.

Corollary 4 A Sasakian manifold with R(X,Y)K=0 where K is the Yano concircular curvature tensor is of constant curvature 1.

Corollary 5 Sasakian manifold with R(X,Y)C=0 and $S=m^2-m$ where C is the Weyl conformal curvature tensor is of constant curvature 1.

Corollary 6 ([4]) A Sasakian manifold conformally flat is of constant curvature (if it has S=m(m-1)).

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