The Analysis of the Evolution of the Gross Domestic Product by Means of Fourier Development

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Abstract. In this article, we will carry out an analysis on the regularity of the Gross Domestic Product of a country, in our case the United States. The method of analysis is based on the consideration of the development in the Fourier series of a function and testing in terms of the average absolute error of the nearest polynomial Fourier of real data are considered. The obtained results show a cycle for 13 years, the average absolute error being 3.69%. The method described allows an prognosis on short-term trends in GDP.

Keywords. GDP; cycle; Fourier

JEL Classification: C65, E17

1. Introduction

In the literature, the economic cycle designate the fluctuations which accompany the evolution of a nation or, sometimes, it simply is associated with the increasing and decreasing of an economy. Throughout history, many states were faced and have experienced economic fluctuations, most tested being the United States.

Given the complexity of economic phenomena, in practice there are as many types of economic cycles or economic fluctuations. We can say that almost any segment of the economic life is subject to the fluctuations that, sometimes, may include periods of more than a year.

According to literature, the theoretical economic cycle is linked on the one hand, by changes in aggregate demand with all components (public consumers, private consumers, investors) or, on the other hand, of the change in supply aggregates (changes in production costs).

A more comprehensive approach to the problem of the economic cycle requires knowledge of all aspects of the market economy.

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AUDŒ, Vol 7, no 4, pp. 142--157

Regardless of the factors that have influenced and favored economic cycles, their approach involves different points of view.

The first analysis of the economic cycle through the prism of the phenomenon of recurrence is due to the French economist Clement Juglar, who has studied the fluctuations of the interest rate and price and on the basis of which was discovered in 1860 an economic cycle with alternate periods of prosperity and depression for 8-11 years.

Economists who have a thorough analysis of Clement Juglar's cycle and, in particular Joseph Schumpeter, have concluded that in it there are four phases: the expansion, the crisis, recession and the renascence.

At the beginning of the 20th century, Joseph Kitchin based on analyses of interest rates and other variables (the analysis being performed on the economies of the United States of America and United Kingdom) discovers a short economic cycle, approximately 40 months.

The economists, after the Great Depression in the years 1929-1933, have focused much more on macroeconomic phenomena that determine the appearance of the economic cycle, looking for patterns of prediction.

In the "The Major Economic Cycles", which appeared in 1925, the Russian Economist Nikolai Kondratieff mark out an economic cycle much longer, about 50-60 years. On the basis of statistical researches on long-term fluctuations in prices (the analysis being performed on the same economies of the United States of America and United Kingdom), Kondratieff observed periods of accelerated growth of branches of Economics, alternate with slower growth. Within this cycle, Kondratieff identified the expansion phase, the phase of stagnation and recession phase. Without finding a universally accepted explanation, he believes that the basis of these cycles long stay technological progress, confirmed later by Schumpeter, which considers "the bunch of related innovations" that generates each cycle.

Other analysis devoted to the economic cycle have been made by Wesley Clair Mitchell in the work "Business Cycle" (1913) and "Measuring Business Cycles" (1927) in which the author discusses some methods of determination and analysis of economic cycle. Mitchell puts emphasis on the differences between the capitalist societies and the pre-capitalist, considering that a course of business would not be possible in a society pre-capitalist, but can occur in one capitalist ([1]).

John Maynard Keynes - the economist of the Great Depression, lay the groundwork for a new economic theory which reveals a close connection between consumption and investment. According to the Keynesian theory and its adherents, any additional expenditure (consumption) generates an income a few times higher than the expenditure itself. This relationship between consumption and investment, known as the investment multiplier, can not produce, considered Keynes, cyclical movements in the economy, but it can lead to an upward trend.

Russian research economist Simon Kuznets, in 1930 put the bases of a cycle lasting on average, over a period of 15-20 years, called “demographic cycle” or “the cycle of investment in infrastructure”. Kuznets considers that a factor that influence the emergence and evolution of an economic cycle is the demographic processes, in particular the phenomenon of migration having disturbing effects in the buildings sector.

The Austrian School sees the economic cycle through its representatives, notably to Ludwig von Mises, as a natural consequence of the massive growth of bank credit, an inappropriate monetary policy conducive to relaxing the conditions of crediting and finally the accumulation of toxic assets. Growth of loans generates, in turn, a rise in prices and a fall in interest rates below the optimum level, and the crisis occurs when manufacturers can’t sell the production because of the very high prices. In the same stream of thought, Friedrich Hayek considers the phenomenon of over-investment as a factor determining the onset of a new economic cycle, while Joseph Schumpeter considers that the emergence and the onset of the economic cycle is based on the existence of investments with high efficiency carried out in a short period and a low demand for new products.

After attempts at explanation of the economic cycle from the early 1970’s of Milton Friedman and Robert Lucas, the work of Finn E. Kydland and Edward C. Prescott “Time to Build And Aggregate Fluctuations” ([3]) launches real business cycle theory, the economic cycles being determined by the fluctuations in the rate of growth of total productivity of factors of production.

Over time, many economists have attempted, through analysis of available statistical data, to develop specific models of foresights of changes taking place in the economy to come to the aid of the decision-makers to act according to actual economic conditions.

The objective of this paper is to determine a possible historical influence on the evolution of GDP in strictly numerical terms. For this, we will consider data sets of given length, then determining the corresponding Lagrange polynomial interpolation. Considering the function resulting from pasting the above functions, we will build the Fourier development of the different values of periodicity and having starting point an arbitrary value. The period appropriate to the smallest average absolute error between the values and actual Fourier will give an indication of a possible periodicity of the phenomenon.

2. Mathematical Considerations on the Fourier Development

Let a function f:**R**→**R**, with f and f' piecewise continous on **R** and periodic with period T, therefore f(x+T)=f(x) ∀x∈ **R**.

Considering Fourier series associated with the function f: F(x)= $\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{2k\pi x}{T}+b_k\sin\frac{2k\pi x}{T}\right)$ we have the following:

Lemma 1 $\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\frac{2n\pi x}{T}dx=\frac{a_nT}{2}$, n≥0, $\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\frac{2n\pi x}{T}dx=\frac{b_nT}{2}$, n≥1.

Proof. $\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\frac{2n\pi x}{T}dx=$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\frac{2m\pi x}{T}+b_m\sin\frac{2m\pi x}{T}\right)\right)\cos\frac{2n\pi x}{T}dx=$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{a_0}{2}\cos\frac{2n\pi x}{T}+\sum_{m=1}^{\infty}\left(a_m\cos\frac{2m\pi x}{T}\cos\frac{2n\pi x}{T}+b_m\sin\frac{2m\pi x}{T}\cos\frac{2n\pi x}{T}\right)dx=$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{a_0}{2}\cos\frac{2n\pi x}{T}dx+\sum_{m=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}a_m\cos\frac{2m\pi x}{T}\cos\frac{2n\pi x}{T}dx+\sum_{m=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}b_m\sin\frac{2m\pi x}{T}\cos\frac{2n\pi x}{T}dx$$

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If n=0, then: $\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\frac{2n\pi x}{T}dx=$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{a_0}{2}dx+\sum_{m=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}a_m\cos\frac{2m\pi x}{T}dx+\sum_{m=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}b_m\sin\frac{2m\pi x}{T}dx=$$

$$\frac{a_0T}{2}+\sum_{m=1}^{\infty}a_m\frac{T}{2m\pi}\sin\frac{2m\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}-\sum_{m=1}^{\infty}b_m\frac{T}{2m\pi}\cos\frac{2m\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}=$$

$$\frac{a_0T}{2}+\sum_{m=1}^{\infty}a_m\frac{T}{2m\pi}2\sin m\pi=\frac{a_0T}{2}.$$

If n≥1 then:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos\frac{2n\pi x}{T}dx=\frac{a_0}{2}\frac{T}{2n\pi}\sin\frac{2n\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}+\sum_{m=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{a_m}{2}\left(\cos\frac{2(m+n)\pi x}{T}+\cos\frac{2(m-n)\pi x}{T}\right)dx+$$

$$\sum_{m=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{b_m}{2}\left(\sin\frac{2(m+n)\pi x}{T}+\sin\frac{2(m-n)\pi x}{T}\right)dx=$$

$$\frac{a_0}{2}\frac{T}{2n\pi}2\sin n\pi+\sum_{m=1}^{\infty}\frac{a_m}{2}\frac{T}{2(m+n)\pi}\sin\frac{2(m+n)\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}+\sum_{\substack{m=1\\m\neq n}}^{\infty}\frac{a_m}{2}\frac{T}{2(m-n)\pi}\sin\frac{2(m-n)\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}+\frac{a_nT}{2}-$$

$$\sum_{m=1}^{\infty}\frac{b_m}{2}\frac{T}{2(m+n)\pi}\cos\frac{2(m+n)\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}-\sum_{\substack{m=1\\m\neq n}}^{\infty}\frac{a_m}{2}\frac{T}{2(m-n)\pi}\cos\frac{2(m-n)\pi x}{T}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}=$$

$$+\sum_{m=1}^{\infty}\frac{a_m}{2}\frac{T}{2(m+n)\pi}2\sin(m+n)\pi+\sum_{\substack{m=1\\m\neq n}}^{\infty}\frac{a_m}{2}\frac{T}{2(m-n)\pi}\sin 2(m-n)\pi+\frac{a_nT}{2}=\frac{a_nT}{2}.$$

The proof is analogous for the other claim. **Q.E.D.**

From Fourier series expression, it is observed that $F(x+T)=F(x)\ \forall x\in\mathbf{R}$ so its sum is also a periodic function of period T.

The Dirichlet's theorem (Spiegel, 1974) states that in the conditions above, the Fourier series converges punctually to f in the points of continuity and to $\frac{f(x+0)+f(x-0)}{2}$ in the discontinuity points.

Considering the partial sum of order n corresponding to the series of function F, the n-th Fourier polynomials are:

$$F_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos\frac{2k\pi x}{T}+b_k\sin\frac{2k\pi x}{T}\right)$$

It is obvious also that $F_n(x)=F_n(x+T)\ \forall x\in \mathbf{R}$.

The Fourier polynomials have the property of approximating the function through one periodical with the observation that the absolute error tends to fall (due to the convergence points) with the rise of n.

Due to the existence of an important number of cyclical phenomena in many scientific fields, we intend, below, to approximate their development by means of Fourier polynomials of degree conveniently chosen.

In the case of the discretized phenomenons, we put the problem in the generation of functions that will pass through a series of data points. A very useful tool is the Lagrange interpolation polynomial. Therefore, considering a set of data (x_i,y_i), i= $\overline{1,k+1}$, the Lagrange interpolation polynomial has the form:

$$L_n(x)=\sum_{i=1}^{k+1}\frac{(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}y_i$$

and is the polynomial of minimum degree (k) passing through the data points.

We will demonstrate, first, the following:

Lemma 2 Let $f(x)=a_kx^k+...+a_0\in \mathbf{R}[X]$. Then:

$$\int f(x)\cos\frac{2n\pi x}{T}dx=2(-1)^n\sum_{\substack{i=1\\ i=\text{impar}}}^{k-1}\left(\frac{T}{2}\right)^i\sum_{j=0}^{\left[\frac{k-i}{2}\right]}(-1)^j(2j+1)!C_{i+1+2j}^{2j+1}\left(\frac{2n\pi}{T}\right)^{2j+2}a_{i+1+2j}$$

$$\int f(x)\sin\frac{2n\pi x}{T}dx=2(-1)^n\sum_{\substack{i=1\\ i=\text{impar}}}^{k}\left(\frac{T}{2}\right)^i\sum_{j=0}^{\left[\frac{k-i}{2}\right]}(-1)^{j+1}(2j)!C_{i+2j}^{2j}\left(\frac{2n\pi}{T}\right)^{2j+1}a_{i+2j}$$

Proof. Let first: $\int\sum_{i=0}^{k}a_ix^i\cos\frac{2n\pi x}{T}dx=\sum_{i=0}^{k}b_ix^i\cos\frac{2n\pi x}{T}+\sum_{i=0}^{k}c_ix^i\sin\frac{2n\pi x}{T}$.

If we derivating in both terms:

$$\sum_{i=0}^{k} a_i x^i \cos\frac{2n\pi x}{T} =$$

$$\sum_{i=0}^{k} b_i\left(ix^{i-1}\cos\frac{2n\pi x}{T} - \frac{2n\pi}{T}x^i \sin\frac{2n\pi x}{T}\right) + c_i\left(ix^{i-1}\sin\frac{2n\pi x}{T} + \frac{2n\pi}{T}x^i\cos\frac{2n\pi x}{T}\right)$$

from where: $\sum_{i=0}^{k} a_i x^i \cos\frac{2n\pi x}{T} =$

$$\sum_{i=-1}^{k-1}\left(b_{i+1}(i+1)x^i\cos\frac{2n\pi x}{T} + c_{i+1}(i+1)x^i\sin\frac{2n\pi x}{T}\right) + \sum_{i=0}^{k}\left(-b_i\frac{2n\pi}{T}x^i\sin\frac{2n\pi x}{T} + c_i\frac{2n\pi}{T}x^i\cos\frac{2n\pi x}{T}\right)$$

After the identification of the coefficients, we obtain:

$$\begin{cases} c_i = \dfrac{T}{2n\pi}\left(a_i - (i+1)b_{i+1}\right), i = \overline{0, k-1} \\ b_i = \dfrac{T}{2n\pi}(i+1)c_{i+1}, i = \overline{0, k-1} \\ b_k = 0 \\ c_k = \dfrac{T}{2n\pi}a_k \end{cases}$$

After induction, follows:

$$b_s = \sum_{j=0}^{\left[\frac{k-s-1}{2}\right]} (-1)^j (2j+1)! C_{s+1+2j}^{2j+1}\left(\frac{T}{2n\pi}\right)^{2j+2} a_{s+1+2j}, \ s = \overline{0, k-1}, \ b_k = 0,$$

$$c_s = \sum_{j=0}^{\left[\frac{k-s}{2}\right]} (-1)^j (2j)! C_{s+2j}^{2j}\left(\frac{T}{2n\pi}\right)^{2j+1} a_{s+2j}, \ s = \overline{0, k}.$$

Analogously, let: $\int \sum_{i=0}^{k} a_i x^i \sin\frac{2n\pi x}{T}dx = \sum_{i=0}^{k} d_i x^i \cos\frac{2n\pi x}{T} + \sum_{i=0}^{k} e_i x^i \sin\frac{2n\pi x}{T}$.

Derivating in both terms:

$$\sum_{i=0}^{k} a_i x^i \sin\frac{2n\pi x}{T} =$$

$$\sum_{i=0}^{k} d_i\left(ix^{i-1}\cos\frac{2n\pi x}{T} - \frac{2n\pi}{T}x^i\sin\frac{2n\pi x}{T}\right) + e_i\left(ix^{i-1}\sin\frac{2n\pi x}{T} + \frac{2n\pi}{T}x^i\cos\frac{2n\pi x}{T}\right)$$

from where: $\sum_{i=0}^{k} a_i x^i \sin\frac{2n\pi x}{T} =$

$$\sum_{i=-1}^{k-1}\left(d_{i+1}(i+1)x^i\cos\frac{2n\pi x}{T}+e_{i+1}(i+1)x^i\sin\frac{2n\pi x}{T}\right)+\sum_{i=0}^{k}\left(-d_i\frac{2n\pi}{T}x^i\sin\frac{2n\pi x}{T}+e_i\frac{2n\pi}{T}x^i\cos\frac{2n\pi x}{T}\right)$$

Also, after the identifying the coefficients, we have:

$$\begin{cases} d_i=\frac{T}{2n\pi}\left(-a_i+(i+1)e_{i+1}\right), i=\overline{0,k-1} \\ e_i=-\frac{T}{2n\pi}(i+1)d_{i+1}, i=\overline{0,k-1} \\ e_k=0 \\ d_k=-\frac{T}{2n\pi}a_k \end{cases}$$

After induction, follows:

$$d_s=\sum_{j=0}^{\left[\frac{k-s}{2}\right]}(-1)^{j+1}(2j)!C_{s+2j}^{2j}\left(\frac{T}{2n\pi}\right)^{2j+1}a_{s+2j}\text{, } s=\overline{0,k}\text{,}$$

$$e_s=\sum_{j=0}^{\left[\frac{k-s-1}{2}\right]}(-1)^{j}(2j+1)!C_{s+2j+1}^{2j+1}\left(\frac{T}{2n\pi}\right)^{2j}a_{s+2j+1}\text{, } s=\overline{0,k-1}\text{, } e_k=0$$

We finally have:

$$\int_\alpha^\beta f(x)\cos\frac{2n\pi x}{T}dx=\sum_{i=0}^{k}b_i x^i\cos\frac{2n\pi x}{T}\Big|_\alpha^\beta+\sum_{i=0}^{k}c_i x^i\sin\frac{2n\pi x}{T}\Big|_\alpha^\beta=$$

$$\sum_{i=0}^{k}b_i\left(\beta^i\cos\frac{2n\pi\beta}{T}-\alpha^i\cos\frac{2n\pi\alpha}{T}\right)+\sum_{i=0}^{k}c_i\left(\beta^i\sin\frac{2n\pi\beta}{T}-\alpha^i\sin\frac{2n\pi\alpha}{T}\right).$$

$$\int_\alpha^\beta f(x)\sin\frac{2n\pi x}{T}dx=\sum_{i=0}^{k}d_i x^i\cos\frac{2n\pi x}{T}\Big|_\alpha^\beta+\sum_{i=0}^{k}e_i x^i\sin\frac{2n\pi x}{T}\Big|_\alpha^\beta=$$

$$\sum_{i=0}^{k}d_i\left(\beta^i\cos\frac{2n\pi\beta}{T}-\alpha^i\cos\frac{2n\pi\alpha}{T}\right)+\sum_{i=0}^{k}e_i\left(\beta^i\sin\frac{2n\pi\beta}{T}-\alpha^i\sin\frac{2n\pi\alpha}{T}\right).$$ **Q.E.D.**

From Lemma 2, we shall study a number of particular cases, i.e. for any $\alpha,\beta\in\mathbf{R}$:

k=0 In this case, the function is considered to be constant over the interval $[\alpha,\beta]$, therefore: $f(x)=a_0$.

From the above, it follows: $b_0=0$, $c_0=\frac{T}{2n\pi}a_0$, $d_0=-\frac{T}{2n\pi}a_0$, $e_0=0$ and finally:

$$\int_\alpha^\beta a_0\cos\frac{2n\pi x}{T}dx=\frac{T}{2n\pi}a_0\left(\sin\frac{2n\pi\beta}{T}-\sin\frac{2n\pi\alpha}{T}\right)$$

$$\int_\alpha^\beta a_0\sin\frac{2n\pi x}{T}dx=-\frac{T}{2n\pi}a_0\left(\cos\frac{2n\pi\beta}{T}-\cos\frac{2n\pi\alpha}{T}\right)$$

k=1 In the case of a linear function: $f(x)=a_1x+a_0$, we have: $b_1=0$, $b_0=\left(\frac{T}{2n\pi}\right)^2a_1$, $c_0=\frac{T}{2n\pi}a_0$,
$c_1=\frac{T}{2n\pi}a_1$, $d_0=-\frac{T}{2n\pi}a_0$, $d_1=-\frac{T}{2n\pi}a_1$, $e_0=a_1$, $e_1=0$ therefore:

$$\int_\alpha^\beta (a_1x+a_0)\cos\frac{2n\pi x}{T}dx=$$

$$\left(\frac{T}{2n\pi}\right)^2a_1\left(\cos\frac{2n\pi\beta}{T}-\cos\frac{2n\pi\alpha}{T}\right)+\frac{T}{2n\pi}\sin\frac{2n\pi\beta}{T}(a_1\beta+a_0)-\frac{T}{2n\pi}\sin\frac{2n\pi\alpha}{T}(a_1\alpha+a_0)$$

$$\int_\alpha^\beta (a_1x+a_0)\sin\frac{2n\pi x}{T}dx=$$

$$a_1\left(\sin\frac{2n\pi\beta}{T}-\sin\frac{2n\pi\alpha}{T}\right)-\frac{T}{2n\pi}\cos\frac{2n\pi\beta}{T}(a_1\beta+a_0)+\frac{T}{2n\pi}\cos\frac{2n\pi\alpha}{T}(a_1\alpha+a_0)$$

k=2 If f is polynomial of second degree: $f(x)=a_2x^2+a_1x+a_0$ follows, analogously: $b_2=0$, $b_1=2\left(\frac{T}{2n\pi}\right)^2a_2$, $b_0=\left(\frac{T}{2n\pi}\right)^2a_1$, $c_2=\frac{T}{2n\pi}a_2$, $c_1=\frac{T}{2n\pi}a_1$, $c_0=\frac{T}{2n\pi}a_0-2\left(\frac{T}{2n\pi}\right)^3a_2$, $d_2=-\frac{T}{2n\pi}a_2$, $d_1=-\frac{T}{2n\pi}a_1$, $d_0=-\frac{T}{2n\pi}a_0+2\left(\frac{T}{2n\pi}\right)^3a_2$, $e_2=0$, $e_1=2a_2$, $e_0=a_1$ therefore:

$$\int_{\alpha}^{\beta} f(x)\cos\frac{2n\pi x}{T}dx =$$

$$\left(\frac{T}{2n\pi}\right)^2(a_1+2a_2\beta)\cos\frac{2n\pi\beta}{T}-\left(\frac{T}{2n\pi}\right)^2(a_1+2a_2\alpha)\cos\frac{2n\pi\alpha}{T}+\frac{T}{2n\pi}\left(a_0+a_1\beta+a_2\beta^2\right)\sin\frac{2n\pi\beta}{T}-$$
$$\frac{T}{2n\pi}\left(a_0+a_1\alpha+a_2\alpha^2\right)\sin\frac{2n\pi\alpha}{T}-2\left(\frac{T}{2n\pi}\right)^3 a_2\left(\sin\frac{2n\pi\beta}{T}-\sin\frac{2n\pi\alpha}{T}\right)$$

$$\int_{\alpha}^{\beta} f(x)\sin\frac{2n\pi x}{T}dx =$$

$$-\frac{T}{2n\pi}\left(a_0+a_1\beta+a_2\beta^2\right)\cos\frac{2n\pi\beta}{T}+\frac{T}{2n\pi}\left(a_0+a_1\alpha+a_2\alpha^2\right)\cos\frac{2n\pi\alpha}{T}+(a_1+2a_2\beta)\sin\frac{2n\pi\beta}{T}-$$
$$(a_1+2a_2\alpha)\sin\frac{2n\pi\alpha}{T}+2\left(\frac{T}{2n\pi}\right)^3 a_2\left(\cos\frac{2n\pi\beta}{T}-\cos\frac{2n\pi\alpha}{T}\right)$$

3. The discrete data analysis using Fourier development

Consider a discrete data set: $Y=(y_1,...,y_n)$. Considering a fixed k, $0\le k\le n-1$, we shall consider sequential data sets: $(y_1,...,y_{k+1})$, $(y_{k+2},...,y_{2k+2})$ etc. and we shall build the corresponding Lagrange interpolation polynomial, where the independent variable would be the sequence number of the corresponding date.We build the partial sum of order n (conveniently chosen) corresponding to the series of Fourier functions determined above, where the intervals $[\alpha,\beta]$ will be of the form: [1,k+1], [k+2,2k+2] etc. After Fourier polynomials determinations, the different values of $n\ge 1$, we will select that polynomial such that the absolute average error between the data calculated by periodicity and the actual is the smallest. In the present analysis, we consider the starting point of the data of any year, a period of polynomial Fourier between 10 and 100 years and an order of between 1 and 9.

4. The Analysis of GDP's Cyclicity

In what follows, we intend to study a possible cycle in the evolution of the gross domestic product of a country.

Considering a period of m consecutive years and GDP_k, $k=\overline{1,m}$ the real value of GDP, consider the real GDP growth rate: $r_k=\frac{GDP_k - GDP_{k-1}}{GDP_{k-1}}$. We then have: $GDP_k=(1+r_k)GDP_{k-1}$, $k=\overline{2,m}$.

Consider now, for analysis, gross domestic product of the U.S. in the period 1792-2010:

Table 1

Year	GDP	r_k	Year	GDP	r_k	Year	GDP	r_k	Year	GDP	r_k
1792	4.58		1847	45.21	0.0680369	1902	468.20	0.0514260	1957	2601.10	0.0201592
1793	4.95	0.0807860	1848	46.73	0.0336209	1903	481.80	0.0290474	1958	2577.60	-0.0090346
1794	5.60	0.1313131	1849	47.38	0.0139097	1904	464.80	-0.0352844	1959	2762.50	0.0717334
1795	5.96	0.0642857	1850	49.59	0.0466442	1905	517.20	0.1127367	1960	2830.90	0.0247602
1796	6.15	0.0318792	1851	53.58	0.0804598	1906	538.40	0.0409899	1961	2896.90	0.0233141
1797	6.27	0.0195122	1852	59.76	0.1153415	1907	552.20	0.0256315	1962	3072.40	0.0605820
1798	6.54	0.0430622	1853	64.65	0.0818273	1908	492.50	-0.1081130	1963	3206.70	0.0437118
1799	7.00	0.0703364	1854	66.88	0.0344934	1909	528.10	0.0722843	1964	3392.30	0.0578788
1800	7.40	0.0571429	1855	69.67	0.0417165	1910	533.80	0.0107934	1965	3610.10	0.0642042
1801	7.76	0.0486486	1856	72.47	0.0401895	1911	551.10	0.0324091	1966	3845.30	0.0651505
1802	8.00	0.0309278	1857	72.84	0.0051056	1912	576.90	0.0468155	1967	3942.50	0.0252776
1803	8.14	0.0175000	1858	75.79	0.0404997	1913	599.70	0.0395216	1968	4133.40	0.0484211
1804	8.45	0.0380835	1859	81.28	0.0724370	1914	553.70	-0.0767050	1969	4261.80	0.0310640
1805	8.90	0.0532544	1860	82.11	0.0102116	1915	568.80	0.0272711	1970	4269.90	0.0019006
1806	9.32	0.0471910	1861	83.57	0.0177810	1916	647.70	0.1387131	1971	4413.30	0.0335839
1807	9.33	0.0010730	1862	93.95	0.1242073	1917	631.70	-0.0247028	1972	4647.70	0.0531122
1808	9.35	0.0021436	1863	101.18	0.0769558	1918	688.70	0.0902327	1973	4917.00	0.0579426
1809	10.07	0.0770053	1864	102.33	0.0113659	1919	694.20	0.0079861	1974	4889.90	-0.0055115
1810	10.63	0.0556107	1865	105.26	0.0286329	1920	687.70	-0.0093633	1975	4879.50	-0.0021268
1811	11.11	0.0451552	1866	100.43	-0.0458864	1921	671.90	-0.0229751	1976	5141.30	0.0536530
1812	11.55	0.0396040	1867	102.15	0.0171264	1922	709.30	0.0556630	1977	5377.70	0.0459806

1813	12.21	0.0571429	1868	106.13	0.0389623	1923	802.60	0.1315381	1978	5677.60	0.0557673
1814	12.72	0.0417690	1869	109.02	0.0272308	1924	827.40	0.0308996	1979	5855.00	0.0312456
1815	12.82	0.0078616	1870	112.30	0.0300862	1925	846.80	0.0234469	1980	5839.00	-0.0027327
1816	12.82	0.0000000	1871	117.60	0.0471950	1926	902.10	0.0653047	1981	5987.20	0.0253811
1817	13.12	0.0234009	1872	127.50	0.0841837	1927	910.80	0.0096442	1982	5870.90	-0.0194248
1818	13.60	0.0365854	1873	138.30	0.0847059	1928	921.30	0.0115283	1983	6136.20	0.0451890
1819	13.86	0.0191176	1874	140.80	0.0180766	1929	977.00	0.0604580	1984	6577.10	0.0718523
1820	14.41	0.0396825	1875	140.60	-0.0014205	1930	892.80	-0.0861822	1985	6849.30	0.0413860
1821	15.18	0.0534351	1876	146.40	0.0412518	1931	834.90	-0.0648522	1986	7086.50	0.0346313
1822	15.76	0.0382082	1877	153.70	0.0498634	1932	725.80	-0.1306743	1987	7313.30	0.0320045
1823	16.33	0.0361675	1878	158.60	0.0318803	1933	716.40	-0.0129512	1988	7613.90	0.0411032
1824	17.30	0.0593999	1879	177.10	0.1166456	1934	794.40	0.1088777	1989	7885.90	0.0357241
1825	18.07	0.0445087	1880	191.80	0.0830040	1935	865.00	0.0888721	1990	8033.90	0.0187677
1826	18.71	0.0354178	1881	215.80	0.1251303	1936	977.90	0.1305202	1991	8015.10	-0.0023401
1827	19.29	0.0309995	1882	227.30	0.0532901	1937	1028.00	0.0512322	1992	8287.10	0.0339359
1828	19.55	0.0134785	1883	233.50	0.0272767	1938	992.60	-0.0344358	1993	8523.40	0.0285142
1829	20.30	0.0383632	1884	229.70	-0.0162741	1939	1072.80	0.0807979	1994	8870.70	0.0407467
1830	22.16	0.0916256	1885	230.50	0.0034828	1940	1166.90	0.0877144	1995	9093.70	0.0251389
1831	23.99	0.0825812	1886	249.20	0.0811280	1941	1366.10	0.1707087	1996	9433.90	0.0374105
1832	25.61	0.0675281	1887	267.30	0.0726324	1942	1618.20	0.1845399	1997	9854.30	0.0445627
1833	26.40	0.0308473	1888	282.70	0.0576132	1943	1883.10	0.1637004	1998	10283.50	0.0435546
1834	26.85	0.0170455	1889	290.80	0.0286523	1944	2035.20	0.0807711	1999	10779.80	0.0482618
1835	28.27	0.0528864	1890	319.10	0.0973177	1945	2012.40	-0.0112028	2000	11226.00	0.0413922
1836	29.11	0.0297135	1891	322.80	0.0115951	1946	1792.20	-0.1094216	2001	11347.20	0.0107964
1837	29.37	0.0089316	1892	339.30	0.0511152	1947	1776.10	-0.0089834	2002	11553.00	0.0181366
1838	30.59	0.0415390	1893	319.60	-0.0580607	1948	1854.20	0.0439727	2003	11840.70	0.0249026
1839	31.37	0.0254985	1894	304.50	-0.0472466	1949	1844.70	-0.0051235	2004	12263.80	0.0357327
1840	31.46	0.0028690	1895	339.20	0.1139573	1950	2006.00	0.0874397	2005	12638.40	0.0305452
1841	32.17	0.0225683	1896	333.60	-0.0165094	1951	2161.10	0.0773180	2006	12976.20	0.0267281
1842	33.19	0.0317066	1897	348.00	0.0431655	1952	2243.90	0.0383138	2007	13254.10	0.0214161
1843	34.84	0.0497138	1898	386.10	0.1094828	1953	2347.20	0.0460359	2008	13312.20	0.0043835
1844	36.82	0.0568312	1899	412.50	0.064000	1954	2332.40	-0.0063054	2009	12990.30	-0.0241808

1845	39.15	0.0632808	1900	422.80	0.0249697	1955	2500.30	0.0719859	2010	13038.70	0.0037259
1846	42.33	0.0812261	1901	445.30	0.0532167	1956	2549.70	0.0197576	-	-	-

* PIB-US $ billion 2005

Source: http://www.usgovernmentrevenue.com

By analysing the data set (k,r_k), corresponding to the period 1793-2010, the minimum average absolute error 4.63% is obtained as a result of applying Fourier Analysis for T=13 years and k=1. The qualitative analysis of the recalculated graph, following the result model, does not allow its acceptance, in the sense of Fourier analysis, the model being totally coherent with real data only occasionally.

For this reason, we have chosen for analysis, the differences $s_k=r_k-r_{k-1}$ which signifies the rate of change of rate of growth of real GDP. In this case, the results are spectacular, gaining for the T = 13 years and k = 3 the minimum mean absolute error of 3.69%.

The recalculated values of s_k are:

Table 2

k	s_k	k	s_k	k	s_k
1	0.0149028	6	0.0093186	11	-0.0026318
2	-0.0000676	7	-0.0097969	12	-0.0048919
3	-0.0143265	8	-0.0144272	13	0.0084500
4	-0.0046977	9	-0.0007267		
5	0.0129542	10	0.0059407		

The comparative graphs of the development of s_k and the indicators recomputing after the Fourier regression are:

The Fourier analysis for the coefficients s(k) between 1794-1850

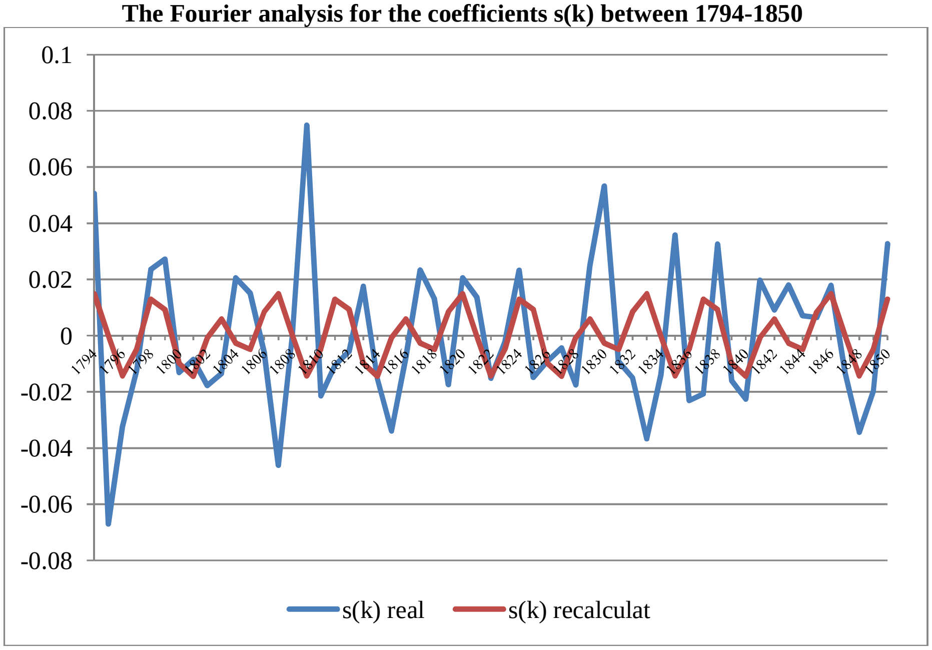

Figure 1

The Fourier analysis for the coefficients s(k) between 1851-1900

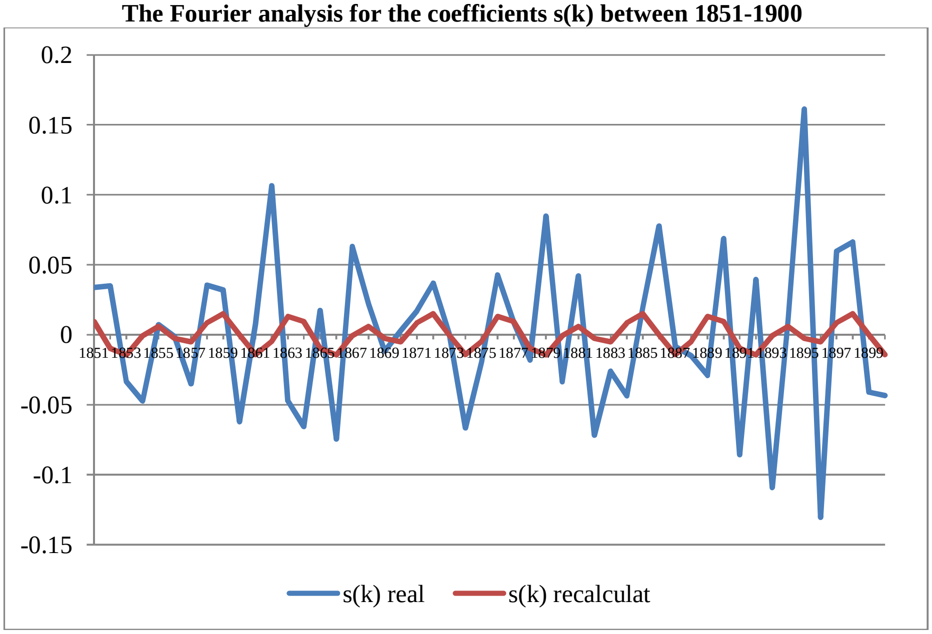

Figure 2

The Fourier analysis for the coefficients s(k) between 1901-1950

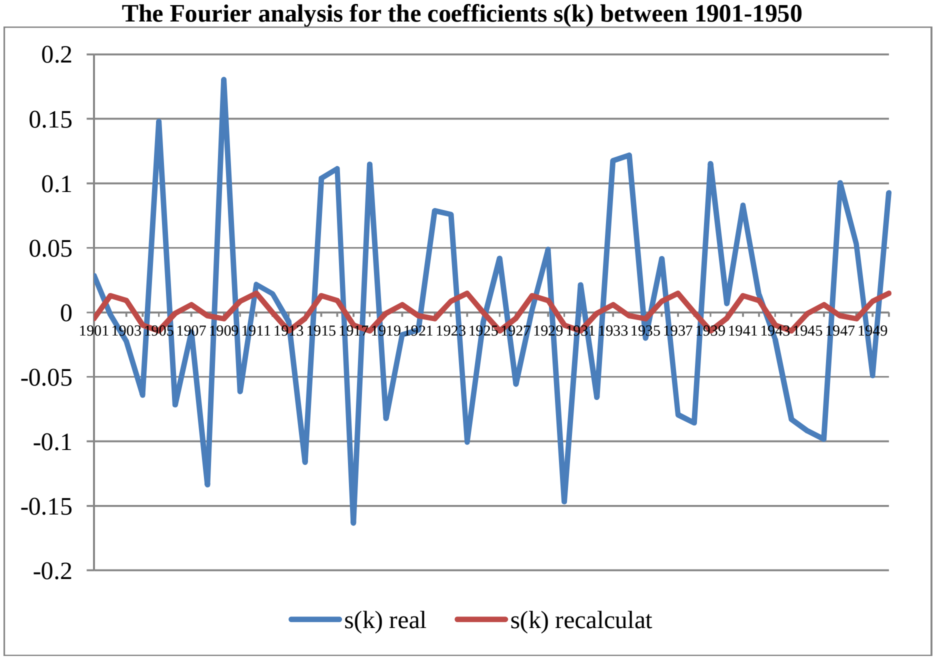

Figure 3

The Fourier analysis for the coefficients s(k) between 1951-2010

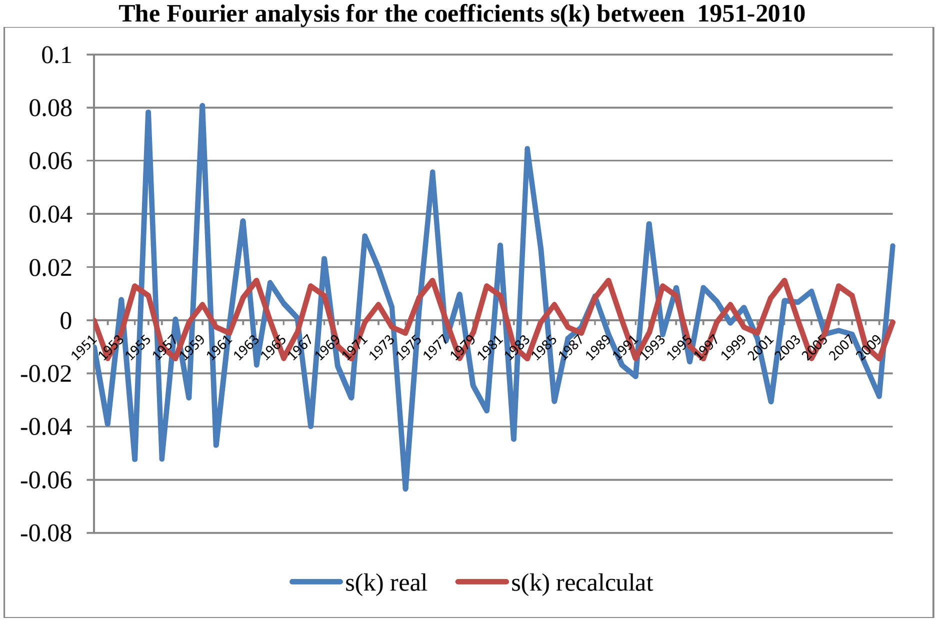

Figure 4

Conclusions

The graphical analysis of data resulting from application of the method of Fourier type, reveals a satisfactory correlation with actual phenomena, extreme points being off, usually with no more than a year against the real phenomenon. Due to the fact that $r_k = r_{k-1}+s_k$, one obtains that: $r_{k+13}=r_k+(s_{k+1}+...+s_{k+13})$. On the other hand, from the values recalculated of s_k it is observed easily that their sum is zero, so $r_{k+13}=r_k$. Therefore, we can assert a tendency of periodicity of the rate of growth of real GDP of 13 years.

References

Howard S. (2001). The Business CycleTheory of Wesley Mitchel. *Journal of Economic* Issues, March 2001.

Jevons W.S. (1878). Commercial Crises and Sun-Spots. *Nature*, Volume 19, Issue 472, pp. 33-37.

Kydland F.E., Prescott E.C. (1982). Time to Build and Aggregate Fluctuations. *Econometrica*, Vol. 50, Issue 6, pp. 1345-1370.

Maddison A. (2007). Fluctuations in the momentum of growth within the capitalist epoch. *Cliometrica.*

Okun A.M. (1970). *The Political Economy of Prosperity*. Norton W.W.&Company Inc.

Samuelson A.P., Nordhaus D.W. (2004). *Economics: An Introductory Analysis*. McGraw-Hill.

Spiegel M.R. (1974). *Fourier Analysis with Applications to Boundary Value Problems*. McGraw-Hill Book Company.