Microeconomics

The Complete Theory of Cobb-Douglas Production Function

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Abstract. The paper treats various aspects concerning the Cobb-Douglas production function. On the one hand were highlighted conditions for the existence of the Cobb-Douglas function. Also were calculated the main indicators of it and short and long-term costs. It has also been studied the dependence of long-term cost of the parameters of the production function. The determination of profit was made both for perfect competition market and maximizes its conditions. Also we have studied the effects of Hicks and Slutsky and the production efficiency problem.

Keywords: production function; Cobb-Douglas; Hicks; Slutsky

1. Introduction

To conduct any economic activity is absolutely indispensable the existence of inputs, in other words of any number of resources required for a good deployment of the production process. We will assume that all resources are indefinitely divisible.

We define on $\mathbf{R}^n$ the production space for n fixed resources as $SP=\{(x_1,...,x_n) \mid x_i \geq 0, i=\overline{1,n}\}$ where $x \in SP$, $x=(x_1,...,x_n)$ is an ordered set of resources and, because inside a production process, depending on the nature of applied technology, not any amount of resources is possible, we will restrict production space to a convex subset $D_p \subset SP$ – called the domain of production.

We will call a production function an application:

$$Q:D_p \to \mathbf{R}_+,\ (x_1,...,x_n) \to Q(x_1,...,x_n) \in \mathbf{R}_+\ \forall (x_1,...,x_n) \in D_p$$

which satisfies the following axioms:

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A1. $Q(0,...,0)=0$;

A2. The production function is of class C^2 on D_p that is it admits partial derivatives of order 2 and they are continuous on D_p;

A3. The production function is monotonically increasing in each variable, that is: $\frac{\partial Q}{\partial x_i} \geq 0$, $i=\overline{1,n}$;

A4. The production function is quasi-concave (*see Appendix*).

Considering a production function $Q:D_p \to \mathbf{R}_+$ and $Q_0 \in \mathbf{R}_+$ - fixed, the set of inputs which generate the production Q_0 called isoquant. An isoquant is therefore characterized by: $\{(x_1,...,x_n)\in D_p \mid Q(x_1,...,x_n)=Q_0\}$ or, in other words, it is the inverse image $Q^{-1}(Q_0)$.

We will say that a production function $Q:D_p \to \mathbf{R}_+$ is constant return to scale if $Q(\lambda x_1,...,\lambda x_n)=\lambda Q(x_1,...,x_n)$, with increasing return to scale if $Q(\lambda x_1,...,\lambda x_n)>\lambda Q(x_1,...,x_n)$ and decreasing return to scale if $Q(\lambda x_1,...,\lambda x_n)<\lambda Q(x_1,...,x_n)$ $\forall \lambda \in (1,\infty)$ $\forall (x_1,...,x_n)\in D_p$.

2. The Cobb-Douglas Production Function

The Cobb-Douglas function has the following expression:

$$Q:D\subset \mathbf{R}_+^n - \{0\} \to \mathbf{R}_+,\ (x_1,...,x_n) \to Q(x_1,...,x_n)= A x_1^{\alpha_1}...x_n^{\alpha_n} \in \mathbf{R}_+\ \forall (x_1,...,x_n)\in D,$$
$$A\in \mathbf{R}_+^*,\ \alpha_1,...,\alpha_n \in \mathbf{R}^*$$

Computing the partial derivatives of first and second order, we get:

$$Q'_{x_i} = \alpha_i A x_1^{\alpha_1}...x_i^{\alpha_i - 1}...x_n^{\alpha_n} = \frac{\alpha_i Q}{x_i}\ \ \forall i=\overline{1,n}$$

$$Q''_{x_i x_j} = \alpha_i \alpha_j A x_1^{\alpha_1}...x_i^{\alpha_i - 1}...x_j^{\alpha_j - 1}...x_n^{\alpha_n} = \frac{\alpha_i \alpha_j Q}{x_i x_j}\ \ \forall i\neq j=\overline{1,n}$$

$$Q''_{x_i x_i} = \alpha_i(\alpha_i - 1) A x_1^{\alpha_1}...x_i^{\alpha_i - 2}...x_n^{\alpha_n} = \frac{\alpha_i(\alpha_i - 1)Q}{x_i^2}\ \ \forall i=\overline{1,n}$$

Let the bordered Hessian matrix:

$$H^B(Q)=\begin{pmatrix} 0 & \frac{\alpha_1 Q}{x_1} & \frac{\alpha_2 Q}{x_2} & \dots & \frac{\alpha_n Q}{x_n} \\ \frac{\alpha_1 Q}{x_1} & \frac{\alpha_1(\alpha_1-1)Q}{x_1^2} & \frac{\alpha_1\alpha_2 Q}{x_1 x_2} & \dots & \frac{\alpha_1\alpha_n Q}{x_1 x_n} \\ \frac{\alpha_2 Q}{x_2} & \frac{\alpha_1\alpha_2 Q}{x_1 x_2} & \frac{\alpha_2(\alpha_2-1)Q}{x_2^2} & \dots & \frac{\alpha_2\alpha_n Q}{x_2 x_n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\alpha_n Q}{x_n} & \frac{\alpha_1\alpha_n Q}{x_1 x_n} & \frac{\alpha_2\alpha_n Q}{x_2 x_n} & \dots & \frac{\alpha_n(\alpha_n-1)Q}{x_n^2} \end{pmatrix}$$

We find (not so easy): $\Delta_k^B=(-1)^k Q^{k+1}\frac{\prod_{i=1}^k \alpha_i \sum_{i=1}^k \alpha_i}{\left(\prod_{i=1}^k x_i\right)^2}$, k=$\overline{1,n}$.

Because $(-1)^k \Delta_k^B=Q^{k+1}\frac{\prod_{i=1}^k \alpha_i \sum_{i=1}^k \alpha_i}{\left(\prod_{i=1}^k x_i\right)^2}$, if $\prod_{i=1}^k \alpha_i \sum_{i=1}^k \alpha_i >0$, k=$\overline{1,n}$ it follows that the function is strictly quasi-concave. Also, if the function is quasi-concave we have that $\prod_{i=1}^k \alpha_i \sum_{i=1}^k \alpha_i \ge 0$.

But from the axiom A3 we must have that $Q'_{x_i}=\frac{\alpha_i Q}{x_i}\ge 0$ that is $\alpha_i>0$. After these considerations we have that if $\alpha_i>0$, i=$\overline{1,n}$ the Cobb-Douglas function is strictly quasi-concave.

We have now: $q(\chi_1,...,\chi_{n-1})=Q(\chi_1,...,\chi_{n-1},1)=A\chi_1^{\alpha_1}...\chi_{n-1}^{\alpha_{n-1}}$ and r=$\sum_{k=1}^n \alpha_k$.

The main indicators are:

- $\eta_{x_i}=A\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{\alpha_i Q}{x_i}$, i=$\overline{1,n}$
- $w_{x_i}=Ax_1^{\alpha_1}...x_i^{\alpha_i-1}...x_n^{\alpha_n}=\frac{Q}{x_i}$, i=$\overline{1,n}$

- RMS(i,j)= $\frac{\alpha_i x_j}{\alpha_j x_i}$, i,j= $\overline{1,n}$
- RMS(i)= $\frac{\alpha_i}{x_i\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n-1}\frac{\alpha_j^2}{x_j^2}}}$, i= $\overline{1,n}$
- $\varepsilon_{x_i} = \alpha_i$, i= $\overline{1,n}$
- σ_{ij}=-1, i,j= $\overline{1,n}$

Reciprocally, if for a homogenous production function of degree r: $\varepsilon_{x_i} = \alpha_i$, i= $\overline{1,n}$ we have that: $\frac{\frac{\partial q}{\partial \chi_i}}{\frac{q}{\chi_i}} = \alpha_i$, i= $\overline{1,n-1}$ and $\frac{rq - \sum_{i=1}^{n-1}\frac{\partial q}{\partial \chi_i}\chi_i}{q} = \alpha_n$.

But now, we have:

$$\frac{\partial q}{\partial \chi_i} = \alpha_i \frac{q}{\chi_i} = q\frac{\partial \ln \chi_i^{\alpha_i}}{\partial \chi_i} \Leftrightarrow \frac{\frac{\partial q}{\partial \chi_i}}{q} = \frac{\partial \ln \chi_i^{\alpha_i}}{\partial \chi_i} \Leftrightarrow \frac{\partial \ln q}{\partial \chi_i} = \frac{\partial \ln \chi_i^{\alpha_i}}{\partial \chi_i} \Leftrightarrow \frac{\partial \ln \frac{q}{\chi_i^{\alpha_i}}}{\partial \chi_i} = 0 \Rightarrow$$

$\ln \frac{q}{\chi_i^{\alpha_i}} = F_i\left(\chi_1,...,\hat{\chi}_i,...,\chi_{n-1}\right)$ (where ^ means that the variable is missing).

We have now: $q = \chi_i^{\alpha_i} e^{F_i(\chi_1,...\hat{\chi}_i,...\chi_{n-1})}$. For j≠i we obtain now: $\alpha_j = \chi_j \frac{\partial F_i}{\partial \chi_j}$ therefore: $\frac{\partial F_i}{\partial \chi_j} = \frac{\alpha_j}{\chi_j}$. Integrating with respect to χ_j:

$F_i = \alpha_j \ln \chi_j + g_i\left(\chi_1,..., \hat{\chi}_i,...\hat{\chi}_j,..., \chi_{n-1}\right)$ therefore: $q = \chi_i^{\alpha_i}\chi_j^{\alpha_j} e^{g_i\left(\chi_1,...\hat{\chi}_i,...\hat{\chi}_j,...\chi_{n-1}\right)}$. Analogously, by recurrence: $q = A\chi_1^{\alpha_1}...\chi_{n-1}^{\alpha_{n-1}}$ with A=constant with respect to $\chi_1,...,\chi_{n-1}$. But: $\alpha_n = \frac{rq - \sum_{i=1}^{n-1}\frac{\partial q}{\partial \chi_i}\chi_i}{q} = r - \sum_{i=1}^{n-1}\alpha_i \Leftrightarrow r = \sum_{i=1}^{n}\alpha_i$. After these considerations it follows that if it is homogenous of degree r, r must be $\sum_{i=1}^{n}\alpha_i$.

Finally: $q = A\chi_1^{\alpha_1}...\chi_{n-1}^{\alpha_{n-1}}$ implies that: $Q(x_1,...,x_n) = x_n^r q(\chi_1,...,\chi_{n-1}) = Ax_n^r \frac{x_1^{\alpha_1}}{x_n^{\alpha_1}}...\frac{x_{n-1}^{\alpha_{n-1}}}{x_n^{\alpha_{n-1}}} = Ax_n^{r-\sum_{k=1}^{n-1}\alpha_k} x_1^{\alpha_1}...x_{n-1}^{\alpha_{n-1}} =$

$Ax_1^{\alpha_1}...x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}$ - the Cobb-Douglas production function.

Considering now again the Cobb-Douglas production: $Q(x_1,...,x_n) = Ax_1^{\alpha_1}...x_n^{\alpha_n}$ let search the dependence of the parameters $\alpha_1,...,\alpha_n$.

We have: $\frac{\partial Q}{\partial \alpha_i} = Ax_1^{\alpha_1}...x_n^{\alpha_n} \ln x_i = Q \ln x_i \geq 0 \ \forall x_i \geq 1,\ i=\overline{1,n}$. From this relation we have that at an increasing of a parameter α_i the production Q will increase also.

In particular, for the Cobb-Douglas function related to capital K and labor L: $Q=AK^{\alpha}L^{\beta}$ we have that the main indicators are:

- $\eta_K = A\alpha K^{\alpha-1}L^{\beta}$
- $\eta_L = A\beta K^{\alpha}L^{\beta-1}$
- $w_K = AK^{\alpha-1}L^{\beta}$
- $w_L = AK^{\alpha}L^{\beta-1}$
- RMS(K,L)=RMS(K)= $\frac{\alpha L}{\beta K}$
- RMS(L,K)= RMS(L)= $\frac{\beta K}{\alpha L}$
- $\varepsilon_K = \alpha$
- $\varepsilon_L = \beta$
- $\sigma = \sigma_{KL} = -1$

3. The Costs of the Cobb-Douglas Production Function

Considering now the problem of minimizing costs for a given production Q_0, where the prices of inputs are p_i, $i=\overline{1,n}$, we have:

$$\begin{cases} \min \sum_{k=1}^{n} p_k x_k \\ A x_1^{\alpha_1} \dots x_n^{\alpha_n} \ge Q_0 \\ x_1, \dots, x_n \ge 0 \end{cases}$$

From the obvious relations: $\begin{cases} \dfrac{\alpha_1}{p_1 x_1} = \dots = \dfrac{\alpha_n}{p_n x_n} \\ A x_1^{\alpha_1} \dots x_n^{\alpha_n} = Q_0 \end{cases}$ we obtain:

$\begin{cases} x_k = \dfrac{\alpha_k p_n}{\alpha_n p_k} x_n, \ k = \overline{1, n-1} \\ A x_1^{\alpha_1} \dots x_n^{\alpha_n} = Q_0 \end{cases}$ and from the second equation:

$A \dfrac{p_n^{\sum_{k=1}^{n-1} \alpha_k} \prod_{k=1}^{n-1} \alpha_k^{\alpha_k}}{\alpha_n^{\sum_{k=1}^{n-1} \alpha_k} \prod_{k=1}^{n-1} p_k^{\alpha_k}} x_n^{\sum_{k=1}^{n} \alpha_k} = Q_0$. Noting $r = \sum_{k=1}^{n} \alpha_k$ we finally obtain:

$$\overline{x}_k = \frac{\left(\prod_{k=1}^{n} p_k^{\alpha_k} \right)^{1/r}}{\left(\prod_{k=1}^{n} \alpha_k^{\alpha_k} \right)^{1/r}} \frac{\alpha_k}{p_k} \frac{Q_0^{1/r}}{A^{1/r}}, \ k = \overline{1, n}$$

The total cost is:

$$TC = \sum_{k=1}^{n} p_k \overline{x}_k = \frac{\left(\prod_{i=1}^{n} p_i^{\alpha_i} \right)^{1/r}}{\left(\prod_{i=1}^{n} \alpha_i^{\alpha_i} \right)^{1/r}} \frac{r Q_0^{1/r}}{A^{1/r}}.$$

At a price change of one factor, i.e. x_k, from the value p_k to $\overline{p}_k$ we have:

$$\overline{TC} = \frac{\left(\prod_{\substack{i=1\\i\neq k}}^{n} p_i^{\alpha_i}\right)^{1/r} \overline{p}_k^{\alpha_k/r}}{\left(\prod_{i=1}^{n} \alpha_i^{\alpha_i}\right)^{1/r}} \frac{rQ_0^{1/r}}{A^{1/r}}$$

where the relative variation of the total cost is: $\frac{\Delta TC}{TC} = \frac{\overline{TC} - TC}{TC} = \left(\frac{\overline{p}_k}{p_k}\right)^{\alpha_k/r} - 1$.

Let us now consider the behavior of the total cost of production function at a parameters variation. We have:

$$\frac{\partial TC}{\partial \alpha_k} = \frac{\left(\frac{\prod_{i=1}^{n} p_i^{\alpha_i}}{\prod_{i=1}^{n} \alpha_i^{\alpha_i}} \frac{Q_0}{A}\right)^{\frac{1}{r}}}{\left(\sum_{j=1}^{n} \alpha_j\right)^2} \left(\ln \frac{A p_k^{\sum_{j=1, j\neq k}^{n} \alpha_j} \prod_{\substack{i=1\\i\neq k}}^{n} \alpha_i^{\alpha_i}}{Q_0 \alpha_k^{\sum_{j=1, j\neq k}^{n} \alpha_j} \prod_{\substack{i=1\\i\neq k}}^{n} p_i^{\alpha_i}} - r \right)$$

Therefore: $\frac{\partial TC}{\partial \alpha_k} \geq 0 \Leftrightarrow e^{\sum_{j=1}^{n} \alpha_j} \alpha_k^{\sum_{j=1, j\neq k}^{n} \alpha_j} \leq \frac{A p_k^{\sum_{j=1, j\neq k}^{n} \alpha_j} \prod_{\substack{i=1\\i\neq k}}^{n} \alpha_i^{\alpha_i}}{Q_0 \prod_{\substack{i=1\\i\neq k}}^{n} p_i^{\alpha_i}}$. If we note: $\Gamma = \sum_{\substack{j=1\\j\neq k}}^{n} \alpha_j > 0$ and

$M = \frac{A p_k^{\sum_{j=1, j\neq k}^{n} \alpha_j} \prod_{\substack{i=1\\i\neq k}}^{n} \alpha_i^{\alpha_i}}{Q_0 \prod_{\substack{i=1\\i\neq k}}^{n} p_i^{\alpha_i}} > 0$ we obtain that: $\frac{\partial TC}{\partial \alpha_k} \geq 0 \Leftrightarrow e^{\alpha_k} \alpha_k^{\Gamma} \leq \frac{M}{e^{\Gamma}}$.

Considering now the function $f:(0,\infty)\to\mathbf{R}$, $f(x)= x^{\Gamma} e^{x}$ we have $f'(x) = x^{\Gamma-1} e^{x} (x + \Gamma) > 0$ therefore f is strictly increasing. Because $\lim_{x\to 0} x^{\Gamma} e^{x} = 0$

and $\lim_{x \to \infty} x^{\Gamma} e^{x} = \infty$ the equation: $x^{\Gamma} e^{x} = \frac{M}{e^{\Gamma}}$ has a unique solution α_k^0 called cost threshold with respect to the k-th parameter. After these considerations we have that for $\alpha_k \leq \alpha_k^0$ the total cost will increase at an increasing of α_k and after it will decrease.

The situation may seem paradoxical that at the growth of the elasticity of one input, total cost increases. Fortunately, due to the sharp rise of f, the values of α_k^0 are very small so it does not significantly affect processes.

Like an example, considering the production function $Q(K,L)=K^{\alpha}L^{\beta}$, $\alpha,\beta>0$ we have that the behavior of β^0 related to α is (for Q=5):

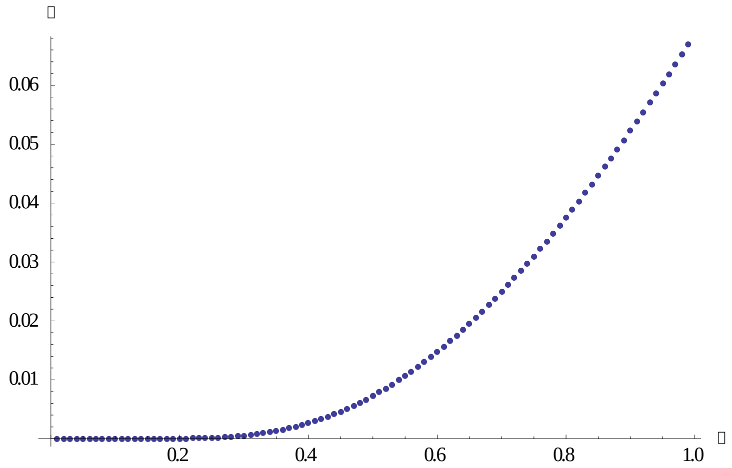

Figure 1

The long-term total cost for α=0.7 and variable β is shown in figure 2:

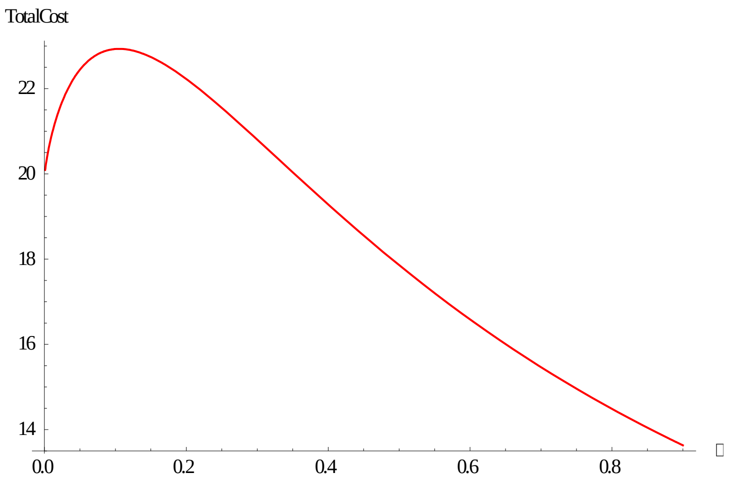

Figure 2

where the maximum value is reached for β=0.025.

If we consider for a given output Q_0, the inputs $x_1,\ldots,x_n$ such that: $Ax_1^{\alpha_1}\ldots x_n^{\alpha_n} = Q_0$

let $x_k = \dfrac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}} x_1^{\frac{\alpha_1}{\alpha_k}} \ldots \hat{x}_k \ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$ where ^ means that the term is missing.

We have $STC_k = \sum_{i=1}^{n} p_i x_i = \sum_{\substack{i=1 \\ i\neq k}}^{n} p_i x_i + p_k \dfrac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}} x_1^{\frac{\alpha_1}{\alpha_k}} \ldots \hat{x}_k \ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$ representing the short-term total cost when factors $x_1,\ldots,\hat{x}_k,\ldots,x_n$ remain constant.

We put now the question of determining the envelope of the family of hypersurfaces:

$$f(Q_0, x_1,\ldots,\hat{x}_k,\ldots,x_n) = \sum_{\substack{i=1 \\ i\neq k}}^{n} p_i x_i + p_k \frac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}} x_1^{\frac{\alpha_1}{\alpha_k}} \ldots \hat{x}_k \ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$$

Conditions to be met are:

$$\begin{cases} TC = f\left(Q_0, x_1, ..., \hat{x}_k, ..., x_n\right) \\ \dfrac{\partial f}{\partial x_i} = 0, i = \overline{1,n}, i \neq k \end{cases}$$

After the elimination of parameters $x_1, ..., \hat{x}_k, ..., x_n$ we have either the locus of singular points of hypersurfaces (which is not the case for the present issue) or envelope sought.

We have therefore:

$$\begin{cases} TC = \sum\limits_{\substack{i=1 \\ i \neq k}}^{n} p_i x_i + p_k \dfrac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}} x_1^{\frac{\alpha_1}{\alpha_k}} ... \hat{x}_k ... x_n^{\frac{\alpha_n}{\alpha_k}}} \\ p_i - \dfrac{\alpha_i}{\alpha_k} \dfrac{p_k Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}} x_1^{\frac{\alpha_1}{\alpha_k}} ... x_i^{\frac{\alpha_i}{\alpha_k}+1} ... \hat{x}_k ... x_n^{\frac{\alpha_n}{\alpha_k}}} = 0, i = \overline{1,n}, i \neq k \end{cases}$$

Noting: $\Psi = \dfrac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}} x_1^{\frac{\alpha_1}{\alpha_k}} ... \hat{x}_k ... x_n^{\frac{\alpha_n}{\alpha_k}}}$ it follows:

$$\begin{cases} TC = \sum\limits_{\substack{i=1 \\ i \neq k}}^{n} p_i x_i + p_k \Psi \\ p_i - \dfrac{\alpha_i}{\alpha_k} \dfrac{p_k \Psi}{x_i} = 0, i = \overline{1,n}, i \neq k \end{cases}$$

from where: $x_i = \dfrac{\alpha_i}{\alpha_k} \dfrac{p_k \Psi}{p_i}, i = \overline{1,n}, i \neq k$. Finally: $\Psi = \dfrac{\alpha_k \left(p_1^{\alpha_1} ... p_n^{\alpha_n}\right)^{1/r} Q_0^{1/r}}{A^{1/r} p_k \left(\alpha_1^{\alpha_1} ... \alpha_n^{\alpha_n}\right)^{1/r}}$ and replacing:

$$x_i = \frac{\alpha_i \left(\prod\limits_{i=1}^{n} p_i^{\alpha_i}\right)^{1/r} Q_0^{1/r}}{A^{1/r} p_i \left(\prod\limits_{i=1}^{n} \alpha_i^{\alpha_i}\right)^{1/r}}, i = \overline{1,n}, i \neq k$$

$$TC=\frac{\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{1/r}}{\left(\prod_{i=1}^{n} \alpha_i^{\alpha_i}\right)^{1/r}}\frac{rQ_0^{1/r}}{A^{1/r}}$$

We obtained so that the envelope of the family of hypersurfaces of the short-term total cost when all inputs are constant except one is just the long-term cost obtained from nonlinear optimization problem with respect to the minimizing of the cost for a given production.

Calculating the costs derived from the (long-term or short-term) total cost now, we have:

$$ATC=\frac{TC}{Q_0}=\frac{\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{1/r}}{\left(\prod_{i=1}^{n} \alpha_i^{\alpha_i}\right)^{1/r}}\frac{rQ_0^{1/r-1}}{A^{1/r}}$$ (*average long-term total cost*)

$$MTC=\frac{\partial TC}{\partial Q_0}=\frac{\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{1/r}}{\left(\prod_{i=1}^{n} \alpha_i^{\alpha_i}\right)^{1/r}}\frac{Q_0^{1/r-1}}{A^{1/r}}=\frac{ATC}{r}$$ (*marginal long-term total cost*)

$$ASTC_k=\frac{STC_k}{Q_0}=\frac{\sum_{\substack{i=1\\ i\neq k}}^{n} p_i x_i}{Q_0}+p_k\frac{Q_0^{\frac{1}{\alpha_k}-1}}{A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$$ (*average short-term total cost*)

$$MC_k=\frac{\partial STC_k}{\partial Q_0}=p_k\frac{Q_0^{\frac{1}{\alpha_k}-1}}{\alpha_k A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$$ (*marginal short-term total cost*)

$$VTC_k=p_k\frac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$$ (*variable short-term total cost*)

$$AVTC_k=\frac{VTC_k}{Q_0}=p_k\frac{Q_0^{\frac{1}{\alpha_k}-1}}{A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$$ (*average variable short-term total cost*)

$$FTC_k=\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i$$ (*fixed short-term total cost*)

$$AFTC_k=\frac{FTC_k}{Q_0}=\frac{\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i}{Q_0}$$ (*average fixed short-term total cost*)

The extreme of the function $ASTC_k(Q_0)=\frac{\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i}{Q_0}+p_k\frac{Q_0^{\frac{1}{\alpha_k}-1}}{A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}$ are given by:

$$ASTC_k'(Q_0)=\frac{p_k\left(\frac{1}{\alpha_k}-1\right)Q_0^{\frac{1}{\alpha_k}}-A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i}{Q_0^2A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}=0$$ from where:

$$Q_{0,d-root}=\frac{Ax_1^{\alpha_1}\ldots\hat{x}_k\ldots x_n^{\alpha_n}\left(\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i\right)^{\alpha_k}}{p_k^{\alpha_k}\left(\frac{1}{\alpha_k}-1\right)^{\alpha_k}}$$ when $\alpha_k<1$ and the minimum value is:

$$ASTC_k(Q_{0-d-root})=\frac{p_k^{\alpha_k}\left(\frac{1}{\alpha_k}-1\right)^{\alpha_k-1}}{A\alpha_k x_1^{\alpha_1}\ldots\hat{x}_k\ldots x_n^{\alpha_n}\left(\sum_{\substack{i=1\\ i\neq k}}^{n}p_i x_i\right)^{\alpha_k-1}}.$$

If $\alpha_k \geq 1$ it follows that $ASTC_k'(Q_0)<0$ therefore the average short-term total cost will decrease.

Finally we have:

$\varepsilon_{p_k} = \dfrac{\frac{\partial CT}{\partial p_k}}{\frac{CT}{p_k}} = \dfrac{\alpha_k}{r}$ - the coefficient of elasticity of long-term total cost with respect to the price factor i

$\varepsilon_{Q} = \dfrac{\frac{\partial CT}{\partial Q_0}}{\frac{CT}{Q_0}} = \dfrac{1}{r}$ - the coefficient of elasticity of long-term total cost with respect to the production Q_0

$\varepsilon_{av,p_k} = \dfrac{\frac{\partial ATC}{\partial p_k}}{\frac{ATC}{p_k}} = \dfrac{\alpha_k}{r}$ - the coefficient of elasticity of average long-term total cost with respect to the price factor i

$\varepsilon_{marg,p_k} = \dfrac{\frac{\partial MTC}{\partial p_k}}{\frac{MTC}{p_k}} = \dfrac{\alpha_k}{r}$ - the coefficient of elasticity of marginal long-term total cost with respect to the price factor i

In particular, for the Cobb-Douglas function related to capital K and labor L: $Q=AK^{\alpha}L^{\beta}$ we have:

$$\overline{K} = \frac{\left(p_K^{\alpha} p_L^{\beta}\right)^{1/(\alpha+\beta)}}{\left(\alpha^{\alpha}\beta^{\beta}\right)^{1/(\alpha+\beta)}} \frac{\alpha}{p_K} \frac{Q_0^{1/(\alpha+\beta)}}{A^{1/(\alpha+\beta)}}$$

$$\overline{L} = \frac{\left(p_K^{\alpha} p_L^{\beta}\right)^{1/(\alpha+\beta)}}{\left(\alpha^{\alpha}\beta^{\beta}\right)^{1/(\alpha+\beta)}} \frac{\beta}{p_L} \frac{Q_0^{1/(\alpha+\beta)}}{A^{1/(\alpha+\beta)}}$$

$$TC=\frac{\left(p_K^{\alpha}p_L^{\beta}\right)^{1/(\alpha+\beta)}}{\left(\alpha^{\alpha}\beta^{\beta}\right)^{1/(\alpha+\beta)}}\frac{(\alpha+\beta)Q_0^{1/(\alpha+\beta)}}{A^{1/(\alpha+\beta)}}$$

On the short-term, we have for constancy of K: $STC_L = p_K K + p_L \frac{Q_0^{\frac{1}{\beta}}}{A^{\frac{1}{\beta}}K^{\frac{\alpha}{\beta}}}$ and

$$ATC=\frac{\left(p_K^{\alpha}p_L^{\beta}\right)^{1/(\alpha+\beta)}}{\left(\alpha^{\alpha}\beta^{\beta}\right)^{1/(\alpha+\beta)}}\frac{(\alpha+\beta)Q_0^{1/(\alpha+\beta)-1}}{A^{1/(\alpha+\beta)}}$$

$$MTC=\frac{\left(p_K^{\alpha}p_L^{\beta}\right)^{1/(\alpha+\beta)}}{\left(\alpha^{\alpha}\beta^{\beta}\right)^{1/(\alpha+\beta)}}\frac{Q_0^{1/(\alpha+\beta)-1}}{A^{1/(\alpha+\beta)}}$$

$$ASTC_L=\frac{p_K K}{Q_0}+p_L\frac{Q_0^{\frac{1}{\beta}-1}}{A^{\frac{1}{\beta}}K^{\frac{\alpha}{\beta}}}$$

$$MC_L=\frac{p_L Q_0^{\frac{1}{\beta}-1}}{\beta A^{\frac{1}{\beta}}K^{\frac{\alpha}{\beta}}}$$

$$VTC_L=p_K\frac{Q_0^{\frac{1}{\alpha}}}{A^{\frac{1}{\alpha}}L^{\frac{\beta}{\alpha}}}$$

$$AVTC_L=p_K\frac{Q_0^{\frac{1}{\alpha}-1}}{A^{\frac{1}{\alpha}}L^{\frac{\beta}{\alpha}}}$$

$$FTC_L=p_L L$$

$$AFTC_L=\frac{p_L L}{Q_0}$$

The extreme of the function: $ASTC_L(Q_0)$ is given by: $Q_{0,d-root} = \dfrac{Ap_K^{\beta}K^{\alpha+\beta}}{p_L^{\alpha}\left(\dfrac{1}{\beta}-1\right)^{\beta}}$ when $\beta<1$ and the minimum value is: $ASTC_L\left(Q_{0-d-root}\right) \dfrac{p_L^{\beta}(1-\beta)^{\beta-1}}{Ap_K^{\beta-1}\beta^{\beta}K^{\alpha+\beta-1}}$.

$$\varepsilon_{p_K} = \frac{\alpha}{\alpha+\beta},\ \varepsilon_{p_L} = \frac{\beta}{\alpha+\beta},\ \varepsilon_Q = \frac{1}{\alpha+\beta},\ \varepsilon_{av,p_K} = \frac{\alpha}{\alpha+\beta},\ \varepsilon_{av,p_L} = \frac{\beta}{\alpha+\beta},$$
$$\varepsilon_{marg,p_K} = \frac{\alpha}{\alpha+\beta},\ \varepsilon_{marg,p_L} = \frac{\beta}{\alpha+\beta}.$$

4. The Profit

Now consider a sale price of output Q_0: $p(Q_0)$. The profit is therefore:

$\Pi(Q_0)= p(Q_0)\cdot Q_0-TC(Q_0)$

It is known that in a market with perfect competition, the price is given and equals marginal cost. The profit on long-term becomes:

$$\Pi(Q_0)=p(Q_0)\cdot Q_0-TC(Q_0)=MTC(Q_0)\cdot Q_0-$$
$$TC(Q_0)= ATC'(Q_0)Q_0^2 = \frac{\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{1/r}}{\left(\prod_{i=1}^{n} \alpha_i^{\alpha_i}\right)^{1/r}} \frac{rQ_0^{1/r+1}}{A^{1/r}}$$

In particular, for the Cobb-Douglas function related to capital K and labor L: $Q=AK^{\alpha}L^{\beta}$ we have: $\Pi(Q_0)= \dfrac{\left(p_K^{\alpha}p_L^{\beta}\right)^{1/(\alpha+\beta)}}{\left(\alpha^{\alpha}\beta^{\beta}\right)^{1/(\alpha+\beta)}} \dfrac{(\alpha+\beta)Q_0^{1/r+1}}{A^{1/r}}$.

On short-term, when factors $x_1,...,\hat{x}_k,...,x_n$ remain constant, we have:

$\Pi(Q_0)=p(Q_0)\cdot Q_0-STC_k(Q_0)=MTC(Q_0)\cdot Q_0-STC_k(Q_0)= AVTC_k'(Q_0)Q_0^2 - FTC_k$

therefore:

$$\Pi(Q_0)=\left(\frac{1}{\alpha_k}-1\right)p_k\frac{Q_0^{\frac{1}{\alpha_k}}}{A^{\frac{1}{\alpha_k}}x_1^{\frac{\alpha_1}{\alpha_k}}\ldots\hat{x}_k\ldots x_n^{\frac{\alpha_n}{\alpha_k}}}-\sum_{\substack{i=1\\i\neq k}}^{n}p_ix_i=\left(\frac{1}{\alpha_k}-1\right)p_kx_k-\sum_{\substack{i=1\\i\neq k}}^{n}p_ix_i$$

Like a conclusion, the company will make a profit in the short-term if, under constancy factors $x_1,\ldots,\hat{x}_k,\ldots,x_n$, the amount of factor x_k will be higher than $\frac{\sum_{\substack{i=1\\i\neq k}}^{n}p_ix_i}{\left(\frac{1}{\alpha_k}-1\right)p_k}$ if $\alpha_k<1$ and less than $\frac{\sum_{\substack{i=1\\i\neq k}}^{n}p_ix_i}{\left(\frac{1}{\alpha_k}-1\right)p_k}$ if $\alpha_k>1$. If $\alpha_k=1$ then the firm will incur losses.

For $Q=AK^{\alpha}L^{\beta}$ we have that if K=constant, the company will make a profit in the short-term in the case $\beta<1$ if $L>\frac{\beta p_K K}{(1-\beta)p_L}$ and in the case $\beta>1$ if $L<\frac{\beta p_K K}{(1-\beta)p_L}$. If $\beta=1$ the firm will incur losses.

The condition of profit maximization for an arbitrarily price p, depending on the factors of production, is: $\max\Pi(x_1,\ldots,x_n)=\max\left(pQ(x_1,\ldots,x_n)-\sum_{i=1}^{n}p_ix_i\right)$ from where $\frac{\partial Q}{\partial x_i}=\frac{p_i}{p}$, $i=\overline{1,n}$ or otherwise: $\alpha_i\frac{Q}{x_i}=\frac{p_i}{p}$ and finally: $\overline{x}_i=\frac{\alpha_i pQ}{p_i}$. Because Q is quasi-concave the solution of the characteristic system is the unique point of maximum. How $Q=Ax_1^{\alpha_1}\ldots x_n^{\alpha_n}$ we obtain that the appropriate production is: $\overline{Q}=A\overline{x}_1^{\alpha_1}\ldots\overline{x}_n^{\alpha_n}=\frac{Ap^r\prod_{k=1}^{n}\alpha_k^{\alpha_k}}{\prod_{k=1}^{n}p_k^{\alpha_k}}\overline{Q}^r$ therefore, if $r\neq1$: $\overline{Q}=\left(\frac{\prod_{k=1}^{n}p_k^{\alpha_k}}{Ap^r\prod_{k=1}^{n}\alpha_k^{\alpha_k}}\right)^{\frac{1}{r-1}}$ and the factors: $\overline{x}_i=\left(\frac{\alpha_i^{r-1-\alpha_i}\prod_{\substack{k=1\\k\neq i}}^{n}p_k^{\alpha_k}}{App_i^{r-1-\alpha_i}\prod_{\substack{k=1\\k\neq i}}^{n}\alpha_k^{\alpha_k}}\right)^{\frac{1}{r-1}}$, $i=\overline{1,n}$.

The maximum profit is: $\Pi(\overline{x}_1,...,\overline{x}_n)=(p-r)\left(\frac{\prod_{k=1}^{n} p_k^{\alpha_k}}{Ap^r\prod_{k=1}^{n}\alpha_k^{\alpha_k}}\right)^{\frac{1}{r-1}}$.

If r=1 the necessary condition for profit maximization is: $\prod_{k=1}^{n} p_k^{\alpha_k}=Ap\prod_{k=1}^{n}\alpha_k^{\alpha_k}$ or p must be: p=$\frac{\prod_{k=1}^{n} p_k^{\alpha_k}}{A\prod_{k=1}^{n}\alpha_k^{\alpha_k}}$ therefore the amount of factors are not independent, that is, for a fixed factor, let say x_s: $\overline{Q}=\frac{Ap_s\prod_{k=1}^{n}\alpha_k^{\alpha_k}}{\alpha_s\prod_{k=1}^{n}p_k^{\alpha_k}}\overline{x}_s$ and: $\overline{x}_i=\frac{\alpha_i p_s}{\alpha_s p_i}\overline{x}_s$, i=$\overline{1,n}$, i≠s. The profit is: $\Pi(\overline{x}_1,...,\overline{x}_n)$=0 for any amount of x_s.

For $Q=AK^{\alpha}L^{\beta}$ we have that, if α+β≠1:

$$\overline{Q}=\left(\frac{p_K^{\alpha}p_L^{\beta}}{Ap^r\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta-1}},\ \overline{K}=\left(\frac{\alpha^{\beta-1}p_L^{\beta}}{App_K^{\beta-1}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta-1}},\ \overline{L}=\left(\frac{\beta^{\alpha-1}p_K^{\alpha}}{App_L^{\alpha-1}\alpha^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}},$$

$$\Pi(\overline{K},\overline{L})=(p-\alpha-\beta)\left(\frac{p_K^{\alpha}p_L^{\beta}}{Ap^{\alpha+\beta}\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta-1}}$$

and if α+β=1 the necessary condition for profit maximization is: $p=\frac{p_K^{1-\beta}p_L^{\beta}}{A(1-\beta)^{1-\beta}\beta^{\beta}}$, $\overline{L}=\frac{\beta p_K}{(1-\beta)p_L}\overline{K}$, $\overline{Q}=\frac{A\beta^{\beta}p_K^{\beta}}{(1-\beta)^{\beta}p_L^{\beta}}\overline{K}$, $\Pi(\overline{K},\overline{L})=0$.

At a variable price p(Q) we have now: Π(Q)=p(Q)·Q-CT(Q) therefore the necessary condition for profit maximization is $\Pi'(Q)=0$ therefore: $p'(Q)Q+p(Q)-MTC(Q)=0$.

Substituting the expression of MTC we obtain: $p'(Q)Q+p(Q)-\Gamma Q^{1/r-1}=0$ where we noted: $\Gamma=\frac{\left(\prod_{i=1}^{n}p_i^{\alpha_i}\right)^{1/r}}{A^{1/r}\left(\prod_{i=1}^{n}\alpha_i^{\alpha_i}\right)^{1/r}}$. But this differential equation gives us: $p(Q)=r\Gamma Q^{1/r-1}+\frac{C}{Q}$, $C\in\mathbf{R}_+$ and the profit Π(Q)=p(Q)·Q-CT(Q)=C. Therefore, for the maximum of the profit Π(Q)=C we must have the price $p(Q)=r\Gamma Q^{1/r-1}+\frac{C}{Q}$, $C\in\mathbf{R}_+$. Because: $p'(Q)=\frac{(1-r)\Gamma Q^{1/r}-C}{Q^2}$ we have that if r>1, the price will decrease with production and if r<1 for $Q\le\left(\frac{C}{(1-r)\Gamma}\right)^r$ the price will decrease and for $Q\ge\left(\frac{C}{(1-r)\Gamma}\right)^r$ the price will increase. If r=1 we have that $p(Q)=\Gamma+\frac{C}{Q}$ and the price will decrease with production.

For $Q=AK^\alpha L^\beta$ we have $\Gamma=\left(\frac{p_K^\alpha p_L^\beta}{A\alpha^\alpha\beta^\beta}\right)^{1/(\alpha+\beta)}$ and for the profit Π(Q)=C we must have the price $p(Q)=(\alpha+\beta)\Gamma Q^{1/(\alpha+\beta)-1}+\frac{C}{Q}$. If α+β>1, the price will decrease with production and if α+β<1 for $Q\le\left(\frac{C}{(1-\alpha-\beta)\Gamma}\right)^{\alpha+\beta}$ the price will decrease and for $Q\ge\left(\frac{C}{(1-\alpha-\beta)\Gamma}\right)^{\alpha+\beta}$ the price will increase. If α+β=1 we have that $p(Q)=\Gamma+\frac{C}{Q}$ and the price will decrease with production.

5. The Hicks and Slutsky Effects for the Cobb-Douglas Production Function

Now consider the production function $Q(x_1,...,x_n)= Ax_1^{\alpha_1}...x_n^{\alpha_n}$ and factor prices $(p_i)_{i=\overline{1,n}}$. The non-linear programming problem relative to maximize production at a given total cost (CT_0) is:

$$\begin{cases} \max Ax_1^{\alpha_1}...x_n^{\alpha_n} \\ \sum_{k=1}^{n} p_k x_k = CT_0 \\ x_1,...,x_n \geq 0 \end{cases}$$

Because the objective function is quasi-concave and also the restriction (being affine) and the partial derivatives are all positive we find that the Karush-Kuhn-Tucker conditions are also sufficient. Therefore, we have:

$$\begin{cases} \dfrac{\frac{\partial Q}{\partial x_1}(\overline{x}_1,...,\overline{x}_n)}{p_1} = ... = \dfrac{\frac{\partial Q}{\partial x_n}(\overline{x}_1,...,\overline{x}_n)}{p_n} \\ \sum_{k=1}^{n} p_k x_k = CT_0 \end{cases}$$

From the first equations we obtain:

$$\begin{cases} \dfrac{\alpha_1}{p_1 x_1} = ... = \dfrac{\alpha_n}{p_n x_n} \\ \sum_{k=1}^{n} p_k x_k = CT_0 \end{cases}$$

therefore:

$$\begin{cases} x_k = \dfrac{\alpha_k p_n}{\alpha_n p_k} x_n, \ k = \overline{1, n-1} \\ \sum_{k=1}^{n} p_k x_k = CT_0 \end{cases}$$

Substituting the first n-1 relations into the last we finally find that: $x_{0,k} = \frac{\alpha_k CT_0}{rp_k}, k = \overline{1,n}$ and the appropriate production:

$$Q_0(x_1,...,x_n) = A\frac{\prod_{i=1}^{n}\alpha_i^{\alpha_i}}{r^r\prod_{i=1}^{n}p_i^{\alpha_i}}CT_0^r.$$

Suppose now that some of the prices of factors of production (possibly after renumbering, we may assume that they are: $x_1,...,x_s$) is modified to values $\bar{p}_1,,,.\bar{p}_s$, the rest remain constant.

From the above, it results:

$$\begin{cases} x_{f,k} = \frac{\alpha_k CT_0}{r\bar{p}_k}, k = \overline{1,s} \\ x_{f,k} = \frac{\alpha_k CT_0}{rp_k}, k = \overline{s+1,n} \\ Q_f = A\frac{\prod_{i=1}^{n}\alpha_i^{\alpha_i}}{r^r\prod_{i=1}^{s}\bar{p}_i^{\alpha_i}\prod_{j=s+1}^{n}p_j^{\alpha_j}}CT_0^r \end{cases}$$

We will apply in the following, the method of Hicks. To an input price change, let consider that it remains unchanged, leading thus to a change of the total cost. We therefore have:

$$A\frac{\prod_{i=1}^{n}\alpha_i^{\alpha_i}}{r^r\prod_{i=1}^{s}\bar{p}_i^{\alpha_i}\prod_{j=s+1}^{n}p_j^{\alpha_j}}\overline{CT}_0^r = A\frac{\prod_{i=1}^{n}\alpha_i^{\alpha_i}}{r^r\prod_{j=1}^{n}p_j^{\alpha_j}}CT_0^r$$

from where:

$$\overline{CT}_0^r = \frac{\prod_{i=1}^{s}\overline{p}_i^{\alpha_i}}{\prod_{j=1}^{s}p_j^{\alpha_j}}CT_0^r$$

With the new total cost, the optimal amounts of inputs become:

$$\begin{cases} x_{int,k} = \dfrac{\alpha_k\left(\prod_{i=1}^{s}\overline{p}_i^{\alpha_i}\right)^{1/r}CT_0}{r\overline{p}_k\left(\prod_{j=1}^{s}p_j^{\alpha_j}\right)^{1/r}}, k=\overline{1,s} \\ x_{int,d} = \dfrac{\alpha_d\left(\prod_{i=1}^{s}\overline{p}_i^{\alpha_i}\right)^{1/r}CT_0}{rp_d\left(\prod_{j=1}^{s}p_j^{\alpha_j}\right)^{1/r}}, d=\overline{s+1,n} \end{cases}$$

The Hicks substitution effect which preserves the production is therefore:

$$\begin{cases} \Delta_{1H}x_k = x_{int,k} - x_{0,k} = \left[\dfrac{p_k}{\overline{p}_k}\prod_{i=1}^{s}\left(\dfrac{\overline{p}_i}{p_i}\right)^{\alpha_i/r} - 1\right]\dfrac{\alpha_k CT_0}{rp_k}, k=\overline{1,s} \\ \Delta_{1H}x_d = x_{int,d} - x_{0,d} = \left[\prod_{i=1}^{s}\left(\dfrac{\overline{p}_i}{p_i}\right)^{\alpha_i/r} - 1\right]\dfrac{\alpha_d CT_0}{rp_d}, d=\overline{s+1,n} \end{cases}$$

The difference caused by the old cost instead the new total cost one is therefore:

$$\begin{cases} \Delta_{2H}x_k = x_{f,k} - x_{int,k} = \left[1 - \prod_{i=1}^{s}\left(\dfrac{\overline{p}_i}{p_i}\right)^{\alpha_i/r}\right]\dfrac{\alpha_k CT_0}{r\overline{p}_k}, k=\overline{1,s} \\ \Delta_{2H}x_d = x_{f,d} - x_{int,d} = \left[1 - \prod_{i=1}^{s}\left(\dfrac{\overline{p}_i}{p_i}\right)^{\alpha_i/r}\right]\dfrac{\alpha_d CT_0}{rp_d}, d=\overline{s+1,n} \end{cases}$$

Let now calculate the new prices influence to the effects of substitution and of new cost in the Hicks effect.

We have:

$$\frac{\partial \Delta_{1H} x_k}{\partial \overline{p}_k} = -\frac{p_k}{\overline{p}_k^2} \prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \frac{\alpha_k CT_0}{r^2 p_k} \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i, \; k = \overline{1, s}$$

$$\frac{\partial \Delta_{2H} x_k}{\partial \overline{p}_k} = -\frac{\alpha_k CT_0}{r^2 \overline{p}_k^2} \left[\prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j - r \right], \; k = \overline{1, s}$$

$$\frac{\partial \Delta_{1H} x_k}{\partial \overline{p}_t} = \prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \frac{\alpha_k \alpha_t CT_0}{r^2 \overline{p}_k \overline{p}_t}, \; k = \overline{1, s}, \; t = \overline{1, s}, \; t \neq k$$

$$\frac{\partial \Delta_{2H} x_k}{\partial \overline{p}_t} = -\prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \frac{\alpha_k \alpha_t CT_0}{r^2 \overline{p}_k \overline{p}_t}, \; k = \overline{1, s}, \; t = \overline{1, s}, \; t \neq k$$

$$\frac{\partial \Delta_{1H} x_d}{\partial \overline{p}_k} = \prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \frac{\alpha_k \alpha_d CT_0}{r^2 p_d \overline{p}_k}, \; d = \overline{s+1, n}, \; k = \overline{1, s}$$

$$\frac{\partial \Delta_{2H} x_d}{\partial \overline{p}_k} = -\prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \frac{\alpha_d \alpha_k CT_0}{r^2 p_d \overline{p}_k}, \; d = \overline{s+1, n}, \; k = \overline{1, s}$$

After these relations, it follows that the effect of substitution at the increase of the price x_k, $k=\overline{1,s}$ is reduced, while the effect of new cost is reduced if $\left[\prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j - r \right] > 0$ or it increase if $\left[\prod_{i=1}^{s} \left(\frac{\overline{p}_i}{p_i} \right)^{\alpha_i / r} \sum_{\substack{j=1 \\ j \neq k}}^{n} \alpha_j - r \right] < 0$.

We shall apply now the Slutsky method for our analysis.

At the modify of the price of the factors $x_1,...,x_s$, the total cost for the same optimal combination of factors is:

$$CT_{int} = \frac{CT_0}{r} \left(\sum_{j=s+1}^{n} \alpha_j + \sum_{i=1}^{s} \alpha_i \frac{\overline{p}_i}{p_i} \right)$$

therefore:

$$\begin{cases} x_{int,k} = \dfrac{\alpha_k CT_0}{r^2 \overline{p}_k}\left(\sum_{j=s+1}^{n}\alpha_j + \sum_{i=1}^{s}\alpha_i \dfrac{\overline{p}_i}{p_i}\right), k = \overline{1,s} \\ x_{int,d} = \dfrac{\alpha_d CT_0}{r^2 p_d}\left(\sum_{j=s+1}^{n}\alpha_j + \sum_{i=1}^{s}\alpha_i \dfrac{\overline{p}_i}{p_i}\right), d = \overline{s+1,n} \end{cases}$$

The appropriate production is:

$$Q_{int}\left(x_{int,1},...,x_{int,n}\right) = A\,\frac{\prod_{i=1}^{n}\alpha_i^{\alpha_i} CT_0^r}{r^{2r}\prod_{k=1}^{s}\overline{p}_k^{\alpha_k}\prod_{d=s+1}^{n}p_d^{\alpha_d}}\left(\sum_{j=s+1}^{n}\alpha_j + \sum_{i=1}^{s}\alpha_i \frac{\overline{p}_i}{p_i}\right)^r$$

The Slutsky substitution effect which not preserves the production is therefore:

$$\begin{cases} \Delta_{1S}x_k = x_{int,k} - x_{0,k} = \dfrac{\alpha_k CT_0}{r^2 \overline{p}_k}\left(\sum_{j=s+1}^{n}\alpha_j + \sum_{i=1}^{s}\alpha_i \dfrac{\overline{p}_i}{p_i} - r\dfrac{\overline{p}_k}{p_k}\right), k = \overline{1,s} \\ \Delta_{1S}x_d = x_{int,d} - x_{0,d} = \dfrac{\alpha_d CT_0}{r^2 p_d}\left(\sum_{j=s+1}^{n}\alpha_j + \sum_{i=1}^{s}\alpha_i \dfrac{\overline{p}_i}{p_i} - r\right), d = \overline{s+1,n} \end{cases}$$

and the difference caused by the old production instead the new production one is therefore:

$$\begin{cases} \Delta_{2S}x_k = x_{f,k} - x_{int,k} = \dfrac{\alpha_k CT_0}{r^2 \overline{p}_k}\sum_{i=1}^{s}\alpha_i\left(1 - \dfrac{\overline{p}_i}{p_i}\right), k = \overline{1,s} \\ \Delta_{2S}x_d = x_{f,d} - x_{int,d} = \dfrac{\alpha_d CT_0}{r^2 p_d}\sum_{i=1}^{s}\alpha_i\left(1 - \dfrac{\overline{p}_i}{p_i}\right), d = \overline{s+1,n} \end{cases}$$

Let us now calculate the influence of the new prices on the effects of substitution and new cost in Slutsky effect.

We have:

$$\frac{\partial \Delta_{1S}x_k}{\partial \overline{p}_k} = -\frac{\alpha_k CT_0}{r^2 \overline{p}_k^2}\left(\sum_{j=s+1}^{n}\alpha_j + \sum_{\substack{i=1 \\ i\neq k}}^{s}\alpha_i \frac{\overline{p}_i}{p_i}\right), k = \overline{1,s}$$

$$\frac{\partial \Delta_{2S}x_k}{\partial \overline{p}_k} = -\frac{\alpha_k CT_0}{r^2 \overline{p}_k^2}\left(\alpha_k + \sum_{\substack{i=1 \\ i\neq k}}^{s}\alpha_i\left(1 - \frac{\overline{p}_i}{p_i}\right)\right), k = \overline{1,s}$$

$$\frac{\partial \Delta_{1S} x_k}{\partial \overline{p}_t} = \frac{\alpha_k \alpha_t CT_0}{r^2 \overline{p}_k p_t}, k = \overline{1,s}, t = \overline{1,s}, t \neq k$$

$$\frac{\partial \Delta_{2S} x_k}{\partial \overline{p}_t} = -\frac{\alpha_k \alpha_t CT_0}{r^2 \overline{p}_k p_t}, k = \overline{1,s}, t = \overline{1,s}, t \neq k$$

$$\frac{\partial \Delta_{1S} x_d}{\partial \overline{p}_k} = \frac{\alpha_d \alpha_k CT_0}{r^2 p_d p_k}, d = \overline{s+1,n}, k = \overline{1,s}$$

$$\frac{\partial \Delta_{2S} x_d}{\partial \overline{p}_k} = -\frac{\alpha_d \alpha_k CT_0}{r^2 p_d p_k}, d = \overline{s+1,n}, k = \overline{1,s}$$

Therefore the effect of substitution at a price increase of the factor x_k, k=$\overline{1,s}$ is reduced, while the effect due to production decrease in case $\alpha_k + \sum_{\substack{i=1 \\ i \neq k}}^{s} \alpha_i \left(1 - \frac{\overline{p}_i}{p_i}\right)$>0 or it increase if $\alpha_k + \sum_{\substack{i=1 \\ i \neq k}}^{s} \alpha_i \left(1 - \frac{\overline{p}_i}{p_i}\right)$<0.

6. Production Efficiency of Cobb-Douglas Production Function

Let now two Cobb-Douglas production functions for two goods Φ, Ψ and a number of n inputs $F_1,...,F_n$ available in quantities $\overline{x}_1,,.\overline{x}_n$. The production functions of Φ or Ψ are:

$$Q_\Phi(x_1,..., x_n) = Ax_1^{\alpha_1}...x_n^{\alpha_n},\ Q_\Psi(x_1,..., x_n) = Bx_1^{\beta_1}...x_n^{\beta_n}$$

appropriate to the consumption of x_k units of factor F_k, k=$\overline{1,n}$.

We have seen that: $\eta_{\phi,x_i} = A\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i - 1}...x_n^{\alpha_n}$, $\eta_{\Psi,x_i} = B\beta_i x_1^{\beta_1}...x_i^{\beta_i - 1}...x_n^{\beta_n}$, i=$\overline{1,n}$.

The production contract curve satisfies:

$$\frac{\eta_{\phi,x_i}}{\eta_{\Psi,x_i}} = \frac{A\alpha_i x_1^{\alpha_1}...x_i^{\alpha_i - 1}...x_n^{\alpha_n}}{B\beta_i(\overline{x}_1 - x_1)^{\beta_1}...(\overline{x}_i - x_i)^{\beta_i - 1}...(\overline{x}_n - x_n)^{\beta_n}} = \mu,\ i=\overline{1,n}$$

Dividing for i≠j: $x_j = \frac{\alpha_j\beta_i\bar{x}_j x_i}{(\alpha_j\beta_i - \alpha_i\beta_j)x_i + \alpha_i\beta_j\bar{x}_i}$ and for i=1: $x_j = \frac{\alpha_j\beta_1\bar{x}_j x_1}{(\alpha_j\beta_1 - \alpha_1\beta_j)x_1 + \alpha_1\beta_j\bar{x}_1}$, j=$\overline{2,n}$. Finally, for $x_i=\lambda$ we have the equation of production contract curve:

$$\begin{cases} x_1 = \lambda \\ x_j = \frac{\alpha_j\beta_1\bar{x}_j\lambda}{(\alpha_j\beta_1 - \alpha_1\beta_j)\lambda + \alpha_1\beta_j\bar{x}_1}, \lambda \in \mathbf{R} \end{cases}$$

If we consider now the input prices: $p_1,...,p_n$ we have that for the production contract curve: $x_1=g_1(\lambda),...,x_n=g_n(\lambda)$, $\lambda\in\mathbf{R}$:

$$\begin{cases} x_1 = g_1(\lambda) = \lambda \\ x_j = g_j(\lambda) = \frac{\alpha_j\beta_1\bar{x}_j\lambda}{(\alpha_j\beta_1 - \alpha_1\beta_j)\lambda + \alpha_1\beta_j\bar{x}_1}, \lambda \in \mathbf{R} \end{cases}$$

and:

$$p_j = \frac{\eta_{\Phi,x_j}(g_1(\lambda),...,g_n(\lambda))}{\eta_{\Phi,x_1}(g_1(\lambda),...,g_n(\lambda))}\nu = \frac{A\alpha_j g_1^{\alpha_1}(\lambda)...g_j^{\alpha_j-1}(\lambda)...g_n^{\alpha_n}(\lambda)}{A\alpha_1 g_1^{\alpha_1-1}(\lambda)g_2^{\alpha_2}(\lambda)...g_n^{\alpha_n}(\lambda)}\nu = \frac{\alpha_j g_1(\lambda)}{\alpha_1 g_j(\lambda)}\nu = \frac{(\alpha_j\beta_1 - \alpha_1\beta_j)\lambda + \alpha_1\beta_j\bar{x}_1}{\alpha_1\beta_1\bar{x}_j}\nu$$

, j=$\overline{1,n}$.

For ν=1 we then obtain: $p_1=1$, $p_j = \frac{(\alpha_j\beta_1 - \alpha_1\beta_j)\lambda + \alpha_1\beta_j\bar{x}_1}{\alpha_1\beta_1\bar{x}_j}$, j=$\overline{2,n}$.

If the initial allocation of factors of production was $x_\Phi = (a_1,...,a_n)$ we have that $\sum_{j=1}^{n} p_j(a_j - x_j) = 0$ therefore:

$$\lambda^* = \frac{a_1\alpha_1\beta_1 + \alpha_1\bar{x}_1\sum_{j=2}^{n}\frac{a_j\beta_j}{\bar{x}_j}}{\alpha_1\beta_1 - \sum_{j=2}^{n}\frac{a_j\alpha_j\beta_1 - a_j\alpha_1\beta_j - \alpha_j\beta_1\bar{x}_j}{\bar{x}_j}} = \frac{a_1\alpha_1\beta_1 + \alpha_1\bar{x}_1\sum_{j=2}^{n}\frac{a_j\beta_j}{\bar{x}_j}}{r_1\beta_1 - \sum_{j=2}^{n}\frac{a_j(\alpha_j\beta_1 - \alpha_1\beta_j)}{\bar{x}_j}}$$ where $r_1=\sum_{k=1}^{n}\alpha_k$.

For this value we find now the final allocation: $\begin{cases} x_1 = \lambda^* \\ x_j = \frac{\alpha_j\beta_1\bar{x}_j\lambda^*}{(\alpha_j\beta_1 - \alpha_1\beta_j)\lambda^* + \alpha_1\beta_j\bar{x}_1} \end{cases}$

If now the two production functions are: $Q_\Phi(K,L)=AK^{\alpha_1}L^{\alpha_2}$, $Q_\Psi(K,L)=AK^{\beta_1}L^{\beta_2}$ we have that for limited quantities of capital $(\overline{K})$ and labor $(\overline{L})$ the equation of production contract curve is:

$$\begin{cases} K=\lambda \\ L=\dfrac{\alpha_2\beta_1\overline{L}\lambda}{(\alpha_2\beta_1-\alpha_1\beta_2)\lambda+\alpha_1\beta_2\overline{K}}, \lambda\in\mathbf{R} \end{cases}$$

and the final allocation for an initial one: $x_\Phi=(K_1,L_1)$:

$$\begin{cases} K=\lambda^* \\ L=\dfrac{\alpha_2\beta_1\overline{L}\lambda^*}{(\alpha_2\beta_1-\alpha_1\beta_2)\lambda+\alpha_1\beta_2\overline{K}} \end{cases} \text{ where } \lambda^*=\frac{K_1\overline{L}\alpha_1\beta_1+\alpha_1\overline{K}L_1\beta_2}{(\alpha_1+\alpha_2)\beta_1\overline{L}-L_1(\alpha_2\beta_1-\alpha_1\beta_2)}.$$

7. The Concrete Determination of the Cobb-Douglas Production Function

Considering an affine function: $f:\mathbf{R}^n\to\mathbf{R}$, $f(x_1,...,x_n)=\beta_1x_1+...+\beta_nx_n+\beta_{n+1}$ and a set of m>n+1 data: $(x_1^k,...,x_n^k,f^k)$, $k=\overline{1,m}$ the problem of determining β_i, $i=\overline{1,n+1}$ using the least square method is to minimize the expression: $\sum_{k=1}^{m}(\beta_1x_1^k+...+\beta_nx_n^k+\beta_{n+1}-f^k)^2$ that is to solve the system:

$$\begin{cases} \beta_1\sum_{k=1}^{m}x_1^kx_i^k+...+\beta_n\sum_{k=1}^{m}x_n^kx_i^k+\beta_{n+1}\sum_{k=1}^{m}x_i^k=\sum_{k=1}^{m}f^kx_i^k, i=\overline{1,n} \\ \beta_1\sum_{k=1}^{m}x_1^k+...+\beta_n\sum_{k=1}^{m}x_n^k+m\beta_{n+1}=\sum_{k=1}^{m}f^k \end{cases}$$

Considering the matrix:

$$\Theta=\begin{pmatrix} \sum_{k=1}^{m}(x_1^k)^2 & \sum_{k=1}^{m}x_1^kx_2^k & ... & \sum_{k=1}^{m}x_1^k \\ \sum_{k=1}^{m}x_1^kx_2^k & \sum_{k=1}^{m}(x_2^k)^2 & ... & \sum_{k=1}^{m}x_2^k \\ ... & ... & ... & ... \\ \sum_{k=1}^{m}x_1^k & \sum_{k=1}^{m}x_2^k & ... & m \end{pmatrix}$$

and Θ_{ij} the cofactor of the (i,j)-element in Θ we will obtain:

$$\beta_i = \frac{\Theta_{1i}\sum_{k=1}^{m} f^k x_1^k + \ldots + \Theta_{ni}\sum_{k=1}^{m} f^k x_n^k + \Theta_{n+1,i}\sum_{k=1}^{m} f^k}{\det\Theta}, i = \overline{1,n+1}$$

Considering now a production function $Q(x_1,...,x_n)= Ax_1^{\alpha_1}...x_n^{\alpha_n}$ we put the problem of concrete determination of the parameters A, α_i, i=$\overline{1,n}$.

Let therefore a set of m>n+1 data: $\left(x_1^k,...,x_n^k,Q^k\right)$, k=$\overline{1,m}$.

Considering the logarithm of Q, we have: $\ln Q = \alpha_1 \ln x_1 + \ldots + \alpha_n \ln x_n + \ln A$ therefore we will modify the data set to the new one: $\left(\ln x_1^k,...,\ln x_n^k, \ln Q^k\right)$, k=$\overline{1,m}$.

From above:

$$\alpha_i = \frac{\Theta_{1i}\sum_{k=1}^{m}\ln Q^k \ln x_1^k + \ldots + \Theta_{ni}\sum_{k=1}^{m}\ln Q^k \ln x_n^k + \Theta_{n+1,i}\sum_{k=1}^{m}\ln Q^k}{\det\Theta}, i = \overline{1,n}$$

$$\ln A = \frac{\Theta_{1,n+1}\sum_{k=1}^{m}\ln Q^k \ln x_1^k + \ldots + \Theta_{n,n+1}\sum_{k=1}^{m}\ln Q^k \ln x_n^k + \Theta_{n+1,n+1}\sum_{k=1}^{m}\ln Q^k}{\det\Theta}$$

where $\Theta = \begin{pmatrix} \sum_{k=1}^{m}\left(\ln x_1^k\right)^2 & \sum_{k=1}^{m}\ln x_1^k \ln x_2^k & \ldots & \sum_{k=1}^{m}\ln x_1^k \\ \sum_{k=1}^{m}\ln x_1^k \ln x_2^k & \sum_{k=1}^{m}\left(\ln x_2^k\right)^2 & \ldots & \sum_{k=1}^{m}\ln x_2^k \\ \ldots & \ldots & \ldots & \ldots \\ \sum_{k=1}^{m}\ln x_1^k & \sum_{k=1}^{m}\ln x_2^k & \ldots & m \end{pmatrix}$.

For the Cobb-Douglas $Q=Q(K,L)=AK^{\alpha}L^{\beta}$ we have therefore, for the set: $\left(K_i, L_i, Q_i\right)_{i=\overline{1,m}}$:

$$\alpha = \frac{\begin{vmatrix} \sum_{i=1}^{m} \ln Q_i \ln K_i & \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln K_i \\ \sum_{i=1}^{m} \ln Q_i \ln L_i & \sum_{i=1}^{m} \ln^2 L_i & \sum_{i=1}^{m} \ln L_i \\ \sum_{i=1}^{m} \ln Q_i & \sum_{i=1}^{m} \ln L_i & m \end{vmatrix}}{\begin{vmatrix} \sum_{i=1}^{m} \ln^2 K_i & \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln K_i \\ \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln^2 L_i & \sum_{i=1}^{m} \ln L_i \\ \sum_{i=1}^{m} \ln K_i & \sum_{i=1}^{m} \ln L_i & m \end{vmatrix}},$$

$$\beta = \frac{\begin{vmatrix} \sum_{i=1}^{m} \ln^2 K_i & \sum_{i=1}^{m} \ln Q_i \ln K_i & \sum_{i=1}^{m} \ln K_i \\ \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln Q_i \ln L_i & \sum_{i=1}^{m} \ln L_i \\ \sum_{i=1}^{m} \ln K_i & \sum_{i=1}^{m} \ln Q_i & m \end{vmatrix}}{\begin{vmatrix} \sum_{i=1}^{m} \ln^2 K_i & \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln K_i \\ \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln^2 L_i & \sum_{i=1}^{m} \ln L_i \\ \sum_{i=1}^{m} \ln K_i & \sum_{i=1}^{m} \ln L_i & m \end{vmatrix}},$$

$$\ln A = \frac{\begin{vmatrix} \sum_{i=1}^{m} \ln^2 K_i & \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln Q_i \ln K_i \\ \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln^2 L_i & \sum_{i=1}^{m} \ln Q_i \ln L_i \\ \sum_{i=1}^{m} \ln K_i & \sum_{i=1}^{m} \ln L_i & \sum_{i=1}^{m} \ln Q_i \end{vmatrix}}{\begin{vmatrix} \sum_{i=1}^{m} \ln^2 K_i & \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln K_i \\ \sum_{i=1}^{m} \ln K_i \ln L_i & \sum_{i=1}^{m} \ln^2 L_i & \sum_{i=1}^{m} \ln L_i \\ \sum_{i=1}^{m} \ln K_i & \sum_{i=1}^{m} \ln L_i & m \end{vmatrix}}$$

8. Conclusions

The above analysis reveals several aspects. On the one hand were highlighted conditions for the existence of the Cobb-Douglas function. Also were calculated the main indicators of its and short and long-term costs. It has also been studied the dependence of long-term cost of the parameters of the production function. The determination of profit was made both for perfect competition market and maximize its conditions. Also we have studied the effects of Hicks and Slutsky and the production efficiency problem.

9. Appendix

A.1. Mathematical concepts

A function $Q:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex set, is quasi-concave if:

$$Q(\lambda x+(1-\lambda)y)\geq\min(Q(x),Q(y))\ \forall\lambda\in[0,1]\ \forall x,y\in D$$

and is strictly quasi-concave if:

$$Q(\lambda x+(1-\lambda)y)>\min(Q(x),Q(y))\ \forall\lambda\in(0,1)\ \forall x,y\in D$$

A function $Q:D\subset\mathbf{R}^n\to\mathbf{R}$, D – convex set, is quasi-convex if:

$$Q(\lambda x+(1-\lambda)y)\leq\max(Q(x),Q(y))\ \forall\lambda\in[0,1]\ \forall x,y\in D$$

and is strictly quasi-convex if:

$$Q(\lambda x+(1-\lambda)y)<\max(Q(x),Q(y))\ \forall\lambda\in(0,1)\ \forall x,y\in D$$

Geometrically speaking, a quasi-concave function has the property to be above the lowest values recorded at the ends of some segment. This property is equivalent with the convexity of the set $Q^{-1}[a,\infty)=\{x\in D \mid Q(x)\geq a\}$ $\forall a\in\mathbf{R}$.

Note also that if f and g are arbitrary functions:

- f – quasi-concave (quasi-convex) implies that -f is quasi-convex (quasi-concave);
- f – strictly quasi-concave (quasi-convex) implies that f is quasi-concave (quasi-convex);
- f - quasi-concave (quasi-convex) implies that αf is quasi-concave (quasi-convex) for any $\alpha\geq 0$;
- f,g - quasi-concave (quasi-convex) imply that $\min(\alpha f,\beta g)$ ($\max(\alpha f,\beta g)$) is quasi-concave (quasi-convex) for any $\alpha,\beta\geq 0$;
- f - quasi-concave (quasi-convex) and $g:\mathbf{R}\to\mathbf{R}$ is increasing imply that $g\circ f:D\to\mathbf{R}$ is quasi-concave (quasi-convex);
- $f\in C^1(D)$ is (strictly) quasi-concave if and only if: $f(x)\geq f(y)$ $\Rightarrow \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(y)(x_i-y_i)\geq(>)0\ \ \forall x,y\in D$;
- $f\in C^1(D)$ is (strictly) quasi-convex if and only if: $f(x)\geq f(y)$ $\Rightarrow \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x)(x_i-y_i)\geq(>)0\ \ \forall x,y\in D$;
- A monotonically function $f:D\subset\mathbf{R}\to\mathbf{R}$ is quasi-concave and quasi-convex;
- Any affine function is quasi-concave and quasi-convex.

Considering now the bordered hessian matrix:

$$H^B(f)=\begin{pmatrix} 0 & f'_{x_1} & f'_{x_2} & \dots & f'_{x_n} \\ f'_{x_1} & f''_{x_1x_1} & f''_{x_1x_2} & \dots & f''_{x_1x_n} \\ f'_{x_2} & f''_{x_1x_2} & f''_{x_2x_2} & \dots & f''_{x_2x_n} \\ \dots & \dots & \dots & \dots & \dots \\ f'_{x_n} & f''_{x_1x_n} & f''_{x_2x_n} & \dots & f''_{x_nx_n} \end{pmatrix}$$

and the bordered principal diagonal determinants:

$$\Delta_k^B = \begin{vmatrix} 0 & f'_{x_1} & f'_{x_2} & \dots & f'_{x_k} \\ f'_{x_1} & f''_{x_1x_1} & f''_{x_1x_2} & \dots & f''_{x_1x_k} \\ f'_{x_2} & f''_{x_1x_2} & f''_{x_2x_2} & \dots & f''_{x_2x_k} \\ \dots & \dots & \dots & \dots & \dots \\ f'_{x_k} & f''_{x_1x_k} & f''_{x_2x_k} & \dots & f''_{x_kx_k} \end{vmatrix}, k=\overline{1,n}$$

we have the following theorems:

Theorem If the function $f:D\subset \mathbf{R}_+^n \to \mathbf{R}$, D – convex, $f\in C^2(D)$ is quasi-concave then $(-1)^k \Delta_k^B \ge 0$, $k=\overline{1,n}$.

Theorem In order that the function $f:D\subset \mathbf{R}_+^n \to \mathbf{R}$, D – convex, $f\in C^2(D)$ be quasi-concave is sufficient that $(-1)^k \Delta_k^B > 0$, $k=\overline{1,n}$.

A.2. The main indicators of production functions

Let a production function:

$Q:D_p\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+$ $\forall(x_1,...,x_n)\in D_p$

We will call the marginal productivity relative to a production factor x_i: $\eta_{x_i} = \frac{\partial Q}{\partial x_i}$ representing the trend of variation of production at the variation of the factor x_i. In particular, for a production function of the form: Q=Q(K,L) we have $\eta_K=\frac{\partial Q}{\partial K}$ - called the marginal productivity of capital and $\eta_L=\frac{\partial Q}{\partial L}$ - called the marginal productivity of labor.

We call the average productivity relative to a production factor x_i: $w_{x_i} = \frac{Q}{x_i}$ representing the value of production at the consumption of a unit of factor x_i. In particular, for a production function of the form: Q=Q(K,L) we have: $w_K=\frac{Q}{K}$ - called the productivity of capital, and $w_L=\frac{Q}{L}$ - the productivity of labor.

Considering the factors i and j with $i\ne j$, we define the restriction of production area: $P_{ij}=\{(x_1,...,x_n) \mid x_k=a_k=\text{const}, k=\overline{1,n}, k\ne i,j, x_i,x_j\in D_p\}$ relative to the two factors when the others have fixed values. Also, let: $D_{ij}=\{(x_i,x_j) \mid (x_1,...,x_n)\in P_{ij}\}$ - the domain of production relative to factors i and j.

We define: $Q_{ij}:D_{ij}\to\mathbf{R}_+$ - the restriction of the production function to the factors i and j, i.e.: $Q_{ij}(x_i,x_j)=Q(a_1,...,a_{i-1},x_i,a_{i+1},...,a_{j-1},x_j,a_{j+1},...,a_n)$. The functions Q_{ij} define a surface in $\mathbf{R}^3$ for every pair of factors (i,j).

We will call partial marginal rate of technical substitution of the factors i and j, relative to D_{ij} (caeteris paribus), the opposite change in the amount of factor j to substitute a variation of the quantity of factor i in the situation of conservation production level.

We will note: $RMS(i,j)=-\frac{dx_j}{dx_i}$ and we have, since $Q_{ij}(x_i,x_j)=Q_0=\text{constant}$: $RMS(i,j)=\frac{\eta_{x_i}\big|_{D_{ij}}}{\eta_{x_j}\big|_{D_{ij}}}$. Obviously $RMS(i,j)=\frac{1}{RMS(j,i)}$. We also define the global marginal rate of substitution between the i-th factor and the others: $RMS(i)=\frac{\eta_{x_i}}{\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{n}\eta_{x_j}^2}}$. The global marginal rate of technical substitution is the minimum (in the meaning of norm) of changes in consumption of factors so that the total production remain unchanged.

In particular, for a production function of the form: $Q=Q(K,L)$ we have: $RMS(K,L)=RMS(K)=\frac{\eta_K}{\eta_L}$, $RMS(L,K)=RMS(L)=\frac{\eta_L}{\eta_K}$.

It is called elasticity of production in relation to a production factor x_i: $\varepsilon_{x_i}=\frac{\frac{\partial Q}{\partial x_i}}{\frac{Q}{x_i}}=\frac{\eta_{x_i}}{w_{x_i}}$ - the relative variation of production at the relative variation of factor x_i.

In particular, for a production function of the form: Q=Q(K,L) we have $\varepsilon_K=\frac{\eta_K}{w_K}$ - called the elasticity of production in relation to the capital and $\varepsilon_L=\frac{\eta_L}{w_L}$ - the elasticity factor of production in relation to the labor.

Let note now for arbitrary factors x_i, x_j: $\xi_{ij}=\frac{x_i}{x_j}$, $i,j=\overline{1,n}$, $i\neq j$ and we call the factor endowment ratio with the factor i relative to factor j.

It is called the elasticity of marginal rate of technical substitution for a production function relative to inputs i and j: $\sigma_{ij}=\frac{\frac{\partial RMS(i,j)}{\partial \xi_{ij}}}{\frac{RMS(i,j)}{\xi_{ij}}}$, $i,j=\overline{1,n}$, $i\neq j$ and represents the relative variation of marginal rate of technical substitution relative to factors i and j at the relative variation of the factor endowment ratio with factor i relative to factor j.

We have therefore: $\sigma_{ij}=\frac{x_i\frac{\partial RMS(i,j)}{\partial x_i}}{RMS(i,j)}=x_i\frac{\partial \ln RMS(i,j)}{\partial x_i}$.

Considering now a production function $Q:D_p\to\mathbf{R}_+$, $(x_1,...,x_n)\to Q(x_1,...,x_n)\in\mathbf{R}_+$ $\forall(x_1,...,x_n)\in D_p$, homogenous of degree r, let note for an arbitrary factor (for example x_n): $\chi_i=\frac{x_i}{x_n}$, $i=\overline{1,n-1}$. Of course: $\xi_{ij}=\frac{\chi_i}{\chi_j}$.

We obviously have:

$$Q(x_1,...,x_n)=x_n^r Q\left(\frac{x_1}{x_n},...,\frac{x_{n-1}}{x_n},\frac{x_n}{x_n}\right)=x_n^r Q(\chi_1,...,\chi_{n-1},1)$$

Considering the restriction of the production function at $D_p\cap\mathbf{R}_+^{n-1}\times\{1\}$: $q(\chi_1,...,\chi_{n-1})=Q(\chi_1,...,\chi_{n-1},1)$ we can write:

$$Q(x_1,...,x_n)=x_n^r q(\chi_1,...,\chi_{n-1})$$

With the new function introduced, the above indicators are:

- $\eta_{x_i} = x_n^{r-1} \frac{\partial q}{\partial \chi_i}$, i=$\overline{1, n-1}$
- $\eta_{x_n} = x_n^{r-1} \left(rq - \sum_{i=1}^{n-1} \frac{\partial q}{\partial \chi_i} \chi_i \right)$
- $w_{x_i} = x_n^{r-1} \frac{q}{\chi_i}$, i=$\overline{1, n-1}$
- $w_{x_n} = x_n^{r-1} q$
- RMS(i,j)=$\frac{\frac{\partial q}{\partial \chi_i}}{\frac{\partial q}{\partial \chi_j}}$, i,j=$\overline{1, n-1}$
- RMS(i,n)=$\frac{\frac{\partial q}{\partial \chi_i}}{rq - \sum_{i=1}^{n-1} \frac{\partial q}{\partial \chi_i} \chi_i}$, i=$\overline{1, n-1}$
- RMS(i)=$\frac{\frac{\partial q}{\partial \chi_i}}{\sqrt{\left(rq - \sum_{j=1}^{n-1} \frac{\partial q}{\partial \chi_j} \chi_j \right)^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \left(\frac{\partial q}{\partial \chi_j} \right)^2}}$, i=$\overline{1, n-1}$
- RMS(n)= $\frac{rq - \sum_{j=1}^{n-1} \frac{\partial q}{\partial \chi_j} \chi_j}{\sqrt{\sum_{j=1}^{n-1} \left(\frac{\partial q}{\partial \chi_j} \right)^2}}$
- $\varepsilon_{x_i} = \frac{\frac{\partial q}{\partial \chi_i}}{\frac{q}{\chi_i}}$, i=$\overline{1, n-1}$
- $\varepsilon_{x_n} = \frac{rq - \sum_{i=1}^{n-1} \frac{\partial q}{\partial \chi_i} \chi_i}{q}$

- $\sigma_{ij}=\chi_i \dfrac{\dfrac{\partial^2 q}{\partial \chi_i^2}\dfrac{\partial q}{\partial \chi_j}-\dfrac{\partial q}{\partial \chi_i}\dfrac{\partial^2 q}{\partial \chi_i \partial \chi_j}}{\dfrac{\partial q}{\partial \chi_i}\dfrac{\partial q}{\partial \chi_j}}$
- $\sigma_{in}=\chi_i \dfrac{rq\dfrac{\partial^2 q}{\partial \chi_i^2}+(1-r)\left(\dfrac{\partial q}{\partial \chi_i}\right)^2+\dfrac{\partial q}{\partial \chi_i}\sum_{\substack{k=1\\k\neq i}}^{n-1}\dfrac{\partial^2 q}{\partial \chi_k \partial \chi_i}\chi_k-\dfrac{\partial^2 q}{\partial \chi_i^2}\sum_{\substack{k=1\\k\neq i}}^{n-1}\dfrac{\partial q}{\partial \chi_k}\chi_k}{\dfrac{\partial q}{\partial \chi_i}\left(rq-\sum_{k=1}^{n-1}\dfrac{\partial q}{\partial \chi_k}\chi_k\right)}$

For a production function of the form: Q=Q(K,L), $\chi=\dfrac{K}{L}$, $q(\chi)=Q(\chi,1)$:

- $\eta_K = L^{r-1}\dfrac{\partial q}{\partial \chi}$
- $\eta_L = L^{r-1}\left(rq-\dfrac{\partial q}{\partial \chi}\chi\right)$
- $w_K = L^{r-1}\dfrac{q}{\chi}$
- $w_L = L^{r-1}q$
- RMS(K,L)=RMS(K)= $\dfrac{\dfrac{\partial q}{\partial \chi}}{rq-\dfrac{\partial q}{\partial \chi}\chi}$
- $\varepsilon_K=\dfrac{\dfrac{\partial q}{\partial \chi}}{\dfrac{q}{\chi}}$
- $\varepsilon_L=\dfrac{rq-\dfrac{\partial q}{\partial \chi}\chi}{q}$

- $\sigma=\sigma_{KL}=\chi\dfrac{rq\dfrac{\partial^2 q}{\partial\chi^2}+(1-r)\left(\dfrac{\partial q}{\partial\chi}\right)^2}{\dfrac{\partial q}{\partial\chi}\left(rq-\dfrac{\partial q}{\partial\chi}\chi\right)}$

A.3. Necessary and sufficient conditions for nonlinear optimization

Considering now the non-linear programming problem:

$$\begin{cases}\max(\min)\ f(x_1,...,x_n)\\ g_i(x_1,...,x_n)\geq 0,\ i=\overline{1,p}\\ x_1,...,x_n\geq 0\end{cases}$$

where f,g$\in C^2(D_p)$ and a solution $\overline{x}=(\overline{x}_1,...,\overline{x}_n)$ the Karush-Kuhn-Tucker conditions occur: $\exists\lambda_i\in\mathbf{R}_+$, $i=\overline{1,p}$ so that:

$$\begin{cases}\varepsilon\nabla f(\overline{x}_1,...,\overline{x}_n)+\sum_{i=1}^{p}\lambda_i\nabla g_i(\overline{x}_1,...,\overline{x}_n)=0\\ g_i(\overline{x}_1,...,\overline{x}_n)\geq 0,\ i=\overline{1,p}\\ \lambda_i g_i(\overline{x}_1,...,\overline{x}_n)=0,\ i=\overline{1,p}\end{cases}$$

where ∇F is the gradient of F defined by: $\nabla F=\left(\dfrac{\partial F}{\partial x_1},...,\dfrac{\partial F}{\partial x_n}\right)$ and ε=1 for the case of maximizing and ε=-1 in the case of minimizing.

If f,g_i, $i=\overline{1,p}$ are of class C^2, from [1] follows, for the maximizing case, the sufficiency of Karush-Kuhn-Tucker conditions takes place in the broader framework of quasi-concavity of functions f and g and, moreover, if for a solution $\overline{x}=(\overline{x}_1,...,\overline{x}_n)$ one of the conditions occurs:

- $\exists k=\overline{1,n}$ such that $\dfrac{\partial f}{\partial x_k}(\overline{x})<0$;
- $\exists k=\overline{1,n}$ such that $\dfrac{\partial f}{\partial x_k}(\overline{x})>0$ and $\overline{x}_k>0$;
- $\nabla f\neq 0$;
- f is concave.

For the problem: $\begin{cases} \min f(x_1,...,x_n) \\ g_i(x_1,...,x_n) \geq 0, i = \overline{1,p} \\ x_1,...,x_n \geq 0 \end{cases}$

replacing f with –f and taking into account that $\min f(x_1,...,x_n) = -\max(-f(x_1,...,x_n))$ follows that Karush-Kuhn-Tucker conditions becomes:

$$\begin{cases} -\nabla f(\overline{x}_1,...,\overline{x}_n) + \sum_{i=1}^{p} \lambda_i \nabla g_i(\overline{x}_1,...,\overline{x}_n) = 0 \\ g_i(\overline{x}_1,...,\overline{x}_n) \geq 0, i = \overline{1,p} \\ \lambda_i g_i(\overline{x}_1,...,\overline{x}_n) = 0, i = \overline{1,p} \end{cases}, \lambda_i \in \mathbf{R}_+, i = \overline{1,p}$$

and sufficiency reduces to one of the cases:

- $\exists k = \overline{1,n}$ such that $\frac{\partial f}{\partial x_k}(\overline{x}) > 0$;
- $\exists k = \overline{1,n}$ such that $\frac{\partial f}{\partial x_k}(\overline{x}) < 0$ and $\overline{x}_k > 0$;
- $\nabla f \neq 0$;
- f is convex.

In the particular case of the problem of minimizing the total cost (TC) relative to a production function $Q=Q(x_1,...,x_n)$ and p_i, $i=\overline{1,n}$ - the prices of inputs: $\begin{cases} \min \sum_{k=1}^{n} p_k x_k \\ Q(x_1,...,x_n) - Q_0 \geq 0 \\ x_1,...,x_n \geq 0 \end{cases}$ the Karush-Kuhn-Tucker conditions are:

$$\begin{cases} -p_k + \lambda \frac{\partial Q}{\partial x_k}(\overline{x}_1,...,\overline{x}_n) = 0, k = \overline{1,n} \\ Q(x_1,...,x_n) \geq Q_0 \\ \lambda(Q(x_1,...,x_n) - Q_0) = 0 \end{cases}$$

. Because $\lambda=0$ implies $p_k=0$ – which is absurd in economic terms, results: $\lambda \neq 0$ then: $\begin{cases} \lambda \dfrac{\partial Q}{\partial x_k}(\overline{x}_1,..., \overline{x}_n) = p_k, k = \overline{1,n} \\ Q(x_1,..., x_n) = Q_0 \end{cases}$ or, with another expression: $\begin{cases} \dfrac{\frac{\partial Q}{\partial x_1}(\overline{x}_1,..., \overline{x}_n)}{p_1} = ... = \dfrac{\frac{\partial Q}{\partial x_n}(\overline{x}_1,..., \overline{x}_n)}{p_n} \\ Q(x_1,..., x_n) = Q_0 \end{cases}$. Because the objective function $f(x_1,...,x_n) = \sum_{k=1}^{n} p_k x_k$ is affine, Q is quasi-concave and, in addition $\frac{\partial f}{\partial x_k}(\overline{x})$=$p_k$>0 follows, from the foregoing, that these conditions are sufficient.

<u>A.4. Production efficiency</u>

Let us consider in the following two goods Φ, Ψ and a number of n inputs F_1,...,F_n available in quantities $\overline{x}_1,,,.\overline{x}_n$, and the production functions of Φ or Ψ as follows:

$$Q = Q_\Phi(x_1,...,x_n),\ Q = Q_\Psi(x_1,...,x_n)$$

appropriate to the consumption of x_k units of factor F_k, k=$\overline{1,n}$. We will assume that the production functions are of class C^2 inside space production SP.

We will build the Edgeworth's box consisting in a n-dimensional parallelepiped: $[0, \overline{x}_1] \times ... \times [0, \overline{x}_n]$ the quantities of Φ being relative to $O(0,...,0)$ and those of Ψ relative to $F(\overline{x}_1,,,.\overline{x}_n)$ on the parallelepiped sides. Let consider an initial allocation of inputs for Φ and Ψ:

$$x_\Phi = (a_1,...,a_n),\ x_\Psi = (b_1,...,b_n)$$

where a_i+b_i=$\overline{x}_i$, i=$\overline{1,n}$. The productions appropriate to the initial allocation are: $Q_{\Phi,0}(a_1,...,a_n)$, $Q_{\Psi,0}(b_1,...,b_n)$ relative to O and F, respectively. Because b_i=$\overline{x}_i$-a_i, i=$\overline{1,n}$ we have: $Q_{\Psi,0}(b_1,...,b_n)$= $Q_\Psi(\overline{x}_1 - a_1,...,\overline{x}_n - a_n)$. The production function of Ψ is therefore: $\hat{Q}_\Psi = Q_\Psi(\overline{x}_1 - x_1,..., \overline{x}_n - x_n)$ and means the production of Ψ relative to the origin of axes. We have now: $\frac{\partial \hat{Q}_\Psi}{\partial x_i} = -\frac{\partial Q_\Psi}{\partial x_i}$, $\frac{\partial^2 \hat{Q}_\Psi}{\partial x_i \partial x_j} = \frac{\partial^2 Q_\Psi}{\partial x_i \partial x_j}$, i,j=$\overline{1,n}$ therefore $\hat{Q}_\Psi$ is also quasi-concave but with negative partial derivatives of order 1. Considering the isoproduction hypersurfaces, it follows that (relative to O) those of Φ is convex, while that of $\hat{\Psi}$ is concave.

Let $PZ_{\Phi,0} = \{(x_1,..., x_n) \in SP | Q_\Phi(x_1,..., x_n) \geq Q_{\Phi,0}\}$ - the production zone of Φ superior to $Q_{\Phi,0}$ and $PZ_{\Psi,0} = \{(x_1,..., x_n) \in SP | Q_\Psi(x_1,..., x_n) \geq Q_{\Psi,0}\}$ - the production zone of Ψ superior to $Q_{\Psi,0}$.

Suppose now that $\text{int}(PZ_{\Phi,0} \cap PZ_{\Psi,0}) \neq \emptyset$ (*int means the interior of the set, i.e. those points for which there is a n-dimensional cube centered in them sufficiently small side and included in the given set*).

Let now a point $C(c_1,...,c_n) \in \text{int}(PZ_{\Phi,0} \cap PZ_{\Psi,0})$ and also let the straight line that passes through the origin and C. Let note $D(d_1,...,d_n)$ the intersection with the isoproduction hypersurface $Q_\Phi(x_1,...,x_n) = Q_{\Phi,0}$ and $E(e_1,...,e_n)$ the intersection with the isoproduction hypersurface $\hat{Q}_\Psi(x_1,...,x_n) = \hat{Q}_{\Psi,0}$. We have now $Q_\Phi(d_1,...,d_n) = Q_{\Phi,0}$ and $\hat{Q}_\Psi(e_1,...,e_n) = \hat{Q}_{\Psi,0}$. Because Φ is convex we obtain that: $Q_\Phi(d_1,...,d_n) < Q_\Phi(c_1,...,c_n)$ and $\hat{Q}_\Psi$ - concave implies that $\hat{Q}_\Psi(e_1,..., e_n) > \hat{Q}_\Psi(c_1,..., c_n)$ or $Q_\Psi(\overline{x}_1 - e_1,...,\overline{x}_n - e_n) > Q_\Psi(\overline{x}_1 - c_1,...,\overline{x}_n - c_n)$. After these inequalities follows that the production of each good can increase, so the initial allocation is not optimal.

We call Pareto's efficiency the situation where new production can not improve without affecting the other's production. From the foregoing, it follows that the Pareto's efficiency is obtained if the isoproduction hypersurfaces are tangent.

The condition of tangency for Q=$Q_\Phi(x_1,...,x_n)$ and Q=$\hat{Q}_\Psi(x_1,..., x_n)$= $Q_\Psi(\overline{x}_1 - x_1,...,\overline{x}_n - x_n)$ is reduced to the determination of those points $(x_1,...,x_n)$ where $\frac{\partial Q_\Phi}{\partial x_i} = \lambda \frac{\partial \hat{Q}_\Psi}{\partial x_i}$, i=$\overline{1,n}$, $\lambda \in \mathbf{R}$ i.e. those points where hypersurfaces intersect and have the same tangent hyperplane (*directors parameters are proportional*). Taking into account that $\hat{Q}_\Psi(x_1,..., x_n)$= $Q_\Psi(\overline{x}_1 - x_1,...,\overline{x}_n - x_n)$ we have that: $\frac{\partial Q_\Phi}{\partial x_i} = \mu \frac{\partial Q_\Psi}{\partial x_i}$, i=$\overline{1,n}$, $\mu=-\lambda \in \mathbf{R}$.

In marginal notation, we have: $\eta_{\Phi,i}(x_1,...,x_n) = \mu\eta_{\Psi,i}(\overline{x}_1 - x_1,...,\overline{x}_n - x_n)$, i=$\overline{1,n}$, $\mu \in \mathbf{R}$.

For two inputs (K and L) the above relations are equivalent with: $\frac{\eta_{\Phi,K}}{\eta_{\Phi,L}} = \frac{\eta_{\Psi,K}}{\eta_{\Psi,L}}$.

On the other hand: $\frac{\eta_{\Phi,K}}{\eta_{\Phi,L}} = \frac{dL}{dK}|_{\Phi} = RMS_{\Phi}(K,L)$ - marginal rate of technical substitution of capital for Φ and $\frac{\eta_{\Psi,K}}{\eta_{\Psi,L}} = \frac{dL}{dK}|_{\Psi} = RMS_{\Psi}(K,L)$ - marginal rate of technical substitution of capital for Ψ. The upper equality becomes: $RMS_{\Phi}(K,L) = RMS_{\Psi}(K,L)$.

All of the points where the allocation is Pareto's efficient generates the production contract curve.

Contract curve represents all combinations of goods for which no party can maximize its production without diminishing the other's production. On the other hand, any point on the curve represents a possible allocation contracts. The problem is this: if one good will be produced in order to reach the maximum level, what will do the other?

Considering now the prices of n inputs as $p_1,...,p_n$ the total cost is: TC=$\sum_{i=1}^{n} p_i x_i$ and it maximize the production if it is tangent to the isoproduction hypersurface. But each good want to be produced in maximum quantity therefore:

$$\frac{\eta_{\Phi,1}(x_1,...,x_n)}{p_1} = ... = \frac{\eta_{\Phi,n}(x_1,...,x_n)}{p_n},$$
$$\frac{\eta_{\Psi,1}(\overline{x}_1 - x_1,...,\overline{x}_n - x_n)}{p_1} = ... = \frac{\eta_{\Psi,n}(\overline{x}_1 - x_1,...,\overline{x}_n - x_n)}{p_n}$$

or, in other words, the cost hyperplane will be tangent to both isoproduction hypersurfaces, that is it will coincide with the common tangent hyperplane.

Considering the production contract curve of the form:

$$x_1=g_1(\lambda),...,x_n=g_n(\lambda),\ \lambda\in\mathbf{R}$$

follows:

$$\frac{\eta_{\phi,1}(g_1(\lambda),...,g_n(\lambda))}{p_1} = ... = \frac{\eta_{\phi,n}(g_1(\lambda),...,g_n(\lambda))}{p_n}$$

from where:

$$p_k = \frac{\eta_{\Phi,k}(g_1(\lambda),...,g_n(\lambda))}{\eta_{\Phi,1}(g_1(\lambda),...,g_n(\lambda))}\nu \text{ , } \nu>0\text{, } k=\overline{1,n}$$

We note that prices are determined up to a multiplicative factor, which does not affect the result of the problem and can therefore consider $\nu=1$. If the initial allocation of factors of production was $x_\Phi = (a_1,...,a_n)$, $x_\Psi = (b_1,...,b_n)$ the total cost of production of Φ is $TC_\Phi = \sum_{k=1}^{n} p_k a_k$. The new amounts of factors (which also satisfy the same total cost) involves: $\sum_{k=1}^{n} p_k (a_k - x_k) = 0$. Replacing the values of p_k into this equation:

$$\sum_{k=1}^{n} \frac{\eta_{\Phi,k}(g_1(\lambda),...,g_n(\lambda))}{\eta_{\Phi,1}(g_1(\lambda),...,g_n(\lambda))}(a_k - g_k(\lambda)) = 0$$

hence we will find $\lambda \in \mathbf{R}$. Substituting in the appropriate expressions, will result p_k and x_k, $k=\overline{1,n}$.

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