Miscellaneous

Two Methods of Determination of an Acquisition Program in Integer Numbers

Associate Professor Catalin Angelo Ioan, PhD
Danubius University of Galati
catalin_angelo_ioan@univ-danubius.ro

Gina Ioan
Danubius University of Galati
gina_ioan@univ-danubius.ro

Abstract: This paper solves the problem of maximization of the total utility in integer numbers, using first the analysis of integer cases and after with the linear programming in integer numbers.

Keywords: utility; maximization; Simplex; Gomory

JEL Classification: C65; C81; C82

1. Introduction

In the process of acquisition of goods, the first principal problem is those of maximizing the utility under a budget constraint. Let therefore two goods A and B with corresponding prices p_A and p_B and the total budget for acquisition V. The utility function is $U:\mathbf{R}_+^2\to\mathbf{R}$, $(x,y)\to U(x,y)$ where x and y are the quantities of A and B respectively. The classical conditions for maximizing the utility are:

$$\begin{cases} \max U(x,y) \\ p_A x + p_B y \le V \\ x, y \ge 0 \end{cases}$$

It is shown that, if the goods are perfectly divisible, the extreme of U is reached on the straight line: $p_A x+p_B y=V$.

The question which arises is those related to the indivisibility of A and B. Before to formulate the new problem, let suppose that the prices p_A, p_B and the budget V are integers. This is not a forced assumption because after a multiply with a convenient factor this supposition becomes true.

The problem is therefore:

$$\begin{cases} \max U(x,y) \\ p_A x + p_B y \le V \\ x, y \in \mathbf{N} \end{cases}$$

For example, let consider the problem of acquision where p_A=11, p_B=10, V=184 and the utility function is one of Cobb-Douglas type: $U(x,y)=x^{0,37}y^{0,63}$. The classical solution is: x=6.189 and y=11.592 with the maximal utility U_t=9.190. At a first sight, we can take the round values: x_1=6 and y_1=12 with U_{t1}=8,243. But the necessary budget is $V_1=11\cdot6+10\cdot12=186>V$ therefore the combination is not good. Another allocation is find on integer values of x and y, that is: x_2=6 and y_2=11 with U_{t2}=8,790 and $V_2=11\cdot6+10\cdot11=176<V$.

In what follows, we shall see that neither this solution is acceptable.

2. The Allocation of an Integer Number of Goods in order to Maximize the Total Utility

Let $R=\{(x,y) \mid x,y\ge0, p_Ax+p_By\le V\}$ the budget region and $R_{\mathbf{N}}=R\cap\mathbf{N}\times\mathbf{N}$ the restriction of R to all pairs of positive integer coordinates.

Because $p_Ax+p_By\le V$ we have: $x\le\frac{V}{p_A}$ and $y\le\frac{V}{p_B}$ therefore $(x,y)\in R_{\mathbf{N}}$ implies that: $x\in\left[0,\left[\frac{V}{p_A}\right]\right]\cap\mathbf{N}$ (*where [a] is the greatest integer less than a*) and $y\in\left[0,\left[\frac{V}{p_B}\right]\right]\cap\mathbf{N}$.

For a value $k\in\mathbf{N}$ such that $0\le k\le\left[\frac{V}{p_A}\right]$, we find that $y\le\left[\frac{V-kp_A}{p_B}\right]$.

Let now $y\le\left[\frac{V-kp_A}{p_B}\right]$ such that $y+1\le\left[\frac{V-kp_A}{p_B}\right]$. Because the utility is an increasing function with respect to x and y we shall have U(x,y+1)>U(x,y) therefore (x,y+1) will be preferred to (x,y).

After these considerations we have the answer for the problem. We must compute all the values $U\left(k,\left[\frac{V-kp_A}{p_B}\right]\right)$, $0\le k\le\left[\frac{V}{p_A}\right]$ and choose the greatest.

Example For the problem presented in introduction, we shall obtain the best solution: x=4, y=14, U_t=8,807 and V=184 which differ essentially from the above.

3. The Simplex Algorithm for the Allocation of an Integer Number of Goods

Let a consumer which has a budget V of acquisition of two goods A and B. The corresponding prices of A and B are p_A and p_B respectively. The utility function is U: $\mathbf{R}_+^2 \to \mathbf{R}$, $(x,y)\to U(x,y)$ where x and y are the quantities of A and B respectively. Let also the marginal utilities $U_{mA}=\frac{\partial U}{\partial x}$ and $U_{mB}=\frac{\partial U}{\partial y}$ which generate for each integer value k of x and p of y the corresponding values $u_{Ak}=\frac{\partial U}{\partial x}(k)$ and $u_{Bp}=\frac{\partial U}{\partial y}(p)$ respectively. Let note also a – the number of goods A and b – the number of goods B taking into account in a consumer's plan.

Because the total utility is the sum of the marginal utilities, we shall search the maximum of the function: $U_t=\sum_{i=1}^{a} u_{Ai} + \sum_{j=1}^{b} u_{Bj}$.

Let note: $x_{Ai} = \begin{cases} 1 \text{ if the i-th dose from the good A is used} \\ 0 \text{ if the i-th dose from the good A is not used} \end{cases}$

and also for B: $x_{Bi} = \begin{cases} 1 \text{ if the i-th dose from the good B is used} \\ 0 \text{ if the i-th dose from the good B is not used} \end{cases}$

Because the impossibility of using the (i-1)-th dose involved the existence's impossibility of the i-th dose, we shall put the condition that: $x_{Ai},x_{Bi}\in\mathbf{N}$, $0\le x_{Ai}\le x_{Ai-1}$, $0\le x_{Bi}\le x_{Bi-1}$ for i>1.

We have also: $\sum_{i=1}^{\left[\frac{V}{p_A}\right]} p_A x_{Ai} + \sum_{j=1}^{\left[\frac{V}{p_B}\right]} p_B x_{Bj} \le V$.

The problem consists in the determination of x_{Ai},x_{Bi} such that $\max\left(\sum_{i=1}^{\left[\frac{V}{p_A}\right]} u_{Ai} x_{Ai} + \sum_{j=1}^{\left[\frac{V}{p_B}\right]} u_{Bj} x_{Bj}\right)$.

The problem is therefore:

$$(1)\qquad \begin{cases} \max\left(\sum_{i=1}^{\left[\frac{V}{p_A}\right]} u_{Ai}x_{Ai} + \sum_{j=1}^{\left[\frac{V}{p_B}\right]} u_{Bj}x_{Bj}\right) \\ \sum_{i=1}^{\left[\frac{V}{p_A}\right]} p_A x_{Ai} + \sum_{j=1}^{\left[\frac{V}{p_B}\right]} p_B x_{Bj} \le V \\ x_{Ai} \le x_{Ai-1},\ x_{Bj} \le x_{Bj-1} \\ x_{Ai} \le 1,\ x_{Bj} \le 1 \\ x_{Ai} \ge 0,\ x_{Bj} \ge 0 \end{cases}$$

Finally we shall have: a= $\sum_{i=1}^{\left[\frac{V}{p_A}\right]} x_{Ai}$ and b= $\sum_{j=1}^{\left[\frac{V}{p_B}\right]} x_{Bj}$.

Because the problem (1) is in integer numbers, we shall apply the algorithm of Gomory.

After the solving of (1), using the Simplex algorithm, we shall have two cases:

Case 1

If $\overline{x}_{Ai}, \overline{x}_{Bj} \in \mathbf{N}$, $i=\overline{1,\left[\frac{V}{p_A}\right]}$, $j=\overline{1,\left[\frac{V}{p_B}\right]}$ the problem is completely solved.

Case 2

For the simplicity, let note the good A with the index 1 and B with 2.

If $\exists\, \overline{x}_{kp} \notin \mathbf{N}$, the variable $\overline{x}_{kp}$ is obvious in the basis.

In this case, let note y_{kpts} the element of the Simplex table at the intersection of x_{kp}-row with x_{ts}-column. In order to simplify the notations, let: $v_{kpts}=\{y_{kpts}\}\in[0,1)$, $v_{kp}=\{\overline{x}_{kp}\}\in[0,1)$ the fractional part of these quantities, $\mathcal{B}=\{(g,h) \mid x_{gh}$ is a basis variable$\}$ and $\mathcal{S}=\{(t,s) \mid x_{ts}$ is not a basis variable$\}$.

We have now, from: $x_{gh} = \overline{x}_{gh} - \sum_{(t,s)\in\mathcal{S}} y_{ghts}x_{ts}$ $\forall(g,h)\in\mathcal{B}$:

$$(2)\qquad x_{kp} = \overline{x}_{kp} - \sum_{(t,s)\in\mathcal{S}} y_{kpts}x_{ts} = \left[\overline{x}_{kp}\right] + v_{kp} - \sum_{(t,s)\in\mathcal{S}} \left[y_{kpts}\right]x_{ts} - \sum_{(t,s)\in\mathcal{S}} v_{kpts}x_{ts}$$

We can write (2) also in the form:

$$(3)\qquad x_{kp} - [\bar{x}_{kp}] + \sum_{(t,s)\in S}[y_{kpts}]x_{ts} = v_{kp} - \sum_{(t,s)\in S} v_{kpts}x_{ts}$$

In order that the problem has integer solution it therefore necessary and sufficient that: $x_{kp} - [\bar{x}_{kp}] + \sum_{(t,s)\in S}[y_{kpts}]x_{ts} \in \mathbf{Z}$ or, in other words: $v_{kp} - \sum_{(t,s)\in S} v_{kpts}x_{ts} \in \mathbf{Z}$.

Let now:

$$(4)\qquad v = v_{kp} - \sum_{(t,s)\in S} v_{kpts}x_{ts}$$

from where:

$$(5)\qquad \sum_{(t,s)\in S} v_{kpts}x_{ts} = v_{kp} - v,\ v \in \mathbf{Z}$$

From the hypothesis, v_{kpts}, $v_{kp} \in [0,1)$ and $\sum_{(t,s)\in S} v_{kpts}x_{ts} \geq 0$ from the positive character of variables.

We have now three cases:

Case 2.1.

If $v>0$ we have $v\in \mathbf{N}^*$ therefore $0 \leq \sum_{(t,s)\in S} v_{kpts}x_{ts} = v_{kp} - v$. From this: $v_{kp} \geq v \geq 1$ – contradiction with the choice of v_{kp}.

Case 2.2.

If $v=0$ we have that $\sum_{(t,s)\in S} v_{kpts}x_{ts} = v_{kp} \geq v_{kp}$.

Case 2.3.

If $v<0$ we have from the condition that v is integer: $v \leq -1$ which implies: $-v \geq 1$. Finally: $\sum_{(t,s)\in S} v_{kpts}x_{ts} = v_{kp} - v \geq v_{kp} + 1 > v_{kp} > 0$.

From these cases, we have that the condition to be integer for x_{kp} is: $\sum_{(t,s)\in S} v_{kpts}x_{ts} \geq v_{kp}$.

After all these considerations, making the notation: $y = \sum_{(t,s)\in S} v_{kpts}x_{ts} - v_{kp}$ we shall obtain the new problem:

$$(6)\quad \begin{cases} \max\left(\sum_{i=1}^{\left[\frac{V}{p_A}\right]} u_{Ai}x_{Ai} + \sum_{j=1}^{\left[\frac{V}{p_B}\right]} u_{Bj}x_{Bj}\right) \\ \sum_{i=1}^{\left[\frac{V}{p_A}\right]} p_A x_{Ai} + \sum_{j=1}^{\left[\frac{V}{p_B}\right]} p_B x_{Bj} \le V \\ y - \sum_{(t,s)\in S} v_{kpts}x_{ts} = -v_{kp} \\ x_{Ai} \le x_{Ai-1},\, x_{Bj} \le x_{Bj-1} \\ x_{Ai} \le 1,\, x_{Bj} \le 1 \\ x_{Ai} \ge 0,\, x_{Bj} \ge 0 \end{cases}$$

If the problem (6) will has at finally an integer solution the problem will be completely solved. If not, we shall resume the upper steps.

4. References

Ioan, C.A. (2009). *A method of the determination of an acquisition program for the maximization of the total utility*, The European Integration – Realities and Perspectives" Galati: Editura Universitara Danubius.

Ioan, C.A. & Ioan, G. (2010). *Applied mathematics in micro and macroeconomics*. Galati: Sinteze.

Ioan, C.A. & Ioan, G. (2010). *Mathematical models and methods in economical activity*. Galati: Sinteze.

Mas-Collel, A.; Whinston, M.D. & Green, J.R. (1995). *Microeconomic Theory*. New York: Oxford University Press.

Pindyck, R.S. & Rubinfeld, D.L. (1996). *Microeconomics*. Prentice-Hall International.